Spatial extent of an outbreak in animal epidemics

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SIR model for epidemics

Three species: susceptibles (S), infected (I), recovered (R)

\[
\frac{dS}{dt} = -\beta IS
\]

\[
\frac{dI}{dt} = \beta IS - \gamma I
\]

\[
\frac{dR}{dt} = \gamma I
\]

- mean field fully connected model
- \( \beta \) rate of infection transmission
- \( \gamma \) rate at which an infected recovers

\[
I(t) + S(t) + R(t) = N
\]

\( N \) being the total population
Outbreak of an epidemic

Initial condition: \( I(0) = 1, S(0) = N - 1 \approx N, R(0) = 0 \)

\[
\begin{align*}
\frac{dS}{dt} &= -\beta IS \\
\frac{dI}{dt} &= \beta IS - \gamma I \\
\frac{dR}{dt} &= \gamma I
\end{align*}
\]

Outbreak regime

\[
\frac{dI}{dt} \approx (\beta N - \gamma) I
\]

Reproduction rate: \( R_0 = \frac{\beta N}{\gamma} \)
Deterministic and stochastic models

SIR is a deterministic model. In the outbreak fluctuations are important

- Stochastic process: Galton-Watson (mean field)
- each infected individual transmits the disease at rate $N\beta$
- each infected individual recovers at rate $\gamma$

Reproduction rate: $R_0 = \frac{\beta N}{\gamma}$

- $R_0 < 1$ epidemics extinction
- $R_0 > 1$ epidemics invasion
- $R_0 = 1$ critical case
How far in space can an epidemic spread?

Problem 1: How to model the space?

The good candidate: Brownian process with branching and death

In $dt$, each infected can:

- recovers with probability $\gamma dt$
- infects with probability $\beta N dt = \gamma R_0 dt$
- otherwise, it diffuses ($D$ diffusion const.)
**Problem 2:** How to quantify the area that needs to be quarantined?

**CONVEX HULL**

**Algorithms:** Graham Scan (Nlog(N))
Monitoring the outbreak

Day 1

Day 2

Day 3
Real applications

[Graph showing number of infected premises over time with a color-coded map of infected area and a legend indicating days till clinical signs.]
How to compute the convex hull of Branching processes?

Cauchy formulas

\[ L = \int_0^{2\pi} M(\theta) \, d\theta \]

\[ A = \frac{1}{2} \int_0^{2\pi} \left[ M^2(\theta) - (M'(\theta))^2 \right] \, d\theta \]
Support Function

\[ M(\theta) = \max_{0 \leq \tau \leq t} [x_\tau \cos \theta + y_\tau \sin \theta] \]

\[ M(0) = x_{\tau=t_m} = x_m(t) \]

- \( x_m(t) \) \( x \)-maximum up to time \( t \)
- \( t_m \) time location of the maximum

\[ M'(\theta = 0) = -x_{t_m} \sin \theta + y_{t_m} \cos \theta|_{\theta=0} = y_{t_m} \]
\[ \langle L(t) \rangle = 2\pi \langle x_m(t) \rangle \]
\[ \langle A(t) \rangle = \pi \left[ \langle x_m^2(t) \rangle - \langle y^2(t_m) \rangle \right] \]

consider a 1d branching process evolving in \((0, t)\)

- \(x_m\) is the global maximum
- \(t_m\) is the location of the maximum
- \(\langle y^2(t_m) \rangle = \ldots = 2D t_m\)
Backward Fokker Planck equation

\[ Q_t(x_m) = \text{Proba}[\text{global max up to } t < x_m] \]

\[ Q_{t+dt}(x_m) = \gamma dt + R_0 \gamma dt Q_t^2(x_m) + [1 - \gamma(R_0 + 1)] dt \langle Q_t(x_m - \Delta x) \rangle \]

- \[ \langle Q_t(x_m - \Delta x) \rangle = Q_t(x_m) - \langle \Delta x \rangle Q'_t(x_m) + \langle \frac{\Delta x^2}{2} \rangle Q''_t(x_m) + \ldots \]
- \[ \langle \Delta x \rangle = 0 \]
- \[ \langle \Delta x^2 \rangle = 2D dt \]

\[ \langle Q_t(x_m - \Delta x) \rangle = Q_t(x_m) + D dt \partial_x^2 Q_t(x_m) + \ldots \]
\[
\frac{\partial}{\partial t} Q = D \frac{\partial^2}{\partial x_m^2} Q - \gamma (R_0 + 1) Q + \gamma R_0 Q^2 + \gamma
\]

- initial condition \( Q_{t=0}(x_m) = \theta(x_m) \)
- boundary condition \( Q_t(x_m < 0) = 0 \)
- boundary condition \( Q_t(x_m \to \infty) = 1 \)

\[
\langle L(t) \rangle = 2\pi \int_0^\infty [1 - Q_t(x_m)] \, dx_m.
\]
Solid lines: our predictions
Blue lines: super-critical
Red lines: critical
Green lines: sub-critical
Symbols: Monte Carlo simulations
Dashed lines: analytical asymptotic results
The critical case

\[ \langle L(t \to \infty) \rangle = 2\pi \sqrt{\frac{6D}{\gamma}} + O(t^{-1/2}) \]

\[ \langle A(t \to \infty) \rangle = \frac{24\pi D}{5\gamma} \ln t + O(1) \]
When $t \to \infty$ the perimeter remains finite, but the area diverges!

How it is possible? ... Fluctuations

$$\text{Prob}(A) \xrightarrow{t=\infty, A \to \infty} \frac{24\pi D}{5\gamma} A^{-2}$$

$$\text{Prob}(L) \xrightarrow{t=\infty, L \to \infty} L^{-3}$$
Out of criticality

When $R_0 \neq 1$, characteristic time $t^* \sim |R_0 - 1|^{-1}$.

For times $t < t^*$ the epidemic behaves as in the critical regime.

In the subcritical regime, for $t > t^*$ the epidemic goes to extinction.

In the supercritical regime, with probability $1 - 1/R_0$ epidemic explodes.
Supercritical

\[
\langle L(t \gg t^*) \rangle = 4\pi \left( 1 - \frac{1}{R_0} \right) \sqrt{D \gamma (R_0 - 1)} t \\
\langle A(t \gg t^*) \rangle = 4\pi \left( 1 - \frac{1}{R_0} \right) D \gamma (R_0 - 1) t^2
\]

\[t^* \sim |R_0 - 1|^{-1}\]
\[
\frac{\partial}{\partial t} W = D \frac{\partial^2}{\partial x_m^2} W + \gamma (R_0 - 1) W - \gamma R_0 W^2
\]

Traveling front solution
Conclusions:

• Branching Brownian motion with death as a model for the spatial extent of animal epidemics

• Using Cauchy Formulas we can map the convex hull problem in the extreme statistic of the 1-dimensional process

• Backward F-P equations for the extreme distributions

• Critical case has very large fluctuations

• Super Critical case: traveling front solution
How far in space can an epidemic spread?

**Problem 1**: How to model the space?

The population is uniformly distributed.

At time $t = 0$ an infected individual appears

... and moves in space.

Brownian motion is the paradigm of animal migration.

While human beings take the plane (even when they are sick).
Similar calculations allows to express the mean area as:

\[
\langle A(t) \rangle = \pi \int_0^{\infty} dx_m \left[ 2x_m(1 - Q_t(x_m)) - T_t(x_m) \right]
\]

Where the evolution of \( T_t(x_m) \) is governed by:

\[
\frac{\partial}{\partial t} T_t + \partial_x Q_t(x_m) = \left[ D \frac{\partial^2}{\partial x_m^2} + 2\gamma R_0 Q_t - \gamma (R_0 + 1) \right] T_t,
\]

Both PDE can be integrated numerically and solved in some asymptotic limit.
Dimensional reduction

\[ L = \int_{0}^{2\pi} M(\theta) \, d\theta \]

\[ A = \frac{1}{2} \int_{0}^{2\pi} \left[ M^2(\theta) - (M'(\theta))^2 \right] \, d\theta \]

If the process is rotationally invariant any average is independent of \( \theta \)

\[ \langle L(t) \rangle = 2\pi \langle M(0) \rangle \]

\[ \langle A(t) \rangle = \pi \left[ \langle M^2(0) \rangle - \langle M'(0)^2 \rangle \right] \]

\[ \langle L(t) \rangle = 2\pi \langle x_m(t) \rangle \]

\[ \langle A(t) \rangle = \pi \left[ \langle x_m^2(t) \rangle - \langle y^2(t_m) \rangle \right] \]

This relation is valid ONLY in average