An excursion from nonequilibrium steady states to extreme order statistics.

**Questions:**

How to construct nonequilibrium steady states for quantum systems?

General and distinct features of the steady states?

How do quantum systems relax to the steady state?

How to understand quantum fronts?

**Latest:**

The front can be described in terms of the statistics of the edge spectrum of random matrices (GUE).
Transverse Ising model with energy flux

\[ \hat{H}_I = -\sum_{n=1}^{N} S_n^x S_{n+1}^x - \frac{h}{2} \sum_{n=1}^{N} S_n^z \]

Find the ground state of

\[ \hat{H} = \hat{H}_I + \lambda \hat{J}_E \]

Energy flux

\[ \hat{J}_E = \frac{h}{4} \sum_{n=1}^{N} (S_n^x S_{n+1}^y - S_n^y S_{n+1}^x) \]

Local energy

\[ \varepsilon_n = -\frac{1}{2} S_n^x (S_{n-1}^x + S_{n+1}^x) - h S_n^z \]

Local energy flux

\[ \dot{\varepsilon}_n = i[\hat{H}, \varepsilon_n] = j_{E,n-\rightarrow n} - j_{E,n-\rightarrow n+1} \]

\[ \langle \sigma_i^x \sigma_{i+n}^x \rangle \sim \frac{1}{\sqrt{n}} \cos k n \]

J = 0

ordered

\langle \sigma_i^x \rangle \neq 0

J \neq 0

k \neq 0

J = 0

disordered

\langle \sigma_i^x \rangle = 0

k = 0

\[ \zeta = h \lambda / 2 \]

PRL 78, 167 (1997)
T. Antal, L. Sasvari, Z. R.
Evolution from natural initial states

\[ \hat{H}_{xx} = -\sum_{n=1}^{N-1} (S_n^x S_{n+1}^x + S_n^y S_{n+1}^y) \]

**Questions:**
Are there steady states in the \( t \to \infty \)?
Can they be described by \( H_\lambda \)?

Time evolution (exact):
\[ m_0 = \frac{1}{2} \]
\[ \langle S_n^z (t) \rangle = m(n, t) = -\frac{1}{2} \sum_{\ell=1-n}^{\ell=1+n} J^2_\ell (t) \]

Scaling limit \( t \to \infty \quad n/t = \nu \)
\[ m(n, t) \to -\frac{1}{\pi} \arcsin\left(\frac{n}{t}\right) \]

- \( J_M \) states are OK.

Problems in case of energy flux:

\[ \hat{H}_\lambda = \hat{H}_{xx} - \lambda \, j_M \]
Scaling structure of the front

Scaling in the front:

\[ m(n,t) - m(t,t) = \frac{1}{t^{1/3}} \Phi\left(\frac{t-n}{t^{1/3}}\right) \]

Front shape is formed early

- Width of the front \( \propto t^{1/3} \)
- Height of the front \( \propto 1/t^{1/3} \)
Quasi-classical picture of quantum transport?

Area under the steps is constant

\[ h \simeq 1 \]

The steps carry one spin

Quasi-classical picture:

Subdiffusive spreading

Fermionic description: Full counting statistics

\[ \hat{H} = -\frac{1}{2} \sum_{n=-\infty}^{\infty} (c_n^+ c_{n+1} + c_n c_{n+1}^+) \]

Questions:

Number of particles in A: \( \langle \hat{N}_A \rangle \)?

Fluctuations in A: \( \langle \hat{N}_A^2 \rangle - \langle \hat{N}_A \rangle^2 \)?

\[ c_n(t) = \sum_{j=-\infty}^{\infty} i^{j-n} J_{j-n}(t)c_j(0) \]

\[ \rho(n,t) = \langle c_n^+(t)c_n(t) \rangle \]

Generating function for full counting statistics:

\[ \chi(\lambda,t) = \langle \exp[i \lambda \hat{N}_A(t)] \rangle \]
Generating function for full counting statistics (FCS)

\[ \chi(\lambda, t) = \langle \exp[i \lambda \hat{N}_A(t)] \rangle \]

Expression in terms of a determinant
K. Schönhammer, PRB 75, 205329 (2007)

\[ \chi(\lambda, t) = \det[\hat{1} + (e^{i\lambda} - 1)\hat{C}] \]

\[ \hat{C}_{mn}(t) = \langle c_m(t)c_n(t) \rangle = \]

\[ -\frac{t^{n-m}}{2(m-n)} [J_{m-1}(t)J_n(t) - J_m(t)J_{n-1}(t)] \]

To get the scaling regime in the front:

\[ m = t + \left(\frac{t}{2}\right)^{1/3} x \quad n = t + \left(\frac{t}{2}\right)^{1/3} y \]

\[ J_{m-1}(t) = \dot{J}_m(t) + \frac{m}{t} J_m(t) \]

\[ J_m(t) = -\left(\frac{2}{t}\right)^{1/3} \text{Ai}(x) \quad \dot{J}_m(t) = -\left(\frac{2}{t}\right)^{1/3} \text{Ai}'(x) \]
Full counting statistics in the front

\[ \chi(\lambda, t) = \langle \exp[i \lambda \hat{N}_A(t)] \rangle \]

\[ \hat{C}_{mn}(t) = \frac{2^{1/3}}{t^{1/3}} K(x, y) + O\left(\frac{1}{t^{2/3}}\right) \]

\[ K(x, y) = \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}'(x)\text{Ai}(y)}{x - y} \]

\[ \chi(\lambda, s) = \det[I + (e^{i\lambda} - 1)K] \]

\[ m = t + \left(\frac{t}{2}\right)^{1/3} x (y) \]

Airy kernel
**Full counting statistics and random matrices**

\[ \chi(\lambda, s) = \det[ I + (e^{i\lambda} - 1)K ] \]

Comparison with \( N \times N \) random matrices from GUE:

\[ -\sqrt{2N} \quad \quad \quad \quad \quad \quad \quad \sqrt{2N} \]

\[ \frac{s}{\sqrt{2N^{1/6}}} \]

Probability that exactly \( n \) eigenvalues lie in the interval \((s, \infty)\):

\[ E(n, s) = \left. \frac{(-1)^n}{n!} \frac{d^n}{dz^n} \det(1-zK) \right|_{z=1} \]

\[ \sum_n e^{i\lambda n} E(n, s) = \]

\[ \sum_n e^{i\lambda n} \left. \frac{(-1)^n}{n!} \frac{d^n}{dz^n} \det(1-zK) \right|_{z=1} = \sum_n \exp \left( -e^{i\lambda} \frac{d}{dz} \right) \left. \det(1-zK) \right|_{z=1} = \det[ I + (e^{i\lambda} - 1)K ] \]

\[ = \chi(\lambda, s) \]
Using the random matrix results

Probability density of the $n$-th largest eigenvalue:

$$ F(n, x) = \sum_{k=0}^{n-1} \frac{dE(k, x)}{dx} $$

Density of eigenvalues = sum of single eigenvalue densities:

$$ \rho(x) = -\sum_{k=0}^{\infty} \frac{dE(k, x)}{dx} = \sum_{k=0}^{\infty} F(k, x) = K(x, x) = [\text{Ai}'(x)]^2 - x\text{Ai}^2(x) $$

Extreme order statistics:

$$ F(3, x) $$

Probability distribution of the 3$^{rd}$ particle

Total particle density

$x = (m-t)/(t/2)^{1/3}$

4 3 2 1
Using the random matrix results

Fluctuations in particle number: \[ \kappa_2 = \langle N_A^2 \rangle - \langle N_A \rangle^2 = \text{Tr} K (1 - K) \]

Entanglement between \((-\infty, s)\) and \((s, \infty)\): \[ S = \ldots = \text{Tr} [ K \ln K - (1 - K) \ln(1 - K) ] \]

Probability that a \(N \times N\) Gaussian random matrix has \(N_L\) eigenvalues in \([-L, L]\).

Gap scaling \((\text{A. Perret and G. Schehr, arXiv: 1312.2966})\).
Applications, problems, conclusions

Control over moving spins

Practically identical profiles, compare with other spin chains.

Phase transition between the edge and bulk eigenvalue spectrum?

$m_0$ determines the number of steps arriving at time $t$.

Control over moving spins?