3/4-Fractional superdiffusion of energy in a harmonic chain with bulk noise

Cédric Bernardin

(joint work with P. Gonçalves and M. Jara)

University of Nice, France

June, 2014

Motivation

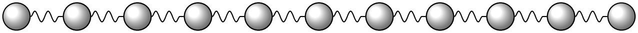
Prepare a macroscopic system at initial time with an inhomogeneous temperature $T_0(x)$. At some macroscopic time t, we expect that the temperature $T_t(x)$ at x is given by the solution of the **heat equation** (Fourier, 1822):

$$\partial_t T = \nabla[\kappa(T)\nabla T].$$

$\kappa(T)$ *is the thermal conductivity.*

- It turns out that one dimensional systems (e.g. carbon nanotubes) can display anomalous energy diffusion *if momentum is conserved*. The heat equation is no longer valid: the conductivity is infinite, energy current correlation function is not integrable...
- What shall replace the heat equation?

Microscopic models



Standard microscopic models of heat conduction are given by very long (=infinite) chains of coupled oscillators, i.e. infinite dimensional Hamiltonian system with Hamiltonian

$$\mathcal{H} = \sum_{x \in \mathbb{Z}} \left\{ \frac{p_x^2}{2} + V(r_x) \right\}, \quad r_x = q_{x+1} - q_x .$$

Conserved quantities

Conserved quantities

Conserved quantities:

Conserved quantities

Conserved quantities:

1. The energy $\mathcal{H} = \sum_x e_x, \quad e_x = \frac{p_x^2}{2} + V(r_x)$,

Conserved quantities

Conserved quantities:

1. The energy $\mathcal{H} = \sum_x e_x, \quad e_x = \frac{p_x^2}{2} + V(r_x)$,
2. The total momentum $\sum_x p_x$,

Conserved quantities

Conserved quantities:

1. The energy $\mathcal{H} = \sum_x e_x, \quad e_x = \frac{p_x^2}{2} + V(r_x)$,
2. The total momentum $\sum_x p_x$,
3. The compression of the chain $\sum_x r_x = \sum_x (q_{x+1} - q_x)$.

The problem of the existence (or not) of other conserved quantities is a highly challenging problem (ergodic problem).

Hydrodynamics: Euler equations

It is expected that in a Euler time scale the empirical energy $\mathfrak{e}(t, x)$, the empirical momentum $\mathfrak{p}(t, x)$ and the empirical compression $\mathfrak{r}(t, x)$ are given by a system of compressible Euler equations (hyperbolic system of conservation laws):

$$\begin{cases} \partial_t \mathfrak{r} = \partial_x \mathfrak{p}, \\ \partial_t \mathfrak{p} = \partial_x \tau, \\ \partial_t \mathfrak{e} = \partial_x (\mathfrak{p} \tau), \end{cases} \qquad \tau := \tau(\mathfrak{r}, \mathfrak{e} - \tfrac{\mathfrak{p}^2}{2}).$$

This can be proved rigorously if the ergodic problem (precisely formulated) can be solved (before the shocks).

Some theoretical approaches

Apart from a huge amount of numerical simulations (see Dhar's review), there are various theoretical approaches to predict the time decay of total energy current correlation function $C(t)$:

- *Renormalization Group analysis* (Narayan-Ramaswamy'02). $C(t) \sim t^{-2/3}$.
- *Mode Coupling Theory* (Delfini-Lepri-Livi-Politi'06): $C(t) \sim t^{-2/3}$ (asymmetric potentials) and $C(t) \sim t^{-1/2}$ (symmetric potentials).
- *Kinetic Theory* (Pereverzev'03, Lukkarinen-Spohn'07): $C(t) \sim t^{-3/5}$ (for FPU β).

Nonlinear fluctuating hydrodynamics predictions

Recently, Spohn (following van Beijeren) developed a theory of *non-linear fluctuating hydrodynamics* (NFH) to predict the behavior of the long time behavior of the time-space correlation functions of *all* the conserved fields $g(x,t) = (r_x(t), p_x(t), e_x(t))$

$$S_{\alpha\alpha'}(x,t) = \langle g_\alpha(x,t) g_{\alpha'}(0,0)\rangle_{\tau,\beta} - \langle g_\alpha\rangle_{\tau,\beta}\langle g_{\alpha'}\rangle_{\tau,\beta}$$

where $\langle\cdot\rangle_{\tau,\beta}$ is the (product) equilibrium Gibbs measure at temperature β^{-1} and pressure τ

$$\langle\cdot\rangle_{\tau,\beta} \sim \exp\{-\beta\sum_x (e_x + \tau r_x)\} drdp.$$

Nonlinear fluctuating hydrodynamics predictions

- The long time behavior of the correlation functions of the conserved fields depends on explicit relations between thermodynamic parameters (KPZ universality class and others).
- It is a *macroscopic* theory based on the validity of the hydrodynamics in the Euler time scale after some corse-graining procedure.
- Mutatis mutandis, it can be applied for any conservative model whose conserved fields evolve in the Euler time scale according to a system of $n = 2, 3 \ldots$ conservation laws. Similar universality classes appear.

Harmonic chain with bulk noise

- A proof of such predictions starting from stochastic Euler equations or from Hamiltonian microscopic dynamics are out of the range of actual mathematical techniques.
- Following ideas of [Olla-Varadhan-Yau'93] and [Fritz-Funaki-Lebowitz'94] we consider chains of oscillators perturbed by a bulk stochastic noise such that in the hyperbolic time scale Euler equations are valid.

- We start with a harmonic chain $\{(r_x(t), p_x(t))\,;\, x \in \mathbb{Z}\}$ and we use an equivalent dynamical variable $\{\eta_x(t)\,;\, x \in \mathbb{Z}\}$ defined by

$$\eta_{2x} = p_x, \quad \eta_{2x+1} = r_x.$$

- Newton's equations are

$$d\eta_x = (\eta_{x+1} - \eta_{x-1})dt, \quad x \in \mathbb{Z}.$$

- **Noise:** On each bond $\{x, x+1\}$ we have a Poisson process (clock). All are independent. When the clock of $\{x, x+1\}$ rings, η_x is exchanged with η_{x+1}. The dynamics between two successive rings of the clocks is given by the Hamiltonian dynamics.

- We obtain in this way a Markov process which conserves the total energy

$$\mathcal{H} = \sum_{x\in\mathbb{Z}} e_x = \sum_{x\in\mathbb{Z}} \eta_x^2 = \sum_{x\in\mathbb{Z}} \left\{ \frac{p_x^2}{2} + \frac{r_x^2}{2} \right\}.$$

- The noise destroys the conservation of the momentum and the conservation of the compression field.
- Nevertheless, the "volume" field $\sum_x \eta_x$ is conserved.

- The energy $\sum_x \eta_x^2$ and the volume $\sum_x \eta_x$ are the **only** conserved quantities of the model (in a suitable sense which can be made precize).
- The Gibbs equilibrium measures $\langle \cdot \rangle_{\tau,\beta}$ are parameterized by two parameters $(\tau, \beta) \in \mathbb{R} \times [0, \infty)$ and are product of Gaussians

$$\langle \cdot \rangle_{\tau,\beta} \sim \exp\{-\beta \sum_x (\eta_x^2 + \tau \eta_x)\} d\eta.$$

Theorem (B., Stoltz'11)

In the Euler time scale, the empirical volume field $\mathfrak{v}(t,x)$ *and the empirical energy field* $\mathfrak{e}(t,x)$ *evolve according to*

$$\begin{cases} \partial_t \mathfrak{v} = 2\partial_x \mathfrak{v}, \\ \partial_t \mathfrak{e} = \partial_x \mathfrak{v}^2. \end{cases}$$

The theorem is clearly false without the presence of the noise.

- We define

$$S_t(x) = \left\langle \left(\eta_0(0)^2 - \tfrac{1}{\beta}\right)\left(\eta_t(x)^2 - \tfrac{1}{\beta}\right) \right\rangle_{\tau=0,\beta}$$

- The case $\tau \neq 0$ can be recovered by considering the dynamics

$$\tilde{\eta}_t(x) = \eta_t(x) - \tau.$$

Theorem (B., Gonçalves, Jara'14)

We have that for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} S_{tn^{3/2}}([nx]) = \tfrac{2}{\beta^2}\, P_t(x),$$

where $\{P_t(x); x \in \mathbb{R}, t \geq 0\}$ is the fundamental solution of the skew fractional heat equation

$$\partial_t u = -\tfrac{1}{\sqrt{2}}\big\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\big\}u.$$

- In fact, we can prove more: the limit of the energy fluctuation field is given by an *infinite dimensional fractional Ornstein Uhlenbeck (Gaussian) process*:

$$\partial_t \mathcal{E} = \mathcal{L}\mathcal{E}\, dt + \sqrt{2T}\left(-\Delta\right)^{3/8} \partial_t \mathcal{W}$$

$$\mathcal{L} = \frac{1}{\sqrt{2}}\left\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\right\}.$$

- These results confirm the predictions of the NFH/MCT for this particular case. The $\sqrt{2}$ is not available in the NFH but it is in the MCT.

- The proof can be adapted to chains of harmonic oscillators with a noise consisting to exchange n.n. momenta at independent random exponential times (3 conserved quantities; Basile, B., Olla'06 model).
- Then, the skew fractional Laplacian has to be replaced by the fractional Laplacian. This is because the two sound modes have opposite velocities and the two drift terms $\pm\nabla(-\Delta)^{1/4}$ annihilate each other.

Related works

- Fractional diffusion has been obtained starting from a linear kinetic phonons equation (Basile-Olla-Spohn'08, Jara-Komorowski-Olla'09).
- Delfini-Lepri-Livi-Mejia-Monasteiro-Politi '08 ... obtained also a fractional Laplacian by considering the NESS of a system of harmonic oscillators with energy conserving noise.
- More recently, Jara, Komorowski and Olla obtained similar results by a very different method (Wigner function). They don't have access to the fractional OU process but their method also work out of equilibrium.

We want to prove

Theorem (B., Gonçalves, Jara'14)

We have that for any $x \in \mathbb{R}$

$$\lim_{n\to\infty} S_{tn^{3/2}}([nx]) = \tfrac{2}{\beta^2}\, P_t(x),$$

where $\{P_t(x); x \in \mathbb{R}, t \geq 0\}$ *is the fundamental solution of the skew fractional heat equation*

$$\partial_t u = -\tfrac{1}{\sqrt{2}}\Big\{(-\Delta)^{3/4} - \nabla(-\Delta)^{1/4}\Big\}u.$$

Ideas of the proof ($\beta = 1$)

- The *energy fluctuation field* is defined as

$$\mathcal{E}_t^n(f) = \frac{1}{\sqrt{n}} \sum_{y \in \mathbb{Z}} f\left(\tfrac{y}{n}\right)\left(\eta_{tn^{3/2}}(y)^2 - \tfrac{1}{\beta}\right).$$

- The *quadratic field* is defined as

$$Q_t^n(h) = \frac{1}{n} \sum_{y \neq z \in \mathbb{Z}} h\left(\tfrac{y}{n}, \tfrac{z}{n}\right) \eta_{tn^{3/2}}(y)\eta_{tn^{3/2}}(z).$$

By Itô calculus,

$$d\mathcal{E}_t^n(f) \approx -2Q_t^n(f' \otimes \delta)dt + \frac{1}{\sqrt{n}}\mathcal{E}_t^n(f'')dt + \mathit{martingale}.$$

$$dQ_t^n(h) \approx Q_t^n\big(L_n h\big)\,dt - 2\mathcal{E}_t^n\big([\mathbf{e}\cdot\nabla h](x,x)\big)\,dt$$
$$+ \frac{2}{\sqrt{n}}Q_t^n\big(\partial_y h(x,x) \otimes \delta\big)\,dt + \mathit{martingale}.$$

where $(\varphi \otimes \delta)(x,y) = \varphi(x)\delta(x=y)$ (distribution) and $\mathbf{e} = (1,1)$.

The linear operator L_n is defined by

$$L_n h = n^{-1/2}\Delta h + 2n^{1/2}(\mathbf{e}\cdot\nabla)h.$$

By Itô calculus,

$$d\mathcal{E}_t^n(f) \approx -2Q_t^n(f' \otimes \delta)dt + \frac{1}{\sqrt{n}}\mathcal{E}_t^n(f'')dt + \mathit{martingale}.$$

$$dQ_t^n(h) \approx Q_t^n\big(L_n h\big)\,dt - 2\mathcal{E}_t^n\big([\mathbf{e}\cdot\nabla h](x,x)\big)\,dt$$

$$+ \frac{2}{\sqrt{n}}Q_t^n\big(\partial_y h(x,x) \otimes \delta\big)\,dt + \mathit{martingale}.$$

where $(\varphi \otimes \delta)(x,y) = \varphi(x)\delta(x=y)$ (distribution) and $\mathbf{e} = (1,1)$.

The linear operator L_n is defined by

$$L_n h = n^{-1/2}\Delta h + 2n^{1/2}(\mathbf{e}\cdot\nabla)h.$$

Choose h_n such that $L_n h_n = 2f' \otimes \delta$ and add the two equations.

Up to small terms and martingale terms, we get

$$d\mathcal{E}_t^n(f) \approx -2\mathcal{E}_t^n\big([\mathbf{e}\cdot\nabla h_n](x,x)\big)\,dt - dQ_t^n(h_n)$$

Integrate in time and use Cauchy-Schwarz inequality to show that $Q_t^n(h_n), Q_0^n(h_n)$ vanish as $n \to \infty$. Then, up to small terms and martingale terms,

$$\mathcal{E}_t^n(f) - \mathcal{E}_0^n(f) \approx -2\int_0^t \mathcal{E}_s^n\big([\mathbf{e}\cdot\nabla h_n](x,x)\big)\,ds$$

Up to small terms and martingale terms, we get

$$d\mathcal{E}_t^n(f) \approx -2\mathcal{E}_t^n\big([\mathbf{e}\cdot\nabla h_n](x,x)\big)\,dt - dQ_t^n(h_n)$$

Integrate in time and use Cauchy-Schwarz inequality to show that $Q_t^n(h_n), Q_0^n(h_n)$ vanish as $n \to \infty$. Then, up to small terms and martingale terms,

$$\mathcal{E}_t^n(f) - \mathcal{E}_0^n(f) \approx -2\int_0^t \mathcal{E}_s^n\big([\mathbf{e}\cdot\nabla h_n](x,x)\big)\,ds$$

Recall that $h_n := h_n(f)$ is the solution of

$$L_n h_n = n^{-1/2}\Delta h_n + 2n^{1/2}(\mathbf{e}\cdot\nabla)h_n = 2f' \otimes \delta$$

The equation for $\mathcal{E}_t^n(\cdot)$ is closed.

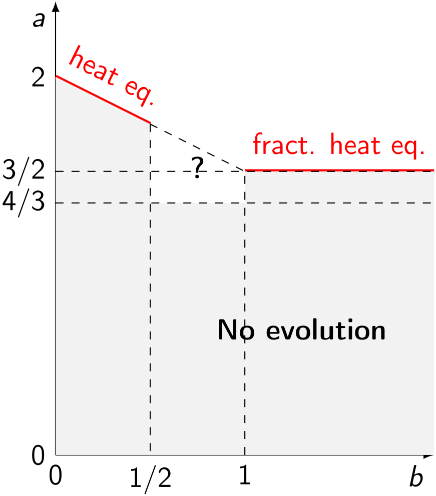
It remains only to show (by Fourier transform, it's easy) that

$$\lim_{n\to\infty}[\mathbf{e}\cdot\nabla h_n](x,x) = \frac{1}{\sqrt{2}}\left[(-\tfrac{d^2}{dx^2})^{3/4} - \tfrac{d}{dx}(-\tfrac{d^2}{dx^2})^{1/4}\right] f.$$

The evanescent flip noise limit

- Consider the same Markov process (harmonic chain + exchange noise) and add a second stochastic perturbation with intensity $\gamma_n = n^{-b}$, $b > 0$, which consists to flip independently on each site at Poissonian times the variable η_x into $-\eta_x$.
- The energy is conserved but the volume $\sum_x \eta_x$ is not (stricto sensu, only if $b = \infty$).
- We look at the system in the time scale tn^a, $a > 0$, such that the energy field has a non-trivial limit.

We have (B., Gonçalves, Jara, Sasada, Simon'14)



Conclusion

- We considered a harmonic chain with a conservative noise (discrete version of the non-linear fluctuating hydrodynamics) and we computed the scaling limit of the energy fluctuation field.
- The limit is given by the stationary solution of the infinite-dimensional fractional Ornstein-Uhlenbeck process.
- Is this limit the same for others nonlinear models (e.g. anharmonic chains with symmetric potentials at zero pressure)? The answer to this question is outside the range of the NFH/RG/MFT predictions.