Validity of spin wave theory for the quantum Heisenberg model

Alessandro Giuliani

Based on joint work with
M. Correggi and R. Seiringer

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Outline

1. Introduction: continuous symmetry breaking and spin waves
2. Main results: free energy at low temperatures

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General question: rigorous understanding of the phenomenon of **spontaneous breaking** of a **continuous symmetry**.

Easier case: **abelian continuous symmetry**.

Several rigorous results based on:

- reflection positivity,
- vortex loop representation
- cluster and spin-wave expansions,

by Fröhlich-Simon-Spencer, Dyson-Lieb-Simon, Bricmont-Fontaine--Lebowitz-Lieb-Spencer, Fröhlich-Spencer, Kennedy-King, ...

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Harder case: **non-abelian symmetry**.

Few rigorous results on:

- classical Heisenberg (Fröhlich-Simon-Spencer by RP)
- quantum Heisenberg *antiferromagnet* (Dyson-Lieb-Simon by RP)
- classical N-vector models (Balaban by RG)

Notably absent: quantum Heisenberg *ferromagnet*

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Quantum Heisenberg ferromagnet

The simplest quantum model for the spontaneous symmetry breaking of a continuous symmetry:

$$H_\Lambda := \sum_{\langle x,y\rangle \subset \Lambda} (S^2 - \vec{S}_x \cdot \vec{S}_y)$$

where:

- Λ is a cubic subset of $\mathbb{Z}^3$ with (say) periodic b.c.
- $\vec{S}_x = (S_x^1, S_x^2, S_x^3)$ and S_x^i are the generators of a $(2S+1)$-dim representation of $SU(2)$, with $S = \frac{1}{2}, 1, \frac{3}{2}, \ldots$:

$$[S_x^i, S_y^j] = i\epsilon_{ijk} S_x^k \delta_{x,y}$$

- The energy is normalized s.t. $\inf \mathrm{spec}(H_\Lambda) = 0$.

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Ground states

One special ground state is

$$|\Omega\rangle := \otimes_{x\in\Lambda}|S_x^3 = -S\rangle$$

All the other ground states have the form

$$(S_T^+)^n|\Omega\rangle, \qquad n = 1, \ldots, 2S|\Lambda|$$

where $S_T^+ = \sum_{x\in\Lambda} S_x^+$ and $S_x^+ = S_x^1 + iS_x^2$.

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Excited states: spin waves

A special class of excited states (**spin waves**) is obtained by raising a spin in a coherent way:

$$|1_k\rangle := \frac{1}{\sqrt{2S|\Lambda|}} \sum_{x \in \Lambda} e^{ikx} S_x^+ |\Omega\rangle \equiv \frac{1}{\sqrt{2S}} \hat{S}_k^+ |\Omega\rangle$$

where $k \in \frac{2\pi}{L}\mathbb{Z}^3$. They are such that

$$H_\Lambda |1_k\rangle = S\epsilon(k)|1_k\rangle$$

where $\epsilon(k) = 2\sum_{i=1}^{3}(1 - \cos k_i)$.

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They can be looked for *in the vicinity* of

$$|\{n_k\}\rangle = \prod_k (2S)^{-n_k/2} \frac{(\hat{S}_k^+)^{n_k}}{\sqrt{n_k!}} |\Omega\rangle$$

If $N = \sum_k n_k > 1$, these are not eigenstates.

They are neither normalized nor orthogonal.

However, H_Λ is almost diagonal on $|\{n_k\}\rangle$ in the low-energy (long-wavelengths) sector.

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Expectation:

low temperatures $\Rightarrow$

$\Rightarrow$ low density of spin waves $\Rightarrow$

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In 3D, it predicts

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\begin{aligned}
f(\beta) &\simeq \frac{1}{\beta}\int \frac{d^3k}{(2\pi)^3}\log(1 - e^{-\beta S\epsilon(k)}) \\
m(\beta) &\simeq S - \int \frac{d^3k}{(2\pi)^3}\frac{1}{e^{\beta S\epsilon(k)} - 1}
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In 3D, it predicts

$$f(\beta) \underset{\beta\to\infty}{\simeq} \beta^{-5/2} S^{-3/2} \int \frac{d^3k}{(2\pi)^3} \log(1 - e^{-k^2})$$

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Holstein-Primakoff representation

A convenient representation:

$$S_x^+ = \sqrt{2S}\, a_x^+ \sqrt{1 - \frac{a_x^+ a_x}{2S}}\ , \quad S_x^3 = a_x^+ a_x - S,$$

where $[a_x, a_y^+] = \delta_{x,y}$ are **bosonic operators**.

Hard-core constraint: $n_x = a_x^+ a_x \leq 2S$.

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In the bosonic language

$$H_\Lambda = S \sum_{\langle x,y \rangle} \left(-a_x^+ \sqrt{1 - \frac{n_x}{2S}} \sqrt{1 - \frac{n_y}{2S}} a_y - a_y^+ \sqrt{1 - \frac{n_y}{2S}} \sqrt{1 - \frac{n_x}{2S}} a_x + n_x + n_y - \frac{1}{S} n_x n_y \right)$$

$$\equiv S \sum_{\langle x,y \rangle} (a_x^+ - a_y^+)(a_x - a_y) - K \equiv T - K$$

The spin wave approximation consists in neglecting K and the on-site hard-core constraint.

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Previous results

$$H_\Lambda = S \sum_{\langle x,y \rangle} (a_x^+ - a_y^+)(a_x - a_y) - K$$

For large S, the interaction K is of relative size $O(1/S)$ as compared to the hopping term.

Easier case: $S \to \infty$ with βS constant (CG 2012)

Harder case: fixed S, say $S = 1/2$. So far, not even a sharp upper bound on the free energy was known. Rigorous upper bounds, off by a constant, were given by Conlon-Solovej and Toth in the early 90s.

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Bosons and random walk

Side remark: the Hamiltonian can be rewritten as

$$H_\Lambda = S \sum_{\langle x,y \rangle} \left(a_x^+ \sqrt{1 - \frac{n_y}{2S}} - a_y^+ \sqrt{1 - \frac{n_x}{2S}} \right) \cdot$$
$$\cdot \left(a_x \sqrt{1 - \frac{n_y}{2S}} - a_y \sqrt{1 - \frac{n_x}{2S}} \right)$$

i.e., it describes a weighted hopping process of bosons on the lattice. The hopping on an occupied site is discouraged (or not allowed).

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Theorem [Correggi-G-Seiringer 2013] (free energy at low temperature).

For any $S \geq 1/2$,

$$\lim_{\beta \to \infty} f(S,\beta)\beta^{5/2} S^{3/2} = \int \log\left(1 - e^{-k^2}\right) \frac{d^3k}{(2\pi)^3} .$$

Remarks

- The proof is based on upper and lower bounds. It comes with explicit estimates on the remainder.

  Relative errors:
  - $O((\beta S)^{-3/8})$ (upper bound)
  - $O((\beta S)^{-1/40+\epsilon})$ (lower bound)
- We do not really need S fixed. Our bounds are uniform in S, provided that $\beta S \to \infty$.
- The case $S \to \infty$ with βS =const. is easier and it was solved by Correggi-G (JSP 2012).

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Remarks

- An important consequence of our proof is an instance of quasi long-range order:

$$\langle S^2 - \vec{S}_x \cdot \vec{S}_y \rangle_\beta \leq \frac{27}{8} |x-y|^2 e(S,\beta) \ ,$$

where $e(S,\beta) = \partial_\beta(\beta f(S,\beta))$ is the energy:

$$e(S,\beta) \underset{\beta\to\infty}{\simeq} \frac{3}{2} S^{-3/2} \beta^{-5/2} \int \frac{dk}{(2\pi)^2} \log \frac{1}{1-e^{-k^2}}$$

Therefore, order persists up to length scales of the order $\beta^{5/4}$. Of course, one expects order to persist at infinite distances, but in absence of a proof this is the best result to date.

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Ideas of the proof

- The proof is based on upper and lower bounds. In both cases we localize the system in boxes of side $\ell = \beta^{1/2+\epsilon}$.
- The upper bound is based on a trial density matrix that is the natural one, i.e., the Gibbs measure associated with the quadratic part of the Hamiltonian projected onto the subspace satisfying the local hard-core constraint.
- The lower bound is based on a preliminary rough bound, off by a log. This uses an estimate on the excitation spectrum

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Summary

- The preliminary rough bound is used to cutoff the energies higher than $\ell^3\beta^{-5/2}(\log\beta)^{5/2}$. In the low energy sector we pass to the bosonic representation.
- In order to bound the interaction energy in the low energy sector, we use a new functional inequality, which allows us to reduce to a 2-body problem. The latter is studied by random walk techniques on a weighted graph.

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Thank you!