Supersonic propagation in long-range lattice models

Michael Kastner

based on:
D. Métivier, R. Bachelard, and M. K., PRL (in press)
J. Eisert, M. van den Worm, S. R. Manmana, and M. K., PRL 111, 260401 (2013)
Stellenbosch
Propagation in spatially extended systems
Propagation in spatially extended systems

Dynamics of spatially extended systems

Nonequilibrium Statistical Physics

Transport theory

Propagation of:
- perturbations / excitations
- information
- correlations / entanglement

A

B
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Propagation of:
- perturbations / excitations
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- basic “building block”
- key quantity: group velocity
Velocity of propagation

In (some) condensed matter systems: propagation velocity is
group velocity \( \frac{\partial \omega(k)}{\partial k} \) obtained from quasi-particle dispersion

General behaviour???

\[ \Rightarrow \quad \text{Lieb-Robinson bound} \]
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Velocity of propagation

In (some) condensed matter systems: propagation velocity is group velocity $\frac{\partial \omega(k)}{\partial k}$ obtained from quasi-particle dispersion.

General behaviour??? $\implies$ Lieb-Robinson bound
Spatio-temporal evolution

**Relativistic theory:** ∃ finite maximum propagation speed

**Nonrelativistic** quantum lattice systems, finite local dimension, finite-range interactions:
∃ finite group velocity, with exponentially small effects outside an effective light cone.
Spatio-temporal evolution

**Relativistic theory**: \( \exists \) finite maximum propagation speed

**Nonrelativistic** quantum lattice systems, finite local dimension, **finite-range interactions**: \( \exists \) finite group velocity, with exponentially small effects outside an effective light cone
**Lieb-Robinson bound**

\[
\| [O_A(t), O_B(0)] \| \leq C \| O_A \| \| O_B \| \min(|A|, |B|) e^{(v|t| - d(A,B))/\xi}
\]

\(\exists\) finite group velocity, with exponentially small effects outside an effective light cone

- physical relevance: transmission of information, growth of entanglement, clustering of correlations, Lieb-Schultz-Mattis in \(D > 1\), finite-size errors of simulations...
- very general result
- restrictions:
  - finite local dimension
  - finite interaction range
Lieb-Robinson bound

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物理意义：信息的传输、纠缠的增长、相关性的聚集体、Lieb-Schultz-Mattis在\( D > 1 \)，有限大小的模拟错误。

非常一般的结果

限制条件：
- 有限局部维数
- 有限相互作用范围
Lieb-Robinson bound


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Long-range lattice models

**Short-range:** finite-range (e.g. nearest-neighbour) or exponentially decaying ($\propto e^{-cr}$ with $c > 0$)

**Long-range:** power law decaying, $\propto 1/r^\alpha$ with $\alpha \geq 0$

Realisations of long-range many-body systems:

- Dipolar materials
- Free Electron Laser
- Rydberg atoms
- Cavity QED
- Crystals of trapped ions: $1/r^\alpha$

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Absence of a finite propagation velocity!

General predictions? Long-range Lieb-Robinson bounds?
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Long-range Lieb-Robinson bounds

Here: classical-mechanical, think of

\[ H = \sum_{i \in \Lambda} \frac{p_i^2}{2} - \frac{J}{2} \sum_{\substack{i,j \in \Lambda \\ i \neq j}} \frac{\cos(q_i - q_j)}{|i - j|^{\alpha}} \]

\[ |\{f_i(0), g_j(t)\}| \leq c \max \left\{ \left| \frac{\partial p_j(t)}{\partial p_i(0)} \right|, \left| \frac{\partial q_j(t)}{\partial p_i(0)} \right|, \left| \frac{\partial p_j(t)}{\partial q_i(0)} \right|, \left| \frac{\partial q_j(t)}{\partial q_i(0)} \right| \right\} \]

“Spreading of a perturbation”

\[ \left| \frac{\partial q_j(t)}{\partial q_i(0)} \right| \leq \sum_{n=1}^{\infty} U_n^{ij} t^{2n} \leq \text{const.} \times \frac{\cosh(v_{\alpha} t) - 1}{|i - j|^{\alpha}} \]

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D. Métivier, R. Bachelard, M. K., PRL (in press);
\(\alpha\)-dependence of the propagation front

\[ \alpha = 1/4 \]

\[ \alpha = 3/4 \]

\[ \alpha = 3/2 \]

Propagation is qualitatively different in the regimes

\[ 0 \leq \alpha < \frac{D}{2} \]

\[ \frac{D}{2} < \alpha < D \]

\[ D < \alpha \]

Two threshold values: \( \alpha = D/2 \) and \( \alpha = D \)

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Application: approach to thermal equilibrium

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Experimental realisation of long-range interactions

Beryllium ions in a Penning trap


- 2d Coulomb crystal on a triangular lattice
- Valence-electron spin states as qubits (Ising spins)
- Spin–spin interactions mediated by crystal’s transverse motional degrees of freedom

Effective (anti-)ferromagnetic Ising Hamiltonian

\[ H = \sum_{i \neq j} J_{ij} \sigma_i^z \sigma_j^z - \sum_i B \cdot \sigma_i \]


\[ J_{ij} \approx -\frac{J}{|i-j|^\alpha} \text{ with } 0.05 \lesssim \alpha \lesssim 1.4 \]
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Experimental results

long-range XY model \[ H = -J \sum_{i,j} \frac{\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y}{|i-j|^{\alpha}}, \]

realised in a linear Paul ion trap

Richerme et al., arXiv1401.5088
Conclusions

- Nonequilibrium dynamics: spreading of whatsoever

- Long-range Lieb-Robinson bounds

\[ \| \cdot \| \leq C \frac{e^{v|t|} - 1}{|i - j|^\alpha} \]

- \( \alpha \)-dependence of the propagation front

\[ \implies \alpha \text{-dependence of thermalisation} \]

- Ion-trap emulation of long-range spin systems

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