Adventures of a Long-Range Walker

Thierry DAUXOIS

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Adventures of a long-range walker, born 13th May 1954

60th birthday

Studying Links between *Statistical Mechanics* and *Nonlinear Dynamics*

Fermi-Pasta-Ulam-*Tsingou* Problem

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2}(x_i - x_{i+1})^2 + \frac{\beta}{12}(x_i - x_{i+1})^4$$

Classical simplification of 1D Heat conduction

Questions:

- Existence of thermodynamic limit for statistical properties of a dynamical system?
- Equipartition threshold in nonlinear Hamiltonian systems
- Role of localized excitations in these systems

Article 1

J. Phys. A: Math. Gen. 19 (1986) 2033–2040. Printed in Great Britain

Distribution of characteristic exponents in the thermodynamic limit

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Abstract. The existence of the thermodynamic limit for the spectrum of the Lyapunov characteristic exponents is numerically investigated for the Fermi–Pasta–Ulam β model. We show that the shape of the spectrum for energy density well above the equipartition threshold ε_c allows the Kolmogorov–Sinai entropy to be expressed simply in terms of the maximum exponent $\lambda_{\max}$. The presence of a power-law behaviour ε^{θ} is investigated. The analogies with similar results obtained from the dynamics of symplectic random matrices seem to indicate the possibility of interpreting chaotic dynamics in terms of some 'universal' properties.

Distribution of characteristic Lyapunov exponents in the thermodynamic limit

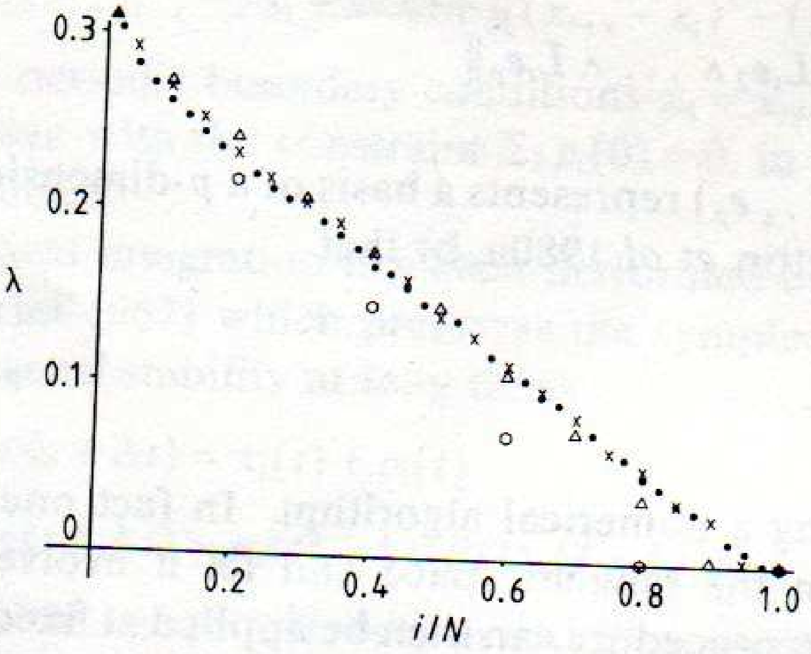

Figure 1. $\lambda(i/N, N)$ plotted against i/N for different values of N (○, 5; △, 10; ×, 20; ●, 40; ▲, 80) and $\varepsilon = 26.4$.

Article 2

PHYSICAL REVIEW E VOLUME 52, NUMBER 3 SEPTEMBER 1995

Clustering and relaxation in Hamiltonian long-range dynamics

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(Received 15 February 1995)

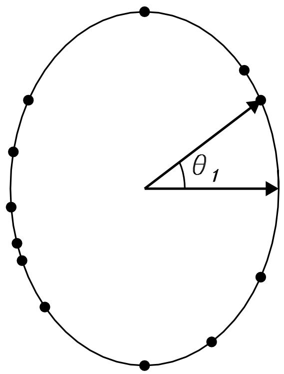
We study the dynamics of a fully coupled network of N classical rotators, which can also be viewed as a mean-field XY Heisenberg (HMF) model, in the attractive (ferromagnetic) and repulsive (antiferromagnetic) cases. The exact free energy and the spectral properties of a Vlasov-Poisson equation give hints on the values of dynamical observables and on time relaxation properties. At high energy (high temperature T) the system relaxes to Maxwellian equilibrium with vanishing magnetization, but the relaxation time to the equilibrium momentum distribution diverges with N as NT^2 in the ferromagnetic case and as $NT^{3/2}$ in the antiferromagnetic case. The N dependence of the relaxation time is suggested by an analogy of the HMF model with gravitational and charged sheets dynamics in one dimension, and is verified in numerical simulations. Below the critical temperature the ferromagnetic HMF model shows a collective phenomenon where the rotators form a drifting cluster; we argue that the drifting speed vanishes as $N^{-1/2}$ but increases as one approaches the critical point (a manifestation of critical slowing down). For the antiferromagnetic HMF model a two-cluster drifting state with zero magnetization forms spontaneously at very small temperatures; at larger temperatures an initial density modulation produces this state, which relaxes very slowly. This suggests the possibility of exciting magnetized states in a mean-field antiferromagnetic system.

Model: Hamiltonian Mean-Field (HMF)

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} - \frac{1}{2N} \sum_{i,j=1}^{N} \cos(\theta_i - \theta_j)$$

Simplification of:

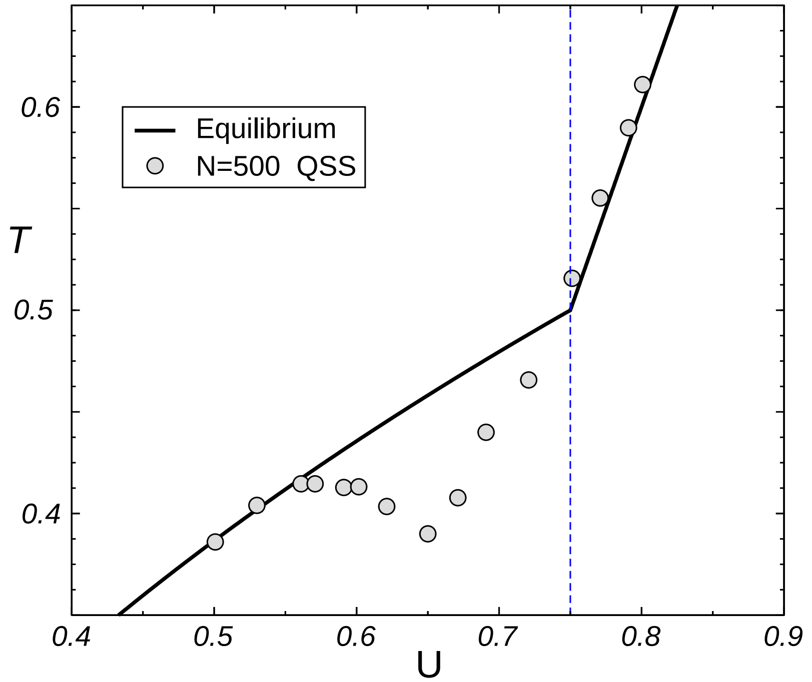
- 1D charged sheets model, 1D gravitation
- Hamiltonian for plasma-wave, or Free Electron Laser



Simple model, Mean Field,

Introducing $\mathbf{m} = \frac{1}{N} \sum_n e^{i\theta_n}$, one obtains $H = K - \frac{N}{2} |\mathbf{m}|^2$

Caloric curve



Solid line: Canonical results at equilibrium

Circles: Microcanonical numerical simulations

Ensemble Inequivalence ?

At Equilibrium

Article 3

VOLUME 87, NUMBER 3 PHYSICAL REVIEW LETTERS 16 JULY 2001

Inequivalence of Ensembles in a System with Long-Range Interactions

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We study the global phase diagram of the infinite-range Blume-Emery-Griffiths model both in the *canonical* and in the *microcanonical* ensembles. The canonical phase diagram shows first-order and continuous transition lines separated by a tricritical point. We find that below the tricritical point, when the canonical transition is first order, the phase diagrams of the two ensembles disagree. In this region the microcanonical ensemble exhibits energy ranges with negative specific heat and temperature jumps at transition energies. These results can be extended to weakly decaying nonintegrable interactions.

Ensemble inequivalence: BEG model

$$H = \Delta \sum_{i=1}^{N} S_i^2 - \frac{J}{2N}\left(\sum_{i=1}^{N} S_i\right)^2 \qquad \text{with} \quad S_i = \pm 1, 0$$

simple model, mean-field, with phase transition, on a lattice.

Ferromagnetic states: $S_i = 1, \forall i$, or $S_i = -1, \forall i \Rightarrow E_F = (\Delta - J/2)N$

Paramagnetic states: $S_i = 0, \forall i \qquad \Rightarrow E_P = 0$

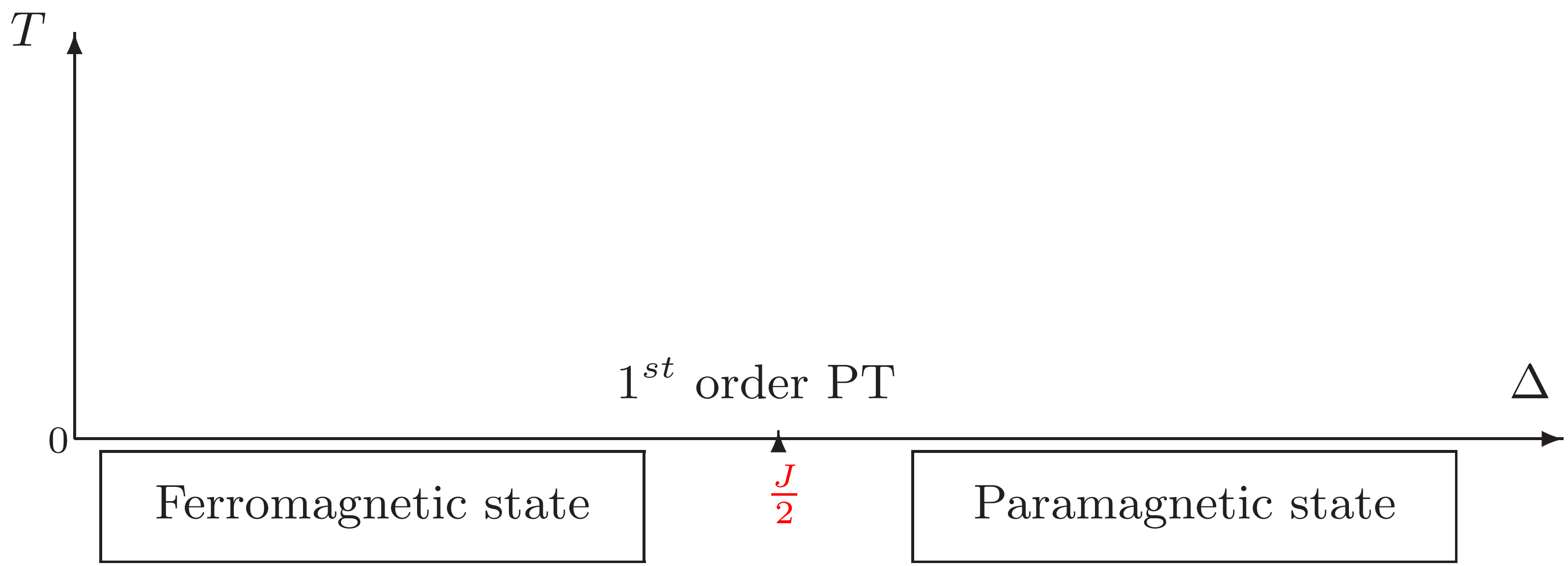
Δ defines the energy difference between ferro. and para states.

Canonical ensemble:
minimization $F = E - TS$ at $T = 0 \rightarrow$ minimization of E

Paramagnetic state is the most favorable if $E_F > E_P \Rightarrow \Delta > J/2$,
Phase transition (PT) at $\Delta = J/2$, which is *first* order since there is a sudden jump of magnetization from ferro. to para. state.

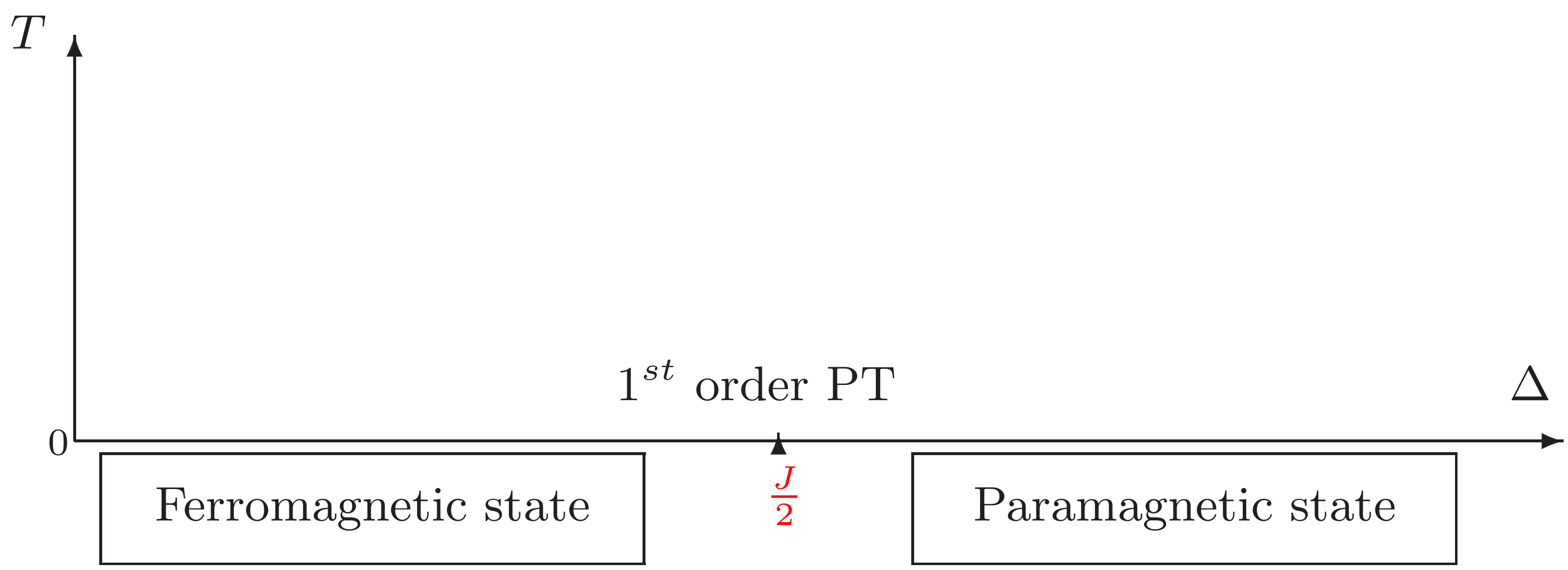
Elementary features of the phase diagram

$$H = \Delta \sum_{i=1}^{N} S_i^2 - \frac{J}{2N}\left(\sum_{i=1}^{N} S_i\right)^2$$



Elementary features of the phase diagram

$$H = \Delta \sum_{i=1}^{N} S_i^2 - \frac{J}{2N} \left(\sum_{i=1}^{N} S_i \right)^2$$



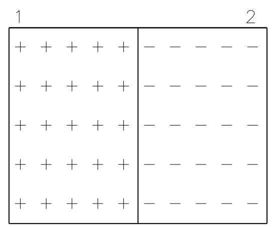
For vanishingly small Δ, one recovers the Curie-Weiss Hamiltonian.

Curie-Weiss Hamiltonian

$$H = -\frac{J}{2N}\left(\sum_{i=1}^{N} S_i\right)^2$$

EXTENSIVITY: For a given *intensive* magnetization $m = \sum_i S_i/N$, if one doubles the number of spins the energy doubles.

ADDITIVITY:



$$E_+ = -\frac{J}{2(N/2)}\left(+\frac{N}{2}\right)^2 = -\frac{JN}{4}$$
$$E_- = -\frac{J}{2(N/2)}\left(-\frac{N}{2}\right)^2 = -\frac{JN}{4}$$

and

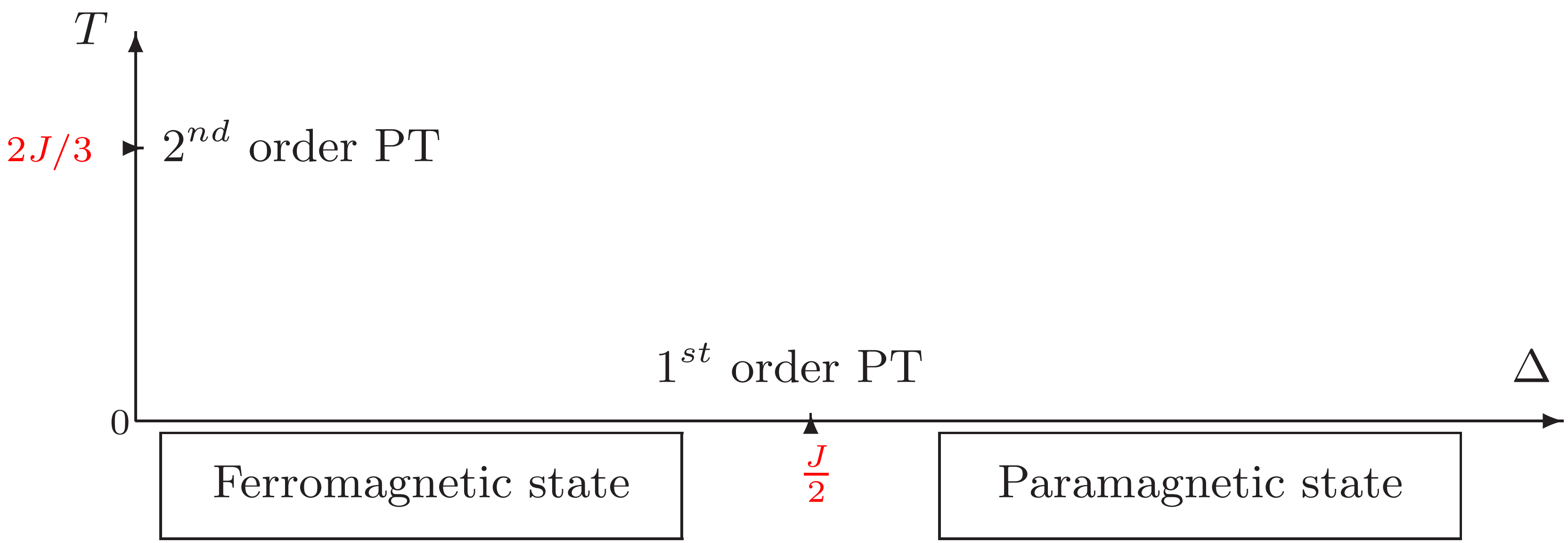
$$E = -\frac{J}{2N}\left(\frac{N}{2} - \frac{N}{2}\right)^2 = 0$$

$$\Rightarrow E_+ + E_- \neq E$$

This model is *extensive* but non *additive*.

Curie-Weiss Hamiltonian

Such a system has a *second* order phase transition when $T_c = 2J/3$



PT of different orders on the T and Δ axis, one expects a *transition line* separating the low T ferro phase from the high T para phase.

Ensemble inequivalence: BEG model

$$H = \Delta \sum_{i=1}^{N} S_i^2 - \frac{J}{2N}\left(\sum_{i=1}^{N} S_i\right)^2 \qquad \text{with} \quad S_i = \pm 1, 0$$

simple model, mean-field, with phase transition, on a lattice.

- Microcanonical: $N_+ + N_- + N_0 = N$

$$\Omega\left(N_+, N_-, N_0\right) = \frac{N!}{N_+! N_-! N_0!} \quad \Rightarrow \quad S = k_B \ln \Omega$$

$$m = \frac{N_+ - N_-}{N} \quad \text{and} \quad q = \frac{N_+ + N_-}{N} \quad \Rightarrow \quad E = \left(\Delta q - \frac{J}{2} m^2\right) N$$

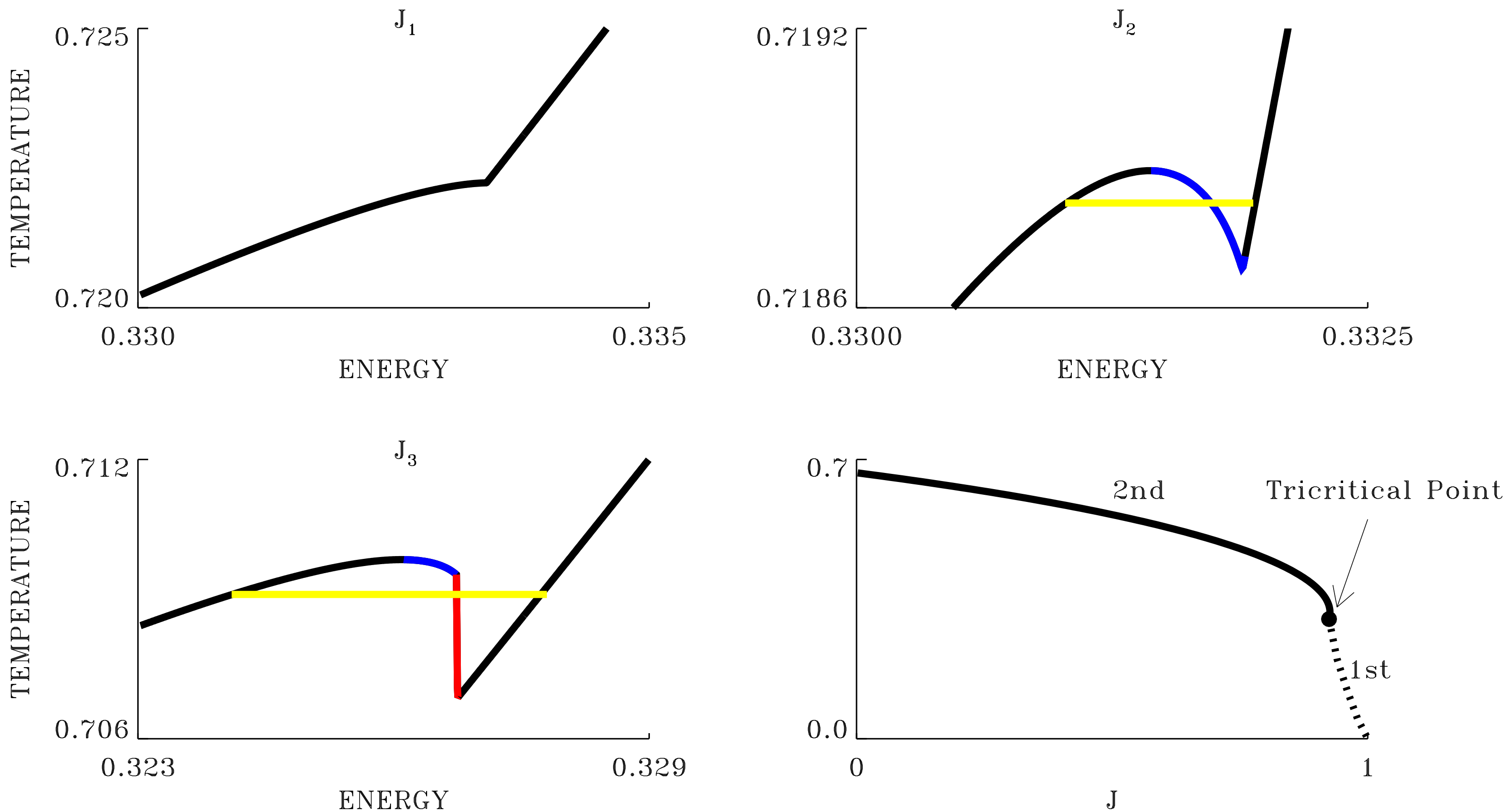
Equilibrium state: maximization of $S(E, m)$ with respect to m.

- Canonical: $Z(\beta, m) = \sum_q \Omega(q, m)\, e^{-\beta E(q,m)}$

Equilibrium state: minimization of $F(\beta, m)$ with respect to m.

Barré, Mukamel, Ruffo, Phys. Rev. Lett. **87**, 030601 (2001).

Caloric Curve



Branches with negative specific heat correspond to local maxima of $F(\beta, m)$, that the constraint of constant energy stabilize in the microcanonical ensemble.

Inequivalence of ensemble

Landau Theory of Phase Transition

Microcanonical ensemble (Power serie expansion of S)

Tricritical point is $\Delta_c = 0.4624$... and $\beta_c = 3.0272$.

Canonical ensemble (Power serie expansion of F)

Tricritical point is $\Delta_c = \ln 4/3$=0.4621... and $\beta_c = 3$.

• Both points although very close do not coincide. The microcanonical critical line extends beyond the canonical one.

• This feature which is a clear indication of ensemble inequivalence was first found in the BEG model (Barré, Mukamel, Ruffo 2001) and later confirmed for gravitational models (Chavanis 2002)

• The non coincidence of microcanonical and canonical tricritical points is a generic feature as proven by Bouchet and Barré (2005)

A wide range of models

Model	Variable	Ensemble Inequivalence	Negative c_v	Ergodicity Breaking	Comput. Entropy
BEG	Discrete	Y	Y	Y	Y
3 states Potts	Discrete	Y	Y	N	Y
Ising L+S	Discrete	Y	Y	Y	Y
α-Ising	Discrete	Y	N	N*	Y
HMF	Continuous	N	N	N	Y
XY L+S	Continuous	Y	Y	Y	Y
α-HMF	Continuous	N	N	N*	N
Generalized XY	Continuous	Y	Y	Y	Y
Mean-Field ϕ^4	Continuous	Y	N	N*	Y
Colson-Bonifacio	Continuous	N	N	N	Y
Point vortex	Continuous	Y	Y	Y	Y
Quasi-geostrophic	Continuous	Y	Y	Y	Y
SGR	Continuous	Y	Y	Y	Y

Stefano was involved in all related studies of these models.

A wide range of models

Model	Variable	Ensemble Inequivalence	Negative c_v	Ergodicity Breaking	Comput. Entropy
BEG	Discrete	Y	Y	Y	Y
3 states Potts	Discrete	Y	Y	N	Y
Ising L+S	Discrete	Y	Y	Y	Y
α-Ising	Discrete	Y	N	N*	Y
HMF	Continuous	N	N	N	Y
XY L+S	Continuous	Y	Y	Y	Y
α-HMF	Continuous	N	N	N*	N
Generalized XY	Continuous	Y	Y	Y	Y
Mean-Field ϕ^4	Continuous	Y	N	N*	Y
Colson-Bonifacio	Continuous	N	N	N	Y
Point vortex	Continuous	Y	Y	Y	Y
Quasi-geostrophic	Continuous	Y	Y	Y	Y
SGR	Continuous	Y	Y	Y	Y

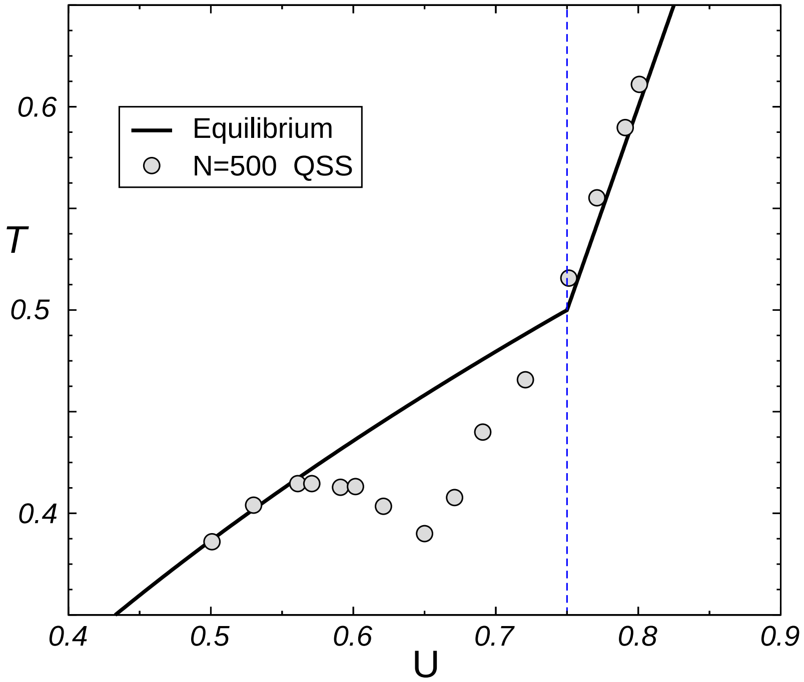
Stefano: chairman of the HMF club.

A wide range of models

Model	Variable	Ensemble Inequivalence	Negative c_v	Ergodicity Breaking	Comput. Entropy
BEG	Discrete	Y	Y	Y	Y
3 states Potts	Discrete	Y	Y	N	Y
Ising L+S	Discrete	Y	Y	Y	Y
α-Ising	Discrete	Y	N	N*	Y
HMF	Continuous	No	N	N	Y
XY L+S	Continuous	Y	Y	Y	Y
α-HMF	Continuous	N	N	N*	N
Generalized XY	Continuous	Y	Y	Y	Y
Mean-Field ϕ^4	Continuous	Y	N	N*	Y
Colson-Bonifacio	Continuous	N	N	N	Y
Point vortex	Continuous	Y	Y	Y	Y
Quasi-geostrophic	Continuous	Y	Y	Y	Y
SGR	Continuous	Y	Y	Y	Y

No ensemble inequivalence for the HMF model?

Caloric curve



Antoni,Ruffo

Phys Rev. E 1995

Solid line: Microcanonical and Canonical results at equilibrium

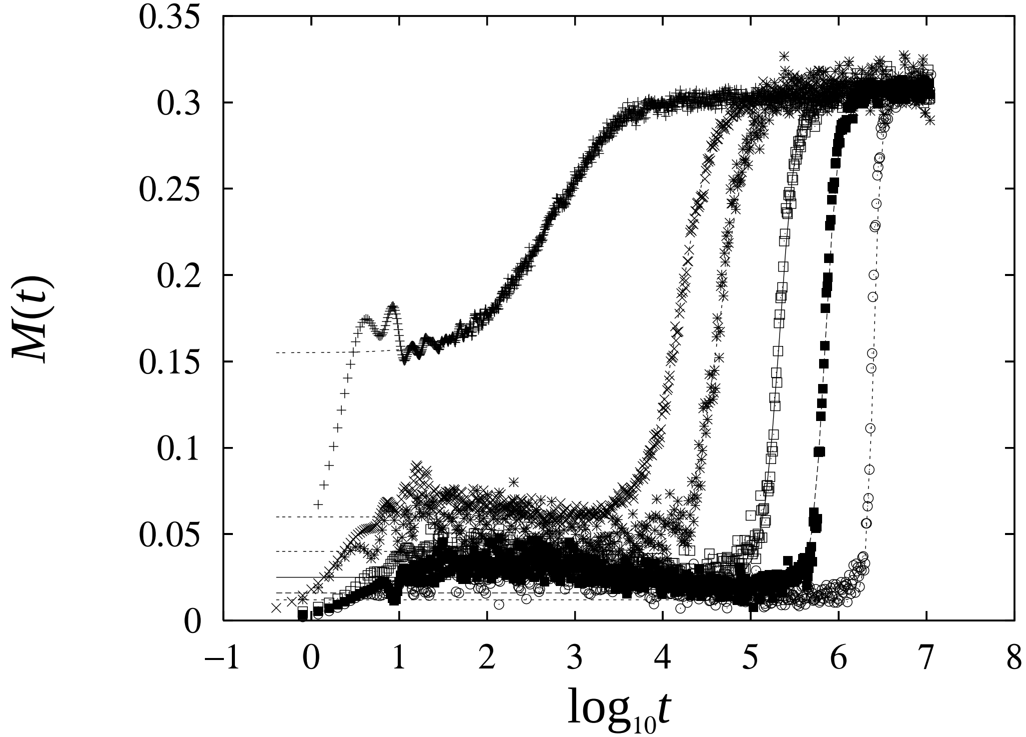
Circles: Microcanonical numerical simulations

Origin of the paradox ?

Dynamics matters

Numerical Simulations

Evolution of the order parameter for different N-values



← Boltzmann Equilibrium

Non trivial scaling law

$t_{QSS} \sim N^{1.7}$

← Quasi-stationary state

$$\lim_{t\to\infty} \text{ before } \lim_{N\to\infty} \neq \lim_{N\to\infty} \text{ before } \lim_{t\to\infty}$$

Questions to be addressed

- Can we explain theoretically these numerical facts ?
  - Dynamical ensemble inequivalence
  - Order of limits
  - Algebraic Relaxation

- Is usual statistical mechanics sufficient ?

Kinetic Theory

For LRI, the single particle time-dependent density function:

$$f_d\left(\theta, p, t\right) = \frac{1}{N} \sum_{j=1}^{N} \delta\left(\theta - \Theta_j\left(t\right)\right) \delta\left(p - P_j\left(t\right)\right),$$

θ, p : Eulerian coordinates of the phase space and
Θ_j, P_j : Lagrangian coordinates of the N-particles

$$\frac{\partial f_d}{\partial t} + p\frac{\partial f_d}{\partial \theta} - \frac{\partial v}{\partial \theta}\frac{\partial f_d}{\partial p} = 0. \qquad \text{Klimontovich Eq.}$$

where $\qquad v(\theta, t) = N \int \mathrm{d}\theta' \mathrm{d}p' \; V(\theta - \theta') f_d(\theta', p', t) \;,$

• Derivation is *exact*, even for a finite number of particles N.

• This equation contains the information about the orbit of every single particle which is far more than necessary but is a useful starting point for approximations.

VLASOV EQUATION

Consider a large number of initial conditions, close to the same macroscopic state.

$$f_d(\theta,p,t) = \underbrace{\langle f_d(\theta,p,t)\rangle}_{f_0(\theta,p,t)} + \frac{1}{\sqrt{N}}\delta f(\theta,p,t).$$

$$\frac{\partial f_0}{\partial t} + p\frac{\partial f_0}{\partial \theta} - \frac{\partial \langle v\rangle}{\partial \theta}\frac{\partial f_0}{\partial p} = \frac{1}{N}\left\langle \frac{\partial \delta v}{\partial \theta}\frac{\partial \delta f}{\partial p}\right\rangle .$$

• For short-range interactions, the r.h.s. leads to the collision term of the Boltzmann equation, while the third term is negligible.

• For long-range interactions, the r.h.s is of order $1/N$ (finite N effects), while the third term is the leading term (*collective* effects).

$2N$ ODE are thus replaced by only 1 PDE.

Next Order: Lenard-Balescu Equation

Restricting to homogeneous f_0, a stable stationary solution of the Vlasov equation, we get

$$\frac{\partial f_0}{\partial t} = \frac{1}{N} \left\langle \frac{\partial \delta v}{\partial \theta} \frac{\partial \delta f}{\partial p} \right\rangle$$

At the level $1/N$, the r.h.s can be determined using solutions for δv and δf of the collisionless dynamics, i.e. *linearized Vlasov equation*.

- For any 1D LRI, Vlasov stable homogeneous distribution functions do not evolve on timescales of order smaller or equal to N
- Explanation of the dynamical ensemble inequivalence

Typical Behavior for Long-Range Systems

Initial Condition

Violent relaxation $\downarrow$ $\tau_v = O(1)$

Vlasov's Equilibrium

Collisional relaxation $\downarrow$ $\tau_c = N^{\delta} \quad (N/\ln N)$

Boltzmann's Equilibrium

Article 4

PHYSICAL REVIEW E **89**, 022123 (2014)

Nonequilibrium first-order phase transition in coupled oscillator systems with inertia and noise

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We study the dynamics of a system of coupled oscillators of distributed natural frequencies, by including the features of both thermal noise, parametrized by a temperature, and inertial terms, parametrized by a moment of inertia. For a general unimodal frequency distribution, we report here the complete phase diagram of the model in the space of dimensionless moment of inertia, temperature, and width of the frequency distribution. We demonstrate that the system undergoes a nonequilibrium first-order phase transition from a synchronized phase at low parameter values to an incoherent phase at high values. We provide strong numerical evidence for the existence of both the synchronized and the incoherent phase, treating the latter analytically to obtain the corresponding linear stability threshold that bounds the first-order transition point from below. In the limit of zero noise and inertia, when the dynamics reduces to the one of the Kuramoto model, we recover the associated known continuous transition. At finite noise and inertia but in the absence of natural frequencies, the dynamics becomes that of a well-studied model of long-range interactions, the Hamiltonian mean-field model. Close to the first-order phase transition, we show that the escape time out of metastable states scales exponentially with the number of oscillators, which we explain to be stemming from the long-range nature of the interaction between the oscillators.

Non-Equilibrium 1^{st} order PT

$$m\frac{d\theta_i}{dt} = v_i \tag{1}$$

$$m\frac{dv_i}{dt} = -\gamma v_i + Kr\sin(\psi - \theta_i) + \gamma\omega_i + \sqrt{\gamma}\,\eta_i(t) \tag{2}$$

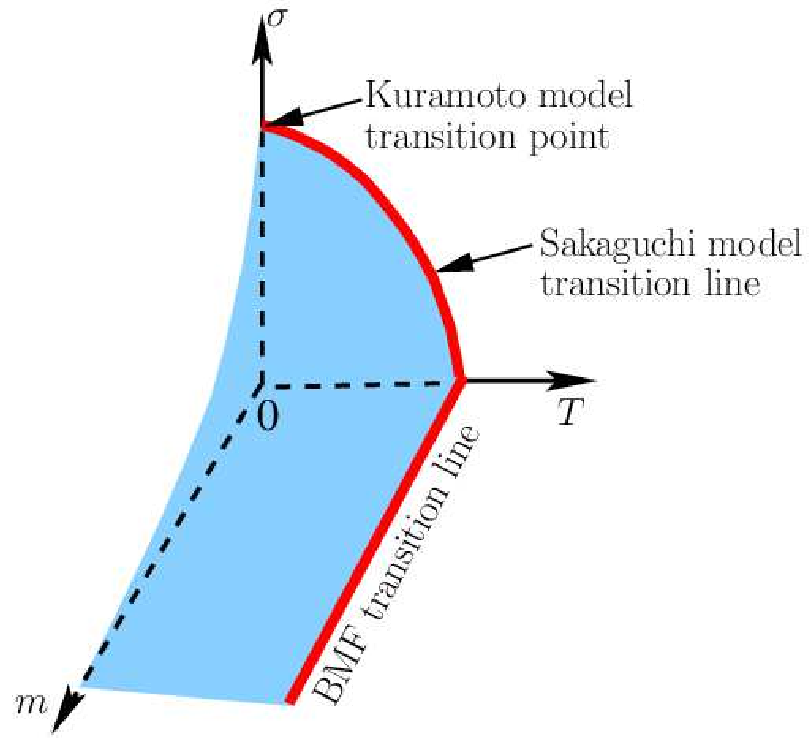
in which

$\eta_i(t)$: Gaussian noise

ω_i :distribution of frequencies

T: Temperature

$$r\ \exp(i\psi(t)) = \frac{1}{N}\sum_{\ell=1}^{N} e^{i\theta_i}$$

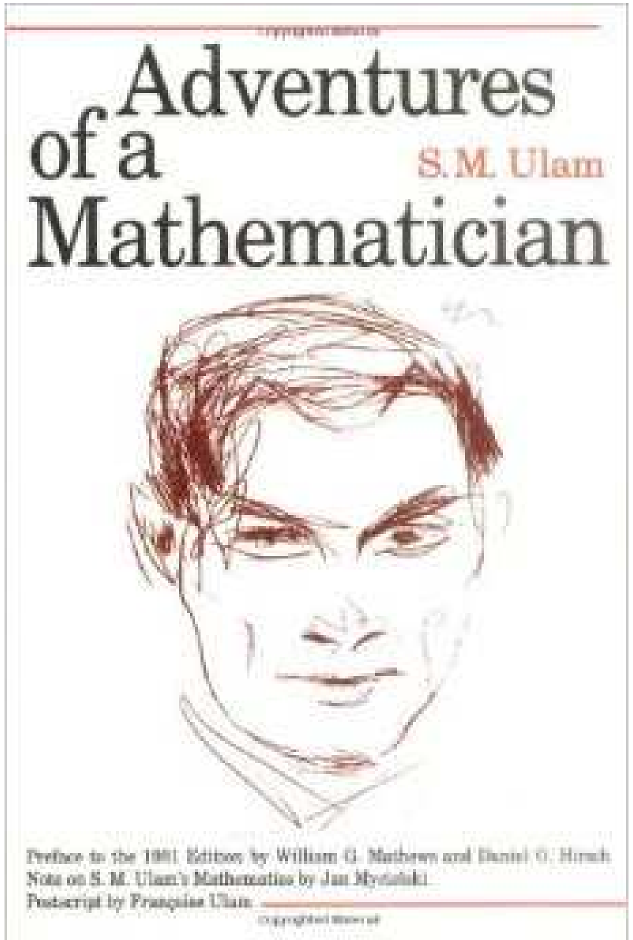


STEFANO'S

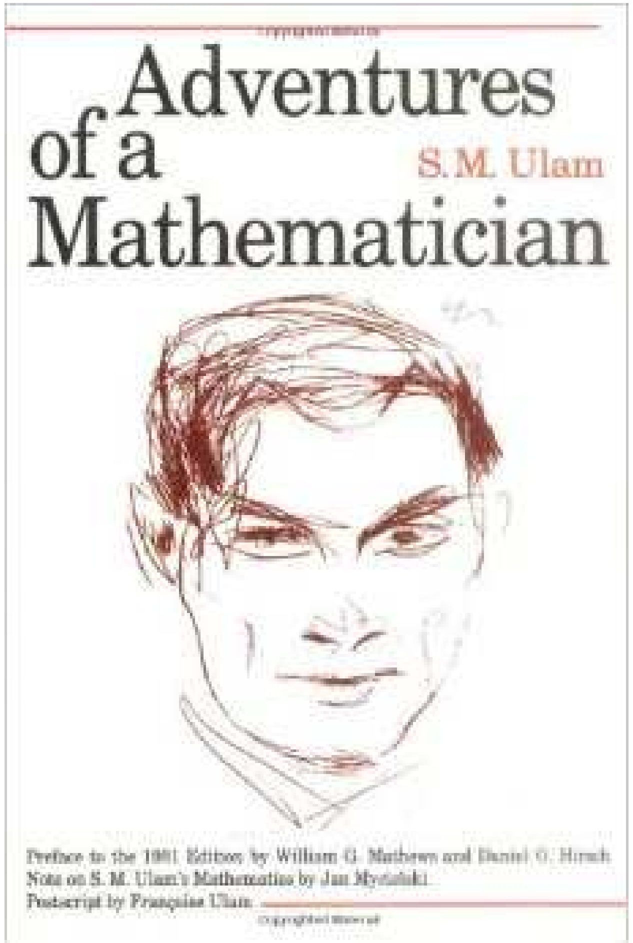
three main qualities

Memory

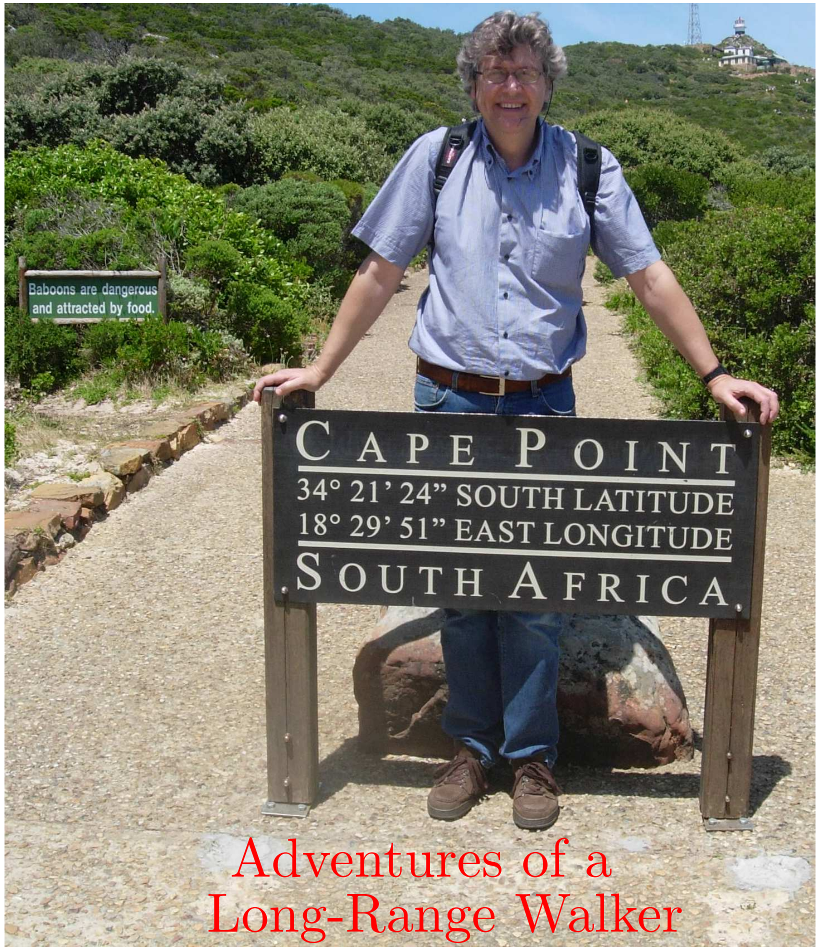
Memory



Memory



Adventures of a Mathematician
S.M. Ulam



Baboons are dangerous and attracted by food.
CAPE POINT
34° 21' 24" SOUTH LATITUDE
18° 29' 51" EAST LONGITUDE
SOUTH AFRICA
Adventures of a
Long-Range Walker

Always young people around him

Always young people around him

Modesty

Modesty

Sometimes it is more difficult not to be seen

Stefano is an excellent Cook

Happy Birthday Stefano

Life begins at 60, Tino Rossi

Thank you very much, Stefano !