Energy Spreading in Strongly Nonlinear Lattices

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Motivation

- Study of nonlinear effects in disordered lattices
- Linear lattices: Anderson localization $\Rightarrow$ no propagation
- Nonlinear lattices: Weak subdiffusive spreading due to chaos
- Problems: Linear modes are only exponentially localized, no clear picture of spreading

Strongly nonlinear lattices

Usual nonlinear lattices

$$H = \sum \frac{p_l^2}{2} + \omega^2 \frac{q_l^2}{2} + \kappa \frac{(q_{l+1} - q_l)^2}{2} + U_{\mathrm{nl}}(q_l) + V_{\mathrm{nl}}(q_{l+1} - q_l)$$

Strongly nonlinear lattice

$$H = \sum \frac{p_l^2}{2} + \omega^2 \frac{q_l^2}{2} + U_{\mathrm{nl}}(q_l) + V_{\mathrm{nl}}(q_{l+1} - q_l)$$

Sonic vacuum

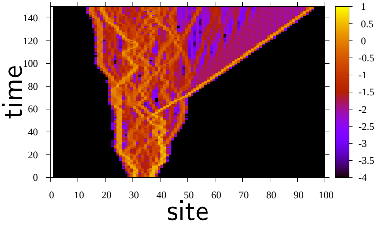
- No phonons, no linear propagating waves and modes
- Localization length $=1$ (minimal possible)
- Only propagating waves are nonlinear ones – typically compactons
- At finite energy density: typically strongly chaotic/turbulent states

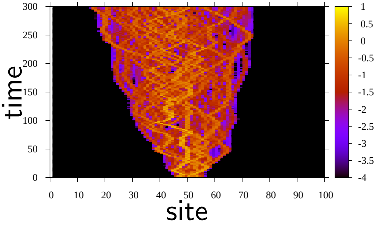
Setup I: Spreading of a localized wave packet in 1-d lattices [with Mario Mulansky, New J. Phys. (2013)]

Strong compactness of the spreading field:
Here "Anderson modes" are one site oscillators $\Rightarrow$ no exponential tails, the packet width L is well-defined at each moment of time
Disorder to prevent ballistic quasi-compactons

Regular lattice

Disordered lattice

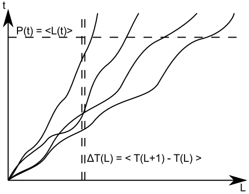




How to average

Traditionally width at fixed time : $\langle \log L(t) \rangle$, but due to large fluctuations one averages here propagation speed at different densities

With sharp edges the averaging of propagation time at fixed width, i.e. at fixed density, is possible: $\log \Delta T = \langle \log(T(L+1) - T(L)) \rangle$



Goal: to describe $\Delta T(L, E)$ for different total energies E

Guiding phenomenology

Use Nonlinear Diffusion Equation (NDE) as a heuristic model

$$\frac{\partial \rho}{\partial t} = D \frac{\partial}{\partial x}\left(\rho^a \frac{\partial \rho}{\partial x}\right), \qquad \text{with} \qquad \int \rho \, dx = E$$

Self-similar solution

$$\rho(x,t) = \frac{1}{[D(t-t_0)]^{1/(2+a)}} \left(E - \frac{ax^2}{2(a+2)[D(t-t_0)]^{2/(a+2)}}\right)^{1/a}$$

yields subdiffusion

$$L = \sqrt{2\frac{2+a}{a}} E^{a/(2+a)} [D(t-t_0)]^{1/(2+a)}$$

One parameter scaling

Reformulate

$$L = \sqrt{2\frac{2+a}{a}} E^{a/(2+a)} (D(t-t_0))^{1/(2+a)}$$

as scaling relaions:

$$\frac{L}{E} \sim \left(\frac{t-t_0}{E^2}\right)^{1/(2+a)} \qquad \frac{1}{E}\frac{\mathrm{d}t}{\mathrm{d}L} \sim \left(\frac{E}{L}\right)^{-(a+1)} \qquad a(w)+1 = -\frac{\mathrm{d}\log\frac{1}{E}\frac{\mathrm{d}t}{\mathrm{d}L}}{\mathrm{d}\log w}$$

where $w = E/L$ is the characteristic density, $\frac{\mathrm{d}t}{\mathrm{d}L} \approx \Delta T$

Spreading in a homogeneously nonlinear lattice

Fully self-similar lattice:
rescaling energy $\Leftrightarrow$ rescaling time

$$H = \sum_k \frac{p_k^2}{2} + W\omega_k^2 \frac{q_k^\kappa}{\kappa} + \beta \frac{(q_{k+1} - q_k)^\kappa}{\kappa}$$

From the rescaling of energy and time it follows

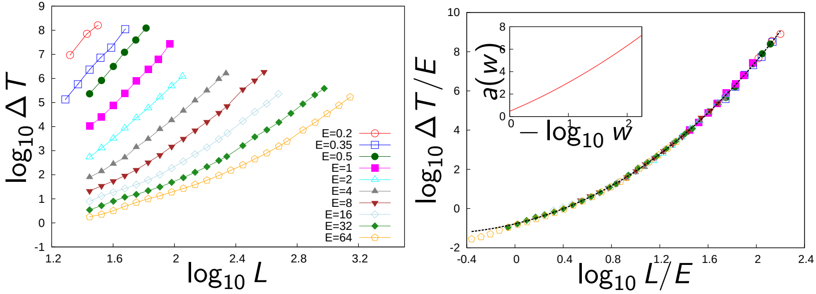
$$t \sim E^{\frac{2\kappa}{2-\kappa}} \quad \Rightarrow \quad a = \frac{\kappa - 2}{2\kappa} \quad \Rightarrow \quad L \sim (t - t_0)^{\frac{2\kappa}{5\kappa - 2}}$$

For the case $\kappa = 4$ we have

$$L \sim (t - t_0)^{4/9} \qquad \Delta T \sim L^{5/4}$$

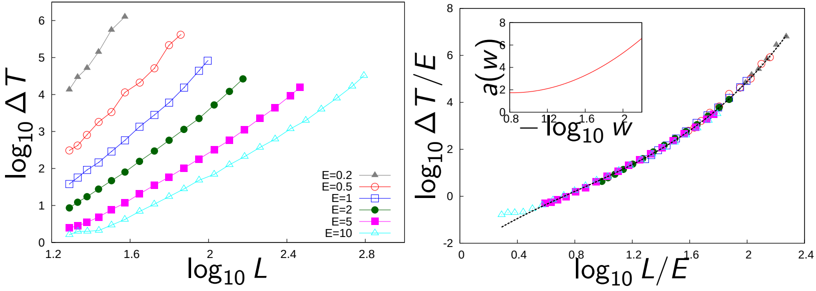
Spreading in a lattice of nonlinearly coupled linear oscillators

$$H = \sum_k \frac{p_k^2 + \omega_k^2 q_k^2}{2} + \frac{(q_{k+1} - q_k)^4}{4}$$



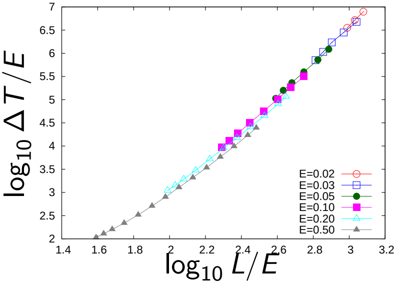
Spreading in a lattice of nonlinearly coupled linear oscillators

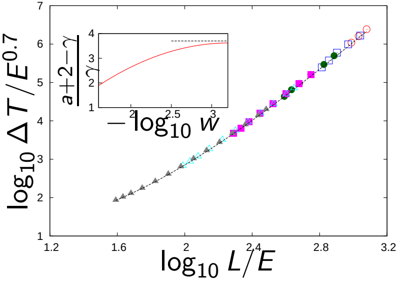
$$H = \sum_k \frac{p_k^2 + \omega_k^2 q_k^2}{2} + \frac{(q_{k+1} - q_k)^6}{6}$$



Nonlinearly coupled nonlinear oscillators

$$H = \sum_k \frac{p_k^2}{2} + \omega_k^2 \frac{q_k^4}{4} + \frac{(q_{k+1} - q_k)^8}{8}$$





Different scaling: $\Delta T/E^{0.7} = F(L/E)$

Fractional nonlinear diffusion equation

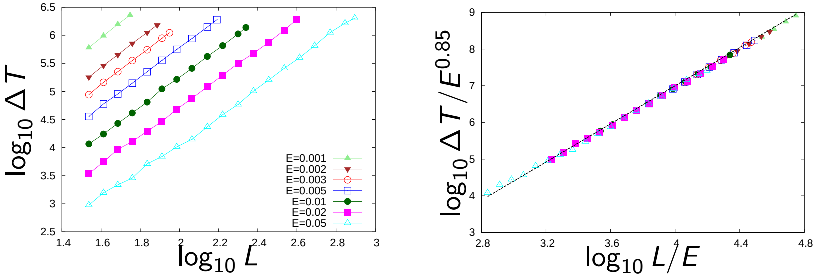
$$\frac{\partial^\gamma \rho}{\partial t^\gamma} = D \frac{\partial}{\partial x}\left(\rho^a \frac{\partial \rho}{\partial x}\right), \qquad \text{with} \qquad \int \rho \, dx = E$$

yields

$$E^{1-2/\gamma} \frac{dt}{dL} \sim \left(\frac{L}{E}\right)^{\frac{a+2-\gamma}{\gamma}}$$

Nonlinearly coupled nonlinear oscillators

$$H = \sum_k \frac{p_k^2}{2} + \omega_k^2 \frac{q_k^4}{4} + \frac{(q_{k+1} - q_k)^6}{6}$$



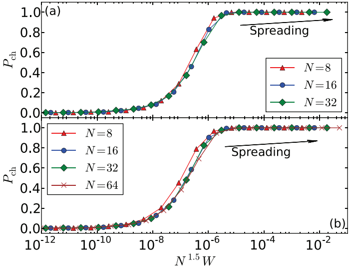
Conclusions for 1-dimensional wavepacket spreading

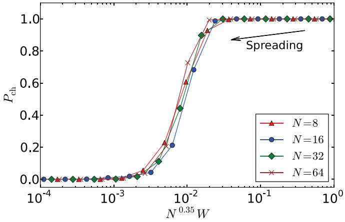
- Nonlinearly coupled linear oscillators:
  NDE scaling works, slowing down of spreading
- Nonlinearly coupled nonlinear oscillators:
  FracNDE scaling works, good power-law

Relation to chaos properties [M. Mulansky, Chaos (2014)]

Probability to observe chaos in a finite lattice in dependence on length and density

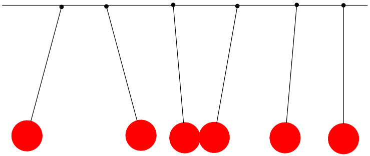
Nonlinear local osc. Linear local osc.





Toy model: Ding-Dong lattice [with S. Roy, CHAOS, v. 22, n. 2, 026118 (2012)]

This is a strongly nonlinear lattice that is easy to model numerically



Ding-Dong model (Prosen, Robnik, 92) is a chain of linear oscillators with elastic collisions

Ding-Dong dynamics

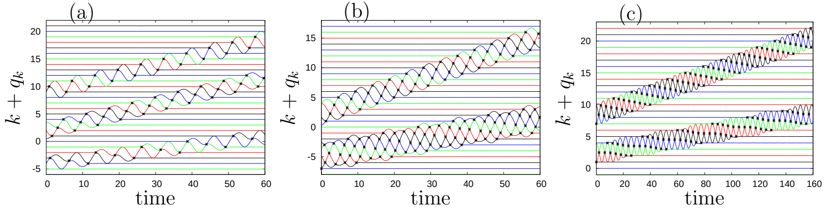
Hamiltonian and collision condition

$$H = \sum_k \frac{p_k^2 + q_k^2}{2} \qquad \text{when } q_k - q_{k+1} = 1 \text{ then } p_k \to p_{k+1},\ p_{k+1} \to p_k$$

Effective calculation of the collision times – simulation on very long times pissible

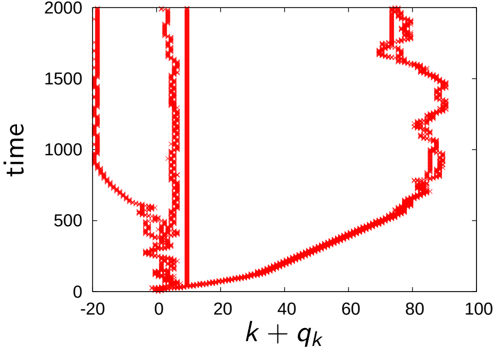
Strongly nonlinear lattice: no linear waves, no phonons, all propagating perturbations are nonlinear

Compactons in a homogeneous Ding-Dong lattice

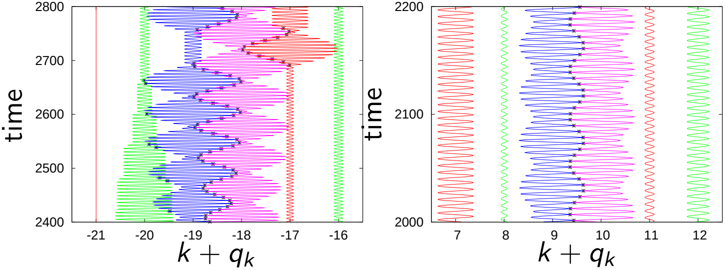


Spreading in a homogeneous lattice

From random initial conditions: chaos, breathers, and (almost)compactons appear



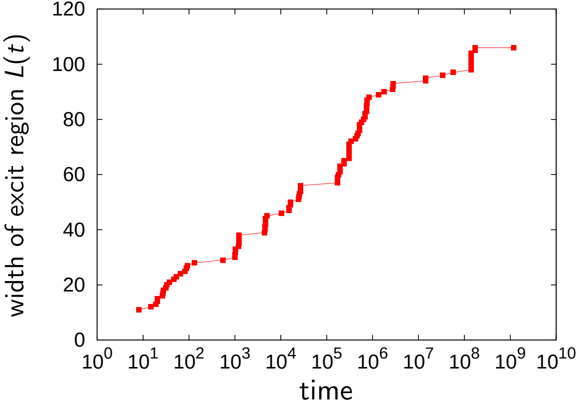
Examples of chaos and breathers



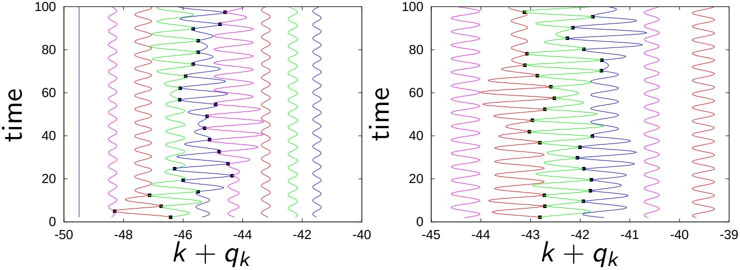
Spreading in a disordered lattice

Disorder in distances or masses destroys compactons

Spreading effectively stops: no spreading events for time interval 10^{10}, a few chaotic spots appear



Chaotic spot at a boundary of spreading range



Spreading in 2-dimensional lattices [M. Mulansky, A.P., Phys. Rev. E, 2012]

Hamiltonian $H = \sum_{i,k} E_{i,k}$ with

$$\begin{aligned} E_{i,k} = \frac{p_{i,k}^2}{2} + \frac{W\omega_{i,k}^2}{\kappa}|q_{i,k}|^\kappa + \\ + \frac{\beta}{2\lambda}(|q_{i+1,k} - q_{i,k}|^\lambda + |q_{i-1,k} - q_{i,k}|^\lambda \\ + |q_{i,k+1} - q_{i,k}|^\lambda + |q_{i,k-1} - q_{i,k}|^\lambda) \ . \end{aligned}$$

Sharply localized field - no compactons even without disorder!

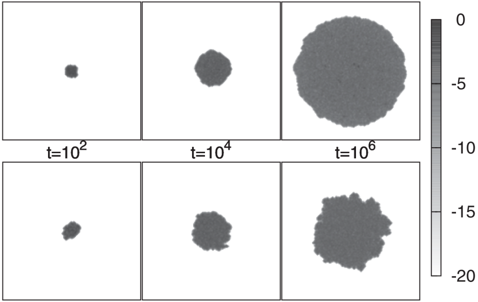


FIG. 1. Instantaneous logarithmic local energy density $\log_{10} w_{i,k}$ for $\kappa = 2$, $\lambda = 4$ at times 10^2, 10^4, and 10^6 (left to right panels). The upper row shows results for a regular lattice ($\omega_{i,k} = 1$) with energy $E = 1$ ($W = \beta = 1$ from variable transformations). The lower row is from simulations of a disordered lattice ($\omega_{i,k} \in [0,1]$) with energy $E = 10$. The total size of the squares is 160×160 lattice sites.

2-dimensional Nonlinear Diffusion Equation

$$\frac{\partial \rho}{\partial t} = \nabla\left(\rho^a \nabla \rho\right), \qquad \text{with} \qquad \int \rho\, d^2\vec{r} = E\ .$$

has a solution with growing radius

$$R^2 = 4B\frac{a+1}{a} \cdot (t - t_0)^{1/(a+1)} \quad \text{and} \quad B = \left(\frac{E}{4\pi}\right)^{\frac{a}{a+1}}\ .$$

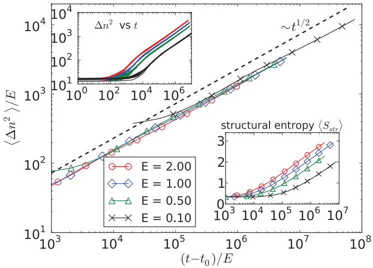
The scaling prediction:

$$\frac{\Delta n^2}{E} \sim \left(\frac{t - t_0}{E}\right)^{\nu} \qquad \nu = \frac{1}{a+1}\ ,$$

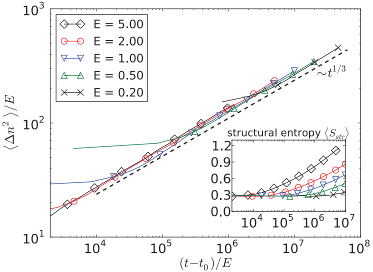
Spreading in regular 2-dimensional lattices

Linear oscillators, coupled via

nonlinearity power 4



nonlinearity power 6



A simple resonance model of spreading

An initially non-excited (linear) site is excited by a neighbor oscillating with amplitude ϵ and frequency $\Omega = 1 + a\epsilon^{\lambda-2}$ (this shift of frequency follows from the nonlinear coupling term).

$$H_1 = \frac{p^2 + q^2}{2} + \frac{|q - \epsilon \sin \Omega t|^\lambda}{\lambda} \ .$$

The resonant averaged Hamiltonian

$$\langle H_1 \rangle = -a\epsilon^{\lambda-2} I + \epsilon^\lambda F(\sqrt{2I\epsilon^{-2}} \cos\theta, \sqrt{2I\epsilon^{-2}} \sin\theta) \ ,$$

can be rescaled by $I \to \epsilon^2 I$, $t \to \epsilon^{\lambda-2} t$ which yields

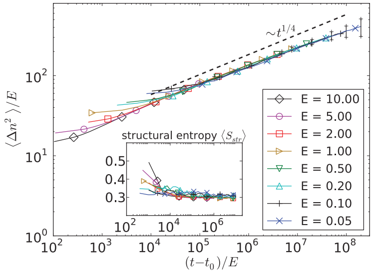
$$\Delta n^2 \sim t^{2/\lambda}$$

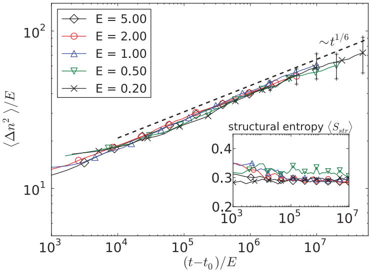
in accordance with numerics

Spreading in disordered 2-dimensional lattices

Linear oscillators, coupled via

nonlinearity power 4 | nonlinearity power 6





Conclusions for 2-dimensional wavepacket spreading

- Nonlinearly coupled linear oscillators:
  NDE $\frac{\partial \rho}{\partial t} = \nabla \left(\rho^a \nabla \rho \right)$ scaling works, good power laws
- Regular lattices:
  effective powers in NDE $a_{4,r} \approx 1, a_{6,r} \approx 2 \Rightarrow a = \frac{\lambda}{2} - 1$
- Irregular lattices:
  effective powers in NDE $a_{4,d} \approx 3, a_{6,d} \approx 5 \Rightarrow a = \lambda - 1$