2014/05/27@The Galileo Galilei Institute for Theoretical Physics
Advances in Nonequilibrium Statistical Mechanics:
large deviations and long-range correlations, extreme value statistics,
anomalous transport and long-range interactions

Nonlinear response theory in long-range Hamiltonian systems

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Main topics

We propose a nonlinear response theory
for long-range Hamiltonian systems.

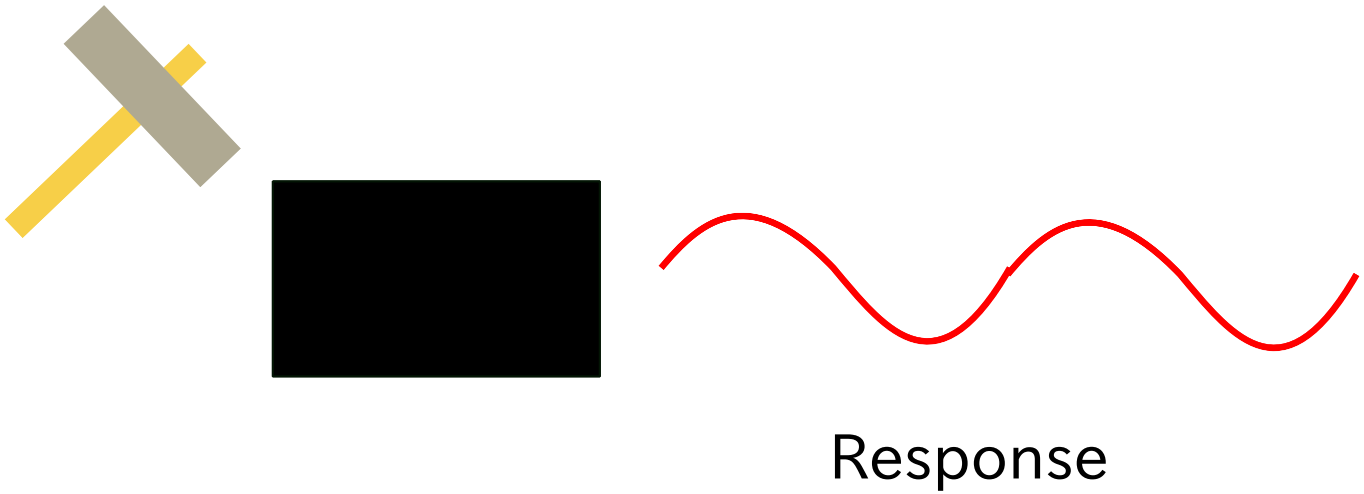
1) Reponse to external field

$\rightarrow$ Strange critical exponents and scaling relation

2) Reponse to perturbation

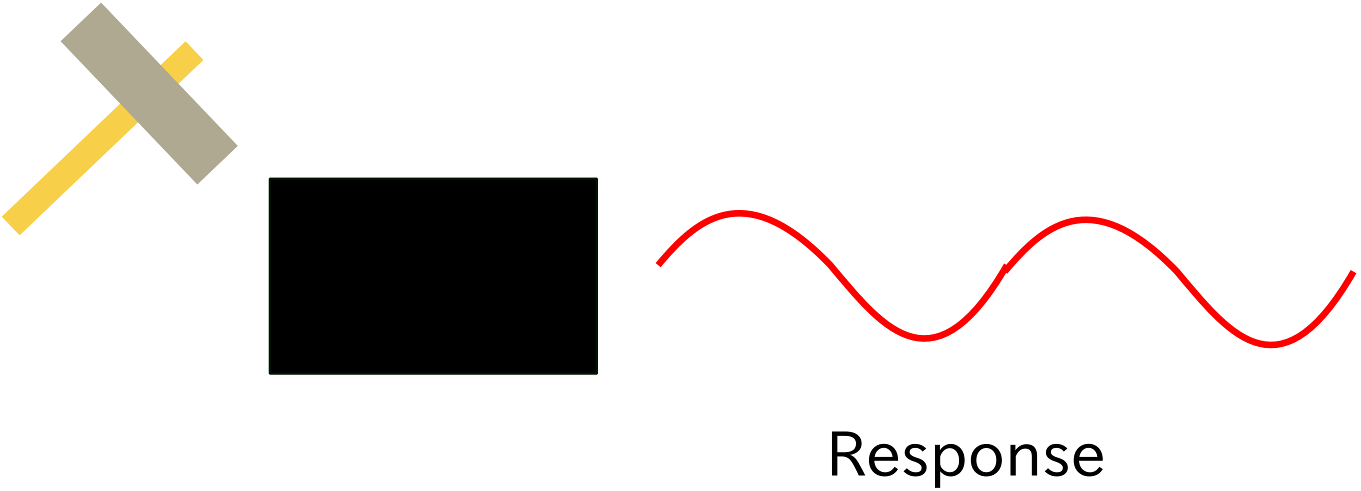
$\rightarrow$ Discussion on limitation of the theory

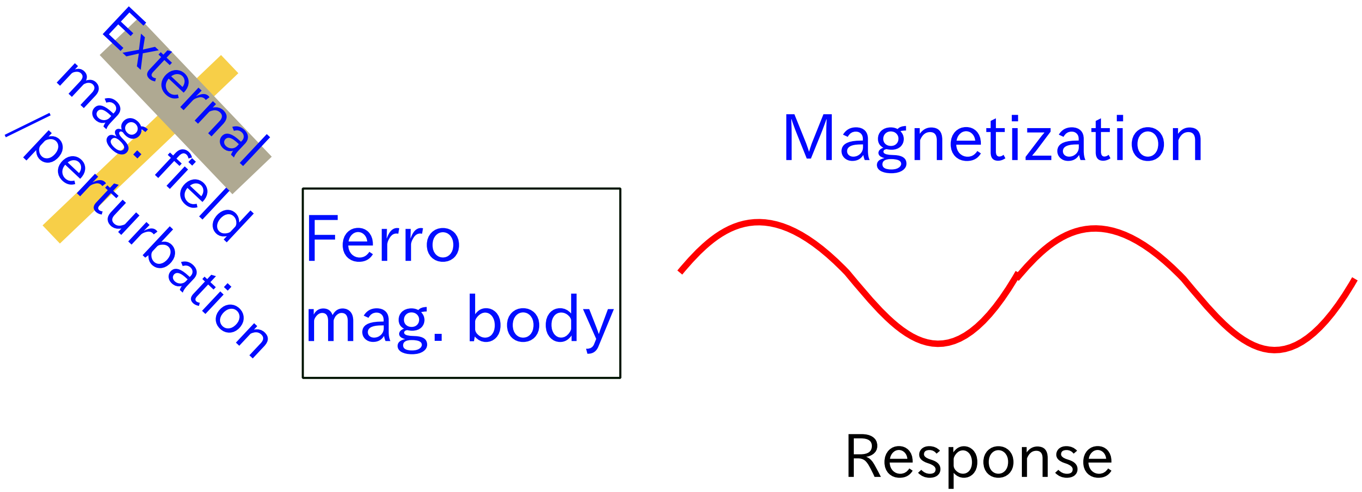
Response



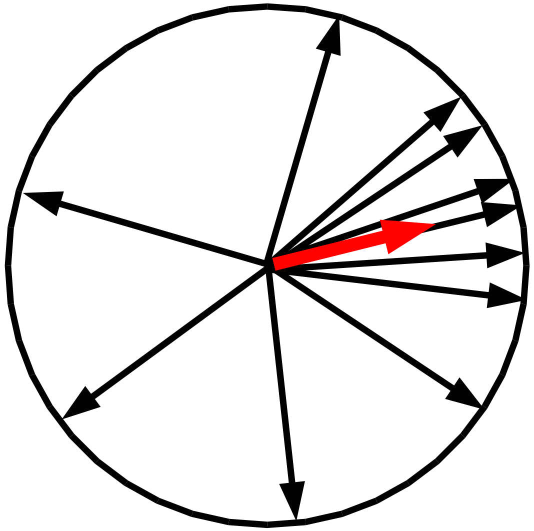
Observing the response, we get information of the black-box.

Response


Response


External
mag. field
/perturbation
Ferro
mag. body
Magnetization
Response

Hamiltonian mean-field model

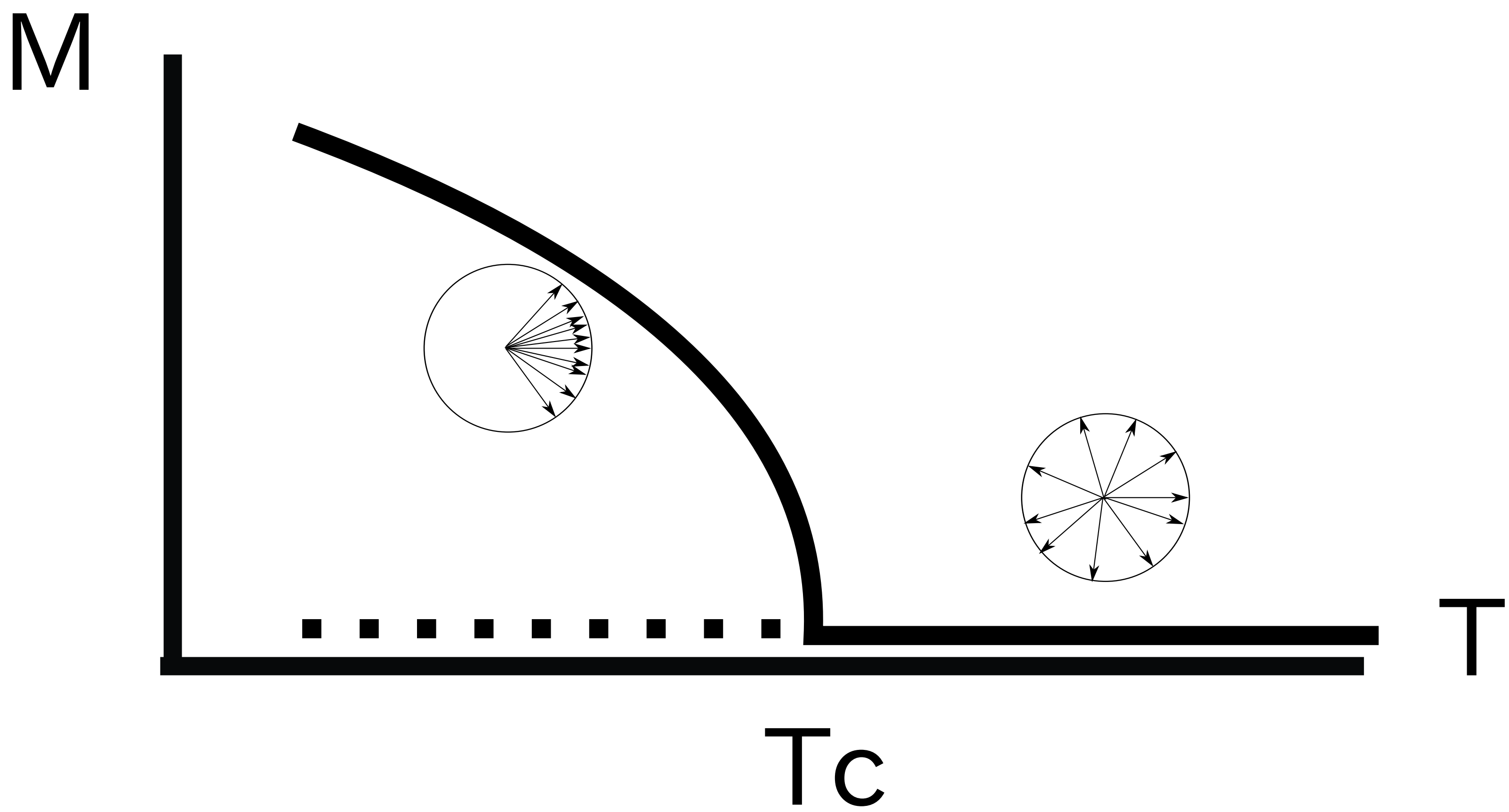


- A paradigmatic toy model of a ferro magnetic body
- Each spin interacts with the other spins attractively
- All interactions are only through the magnetization (mean-field) $\vec{M}$

$$H = \sum_{j=1}^{N} \frac{p_j^2}{2} - \frac{1}{2N} \sum_{j,k=1}^{N} \cos(q_j - q_k) - h \sum_{j=1}^{N} \cos q_j$$

h: external mag. field

Critical phenomena in HMF ($h = 0$)



Critical phenomena of mean-field systems
are analysed by Landau theory

Landau theory

Free energy: $$F(M) = \frac{a}{2}(T - T_\mathrm{c})M^2 + \frac{b}{4}M^4 + \cdots - hM$$

Realized M: $$\frac{\mathsf{d}F}{\mathsf{d}M} = a(T - T_\mathrm{c})M + bM^3 - h = 0$$

Landau theory

Realized M:
$$\frac{\mathrm{d}F}{\mathrm{d}M} = a(T - T_{\mathrm{c}})M + bM^3 - h = 0$$

Critical exponents

Landau theory

Realized M: $$\frac{\mathrm{d}F}{\mathrm{d}M} = a(T - T_\mathrm{c})M + bM^3 - h = 0$$

Critical exponents

$h = 0$: $$M \propto (T_\mathrm{c} - T)^{\beta} \qquad \beta = \frac{1}{2}$$

Landau theory

Realized M: $$\frac{\mathrm{d}F}{\mathrm{d}M} = a(T - T_\mathrm{c})M + bM^3 - h = 0$$

Critical exponents

$h = 0$: $$M \propto (T_\mathrm{c} - T)^{\beta} \qquad \beta = \frac{1}{2}$$

$h \neq 0$: $$\left.\frac{\mathrm{d}M}{\mathrm{d}h}\right|_{h\to 0} \propto (T - T_\mathrm{c})^{-\gamma_+} \qquad \gamma_+ = 1 \qquad T > T_\mathrm{c}$$

Landau theory

Realized M: $$\frac{\mathrm{d}F}{\mathrm{d}M} = a(T - T_\mathrm{c})M + bM^3 - h = 0$$

Critical exponents

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$$\propto (T_\mathrm{c} - T)^{-\gamma_-} \quad \gamma_- = 1 \quad T < T_\mathrm{c}$$

Landau theory

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$$\propto (T_\mathrm{c} - T)^{-\gamma_-} \qquad \gamma_- = 1 \qquad T < T_\mathrm{c}$$

$T = T_\mathrm{c}$: $$M \propto h^{1/\delta} \qquad \delta = 3$$

Landau theory

Scaling relation

$$\gamma_\pm = \beta(\delta - 1)$$

Critical exponents

$h = 0$: $\quad M \propto (T_\mathrm{c} - T)^{\beta} \qquad \beta = \frac{1}{2}$

$h \neq 0$: $\quad \left.\frac{\mathrm{d}M}{\mathrm{d}h}\right|_{h\to 0} \propto (T - T_\mathrm{c})^{-\gamma_+} \qquad \gamma_+ = 1 \qquad T > T_\mathrm{c}$

$\propto (T_\mathrm{c} - T)^{-\gamma_-} \qquad \gamma_- = 1 \qquad T < T_\mathrm{c}$

$T = T_\mathrm{c}$: $\quad M \propto h^{1/\delta} \qquad \delta = 3$

Question

Landau theory gives critical exponents in the context of statistical mechanics.

Q. Does dynamics give the same critical exponents ?

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j}$$

For simplicity, we start from themal equilibrium states:
$\rightarrow \beta = 1/2$.

Vlasov approach

N-body:

$$H = \sum_{j=1}^{N} \left[\frac{p_j^2}{2} - \frac{1}{2N} \sum_{k=1}^{N} \cos(q_j - q_k) - h \cos q_j \right]$$

1-body:

$$\mathcal{H}[f] = \frac{p^2}{2} - \int \cos(q - q') f(q', p', t) dq' dp' - h \cos q$$

Vlasov equation:

$$\frac{\partial f}{\partial t} = \frac{\partial \mathcal{H}[f]}{\partial q} \frac{\partial f}{\partial p} - \frac{\partial \mathcal{H}[f]}{\partial q} \frac{\partial f}{\partial q} = \{\mathcal{H}[f], f\}$$

Linear response theory

- Patelli et al., PRE **85**, 021133 (2012)
- Ogawa-YYY, PRE **85**, 061115 (2012)

- Ogawa-Patelli-YYY, PRE **89**, 032131 (2014)

Critical exponents

$h = 0$: $M \propto (T_c - T)^{\beta}$ $\quad \beta = \frac{1}{2}$

$h \neq 0$: $\left.\frac{dM}{dh}\right|_{h \to 0} \propto (T - T_c)^{-\gamma_+}$ $\quad \gamma_+ = 1$ $\quad \gamma_+ = 1$

$\propto (T_c - T)^{-\gamma_-}$ $\quad \gamma_- = 1$ $\quad \gamma_- = \frac{1}{4}$

$T = T_c$: $M \propto h^{1/\delta}$ $\quad \delta = 3$

Nonlinear response theory

- We need a nonlinear response theory for δ.
- Check the scaling relation $\gamma = \beta(\delta - 1)$.

Critical exponents

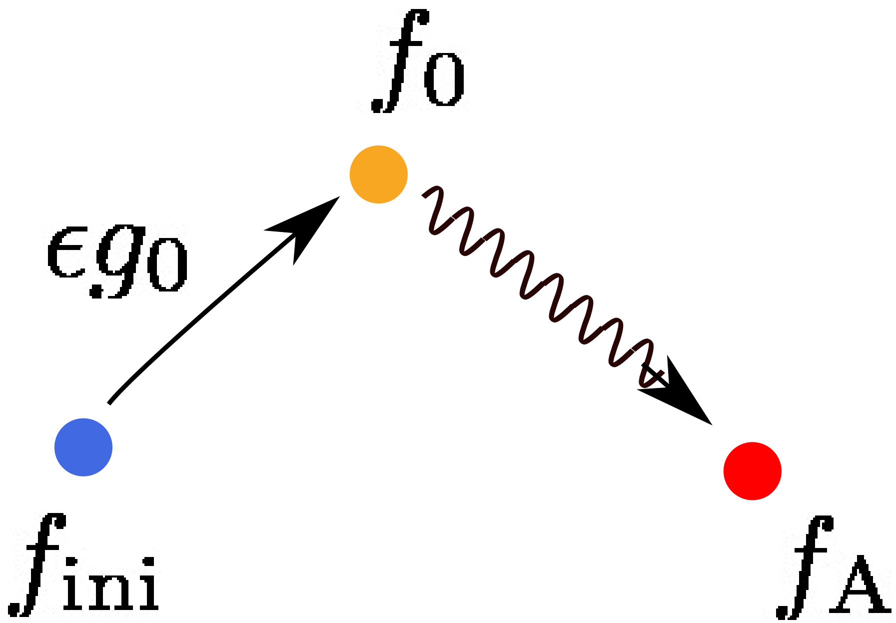
$h = 0$: $\quad M \propto (T_\mathrm{c} - T)^{\beta} \quad \beta = \frac{1}{2}$

$$h \neq 0: \quad \left.\frac{\mathrm{d}M}{\mathrm{d}h}\right|_{h\to 0} \propto (T - T_\mathrm{c})^{-\gamma_+} \quad \gamma_+ = 1 \quad \gamma_+ = 1$$

$$\propto (T_\mathrm{c} - T)^{-\gamma_-} \quad \gamma_- = 1 \quad \gamma_- = \frac{1}{4}$$

$T = T_\mathrm{c}$: $\quad M \propto h^{1/\delta} \quad \delta = 3 \quad \delta = ?$

Idea

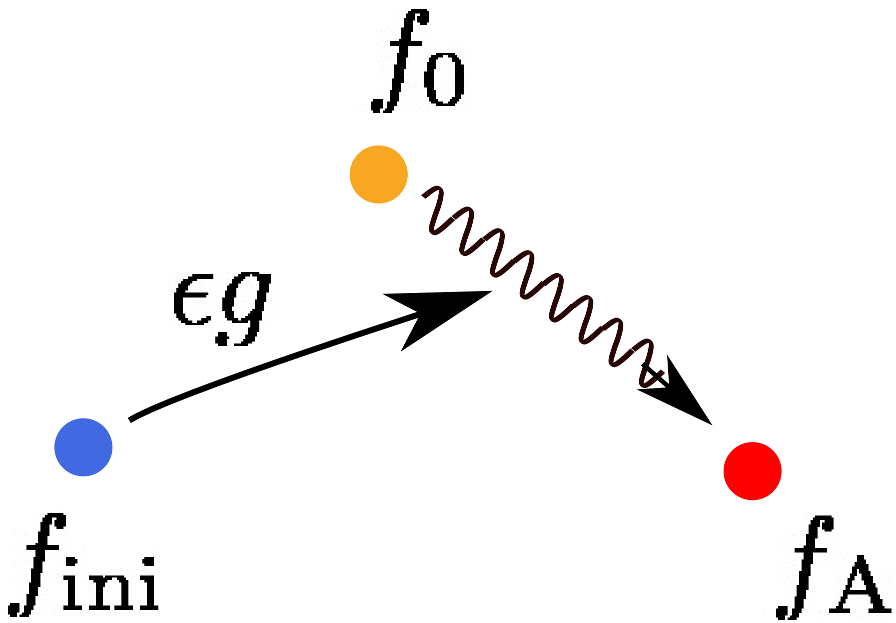


$f_{\rm ini}$: Initial stationary state

f_0 : Initial state with perturbation ϵg_0

$f_{\rm A}$: Asymptotic state

Idea

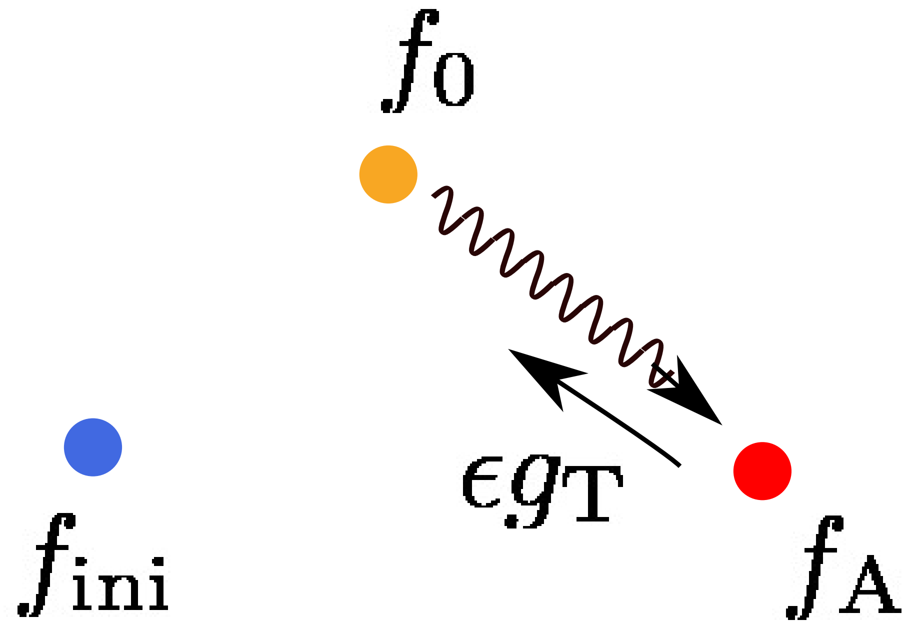


Normal decomposition: $f = f_{\rm ini} + \epsilon g$

$\mathcal{H}[f_{\rm ini}]$ drives the system

(cf. Landau damping)

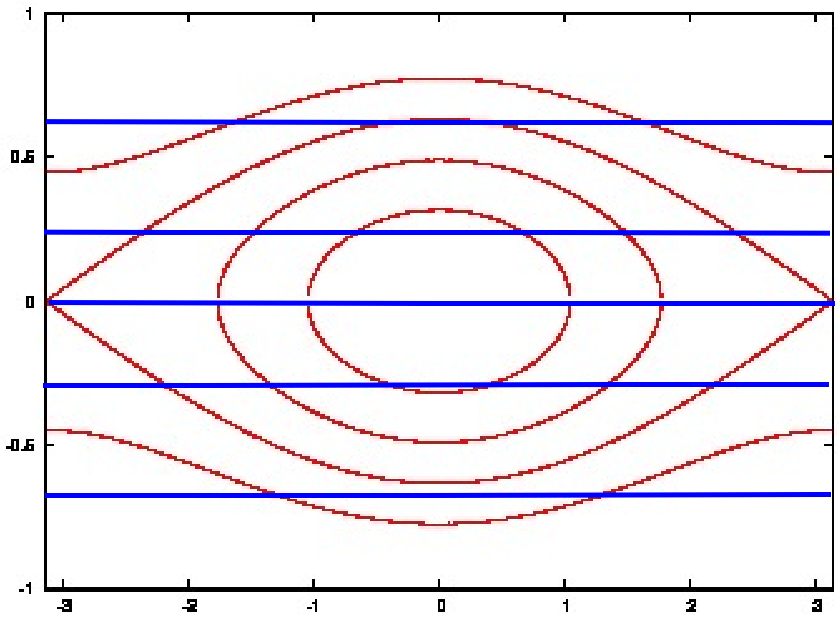
Idea



Our decomposition: $f = f_{\mathrm{A}} + \epsilon g_{\mathrm{T}}$

$\mathcal{H}[f_{\mathrm{A}}]$ drives the system

Asymptotic state

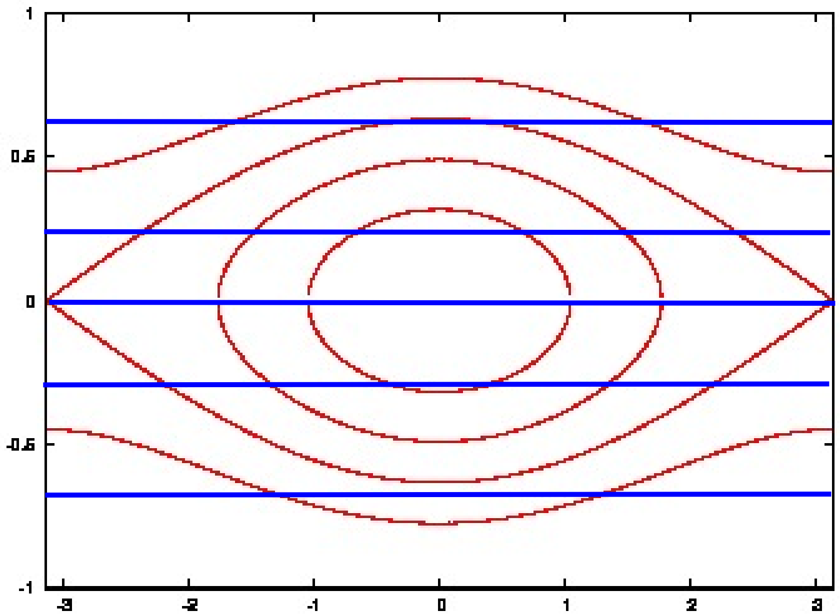


Contours of f_0

Contours of $f_{\rm A}$

$$f_{\rm A} = \text{(average of } f_0 \text{ over iso-}\mathcal{H}[f_{\rm A}] \text{ curve)}$$

Asymptotic state



Contours of f_0

Contours of f_{A}

$f_{\mathrm{A}} =$ (average of f_0 over iso-$\mathcal{H}[f_{\mathrm{A}}]$ curve)

$\Downarrow \quad (\theta, J)$: Angle-action associated with $\mathcal{H}[f_{\mathrm{A}}]$

$f_{\mathrm{A}} = \langle f_0 \rangle_J$: Average over θ (iso$-J$ curve)

Idea of re-arrangement itself is not new

$f_{\mathrm{A}} = \langle f_0 \rangle_J$: Re-arrangement of f_0 along iso-J curve

1-level waterbag initial distribution

- Leoncini-Van Den Berg-Fanelli, EPL **86**, 20002 (2009)
- de Buyl-Mukamel-Ruffo, PRE **84**, 061151 (2011)

multi-level waterbag initial distribution

- Ribeiro-Teixeira et al., PRE **89**, 022130 (2014)

What's new

- Landau like equation for asymptotic M

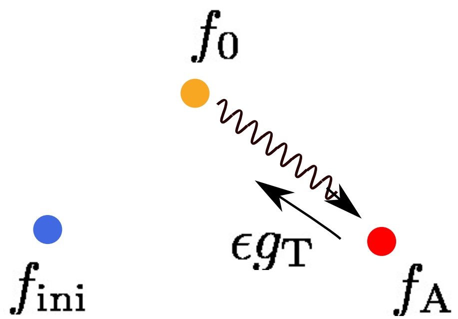
$\Longrightarrow$ Critical exponents

- Justification of theory (omitting ϵg_{T}) by the hypotheses

H0. The asymptotic state f_{A} is stationary.

H1. $f(t)$ is in a $O(\epsilon)$ neighbourhood of f_{ini}.

H2. We may omit $O(\epsilon^2)$.



$\Longrightarrow$ Discussion on limitation of the theory

Self-consistent equation for M

$$f_{\mathrm{A}} = \langle f_0 \rangle_J \qquad \Longrightarrow \qquad M = \int \cos\theta \, \langle f_0 \rangle_J \, dqdp$$

↑

J depends on M through $\mathcal{H}[f_{\mathrm{A}}]$

$$\mathcal{H}[f_{\mathrm{A}}] = \frac{p^2}{2} - (M + h)\cos q$$

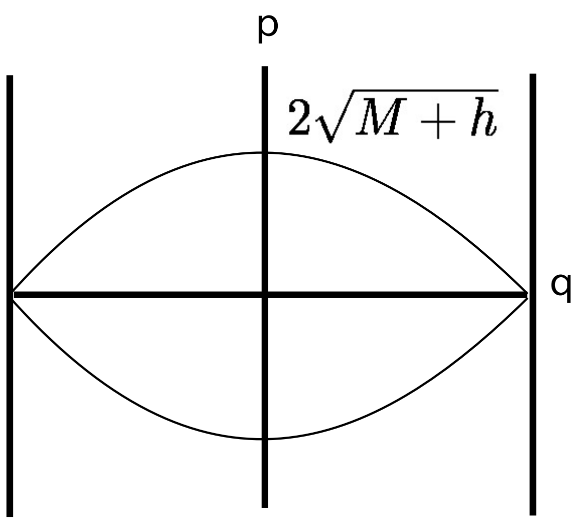
We expand the self-consistent equation for small M.

Expansion of self-consistent equation

- We focus on homogeneous $f_{\mathrm{ini}}(p)$.

$$M = \int \cos q \left\langle f_0 \right\rangle_J dqdp$$

$\longrightarrow$ Expansion — power series of $\sqrt{M+h}$



Separatrix width is of $O(\sqrt{M+h})$

Initial condition

$$f_0(q,p) = Ae^{-p^2/2T}(1+\epsilon\cos q)$$

Homogeneous Maxwellian

Perturbation

After long computations...

Landau like equation

$$-\epsilon a(M+h)^{1/2} + b(T-T_{\mathrm{c}})(M+h) + c(M+h)^{3/2} - h = 0$$

$$a, b, c > 0$$

cf. Landau theory:

$$a(T-T_{\mathrm{c}})M + bM^3 - h = 0$$

Landau like equation

$$-\epsilon a(M+h)^{1/2} + b(T-T_{\mathrm{c}})(M+h) + c(M+h)^{3/2} - h = 0$$

$\epsilon = 0:$

$T > T_{\mathrm{c}}$: $M \propto (T-T_c)^{-1} h$ Linear response

Landau like equation

$$-\epsilon a(M+h)^{1/2} + b(T-T_{\rm c})(M+h) + c(M+h)^{3/2} - h = 0$$

$\epsilon = 0:$

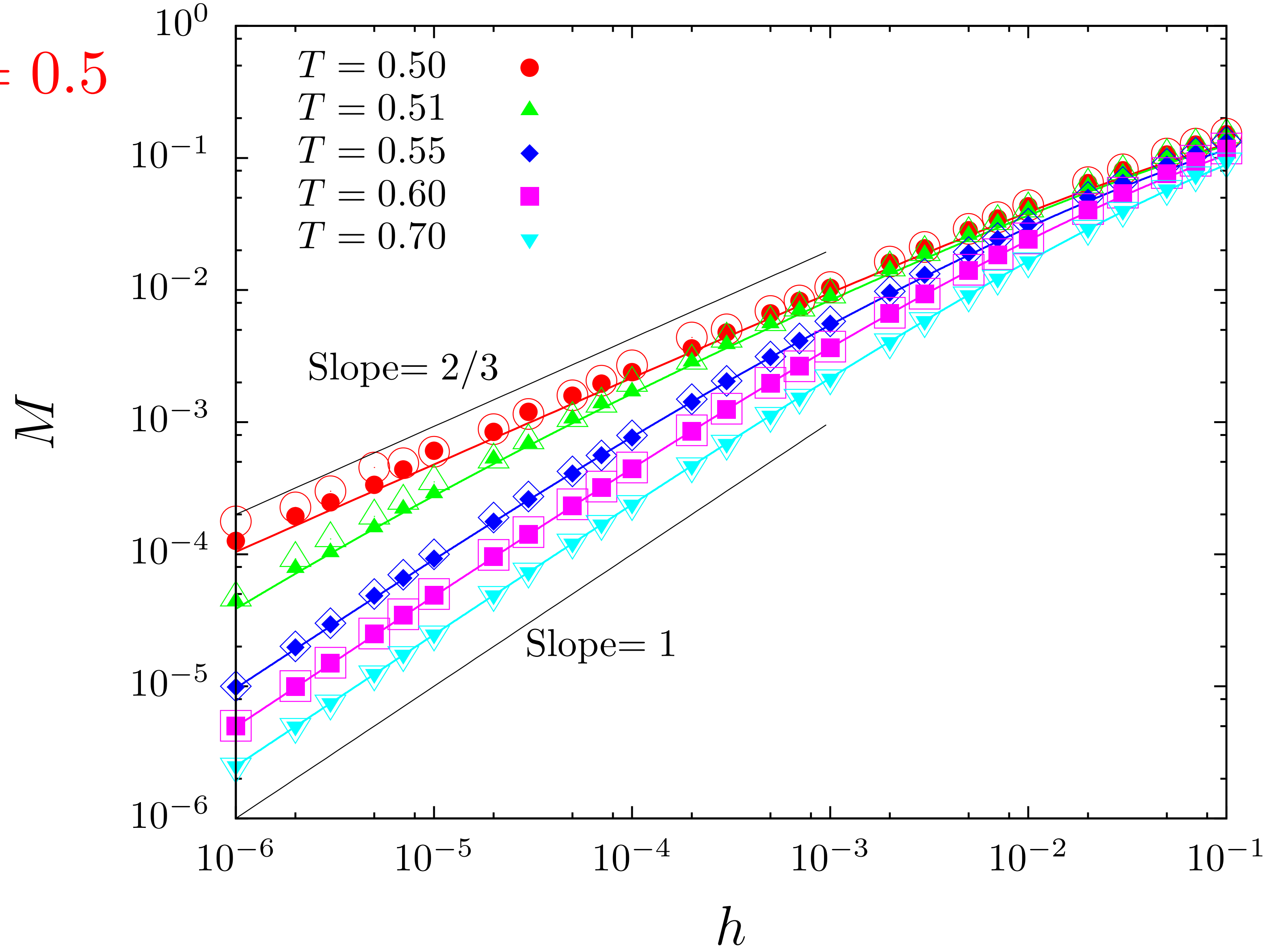
$T > T_{\rm c}$: $\quad M \propto (T-T_c)^{-1} h \quad$ Linear response

$T = T_{\rm c}$: $\quad M \propto h^{2/3} \quad$ Nonlinear response

$\delta = 3/2$

Response to external field (numerical test)

$T_c = 0.5$



Ogawa-YYY, PRE **89**, 052114 (2014) [slightly modified]

Scaling relation in Vlasov dynamics

- Scaling relation holds even in the Vlasov dynamics !

$$\gamma_- = \beta(\delta - 1)$$

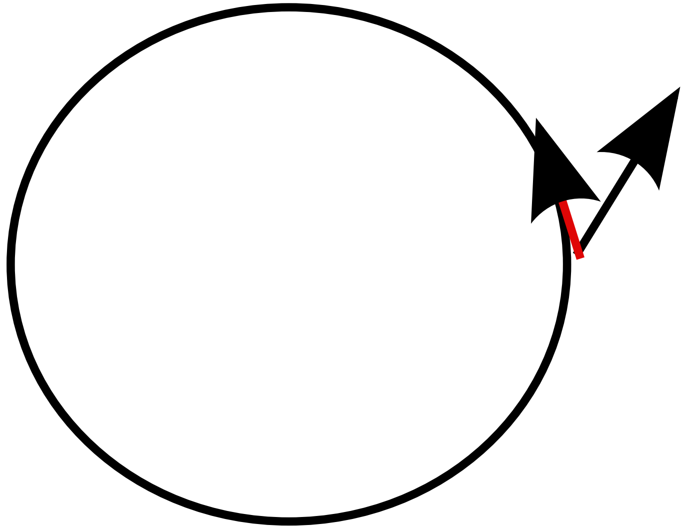
Critical exponents

$h = 0$:	$M \propto (T_c - T)^{\beta}$	$\beta = \frac{1}{2}$	$\beta = \frac{1}{2}$
$h \neq 0$:	$\left.\frac{\mathrm{d}M}{\mathrm{d}h}\right\vert_{h\to 0} \propto (T - T_c)^{-\gamma_+}$	$\gamma_+ = 1$	$\gamma_+ = 1$
	$\propto (T_c - T)^{-\gamma_-}$	$\gamma_- = 1$	$\gamma_- = \frac{1}{4}$
$T = T_c$:	$M \propto h^{1/\delta}$	$\delta = 3$	$\delta = \frac{3}{2}$

Origin of the strange exponents

The Vlasov equation has infinite invariants called Casimirs:

$$C[f] = \int c(f(q,p))dqdp \qquad \forall c \text{ smooth}$$



Response to perturbation

$$-\epsilon a(M+h)^{1/2} + b\Delta T(M+h) + c(M+h)^{3/2} - h = 0$$

$$\Delta T = T - T_{\mathrm{c}}$$

$h = 0:$

$T > T_{\mathrm{c}}$:
$$M = \left(\frac{-b\Delta T + \sqrt{(b\Delta T)^2 + 4\epsilon ac}}{2c}\right)^2$$

Response to perturbation

$$-\epsilon a(M+h)^{1/2}+b\Delta T(M+h)+c(M+h)^{3/2}-h=0$$

$$\Delta T = T - T_{\mathrm{c}}$$

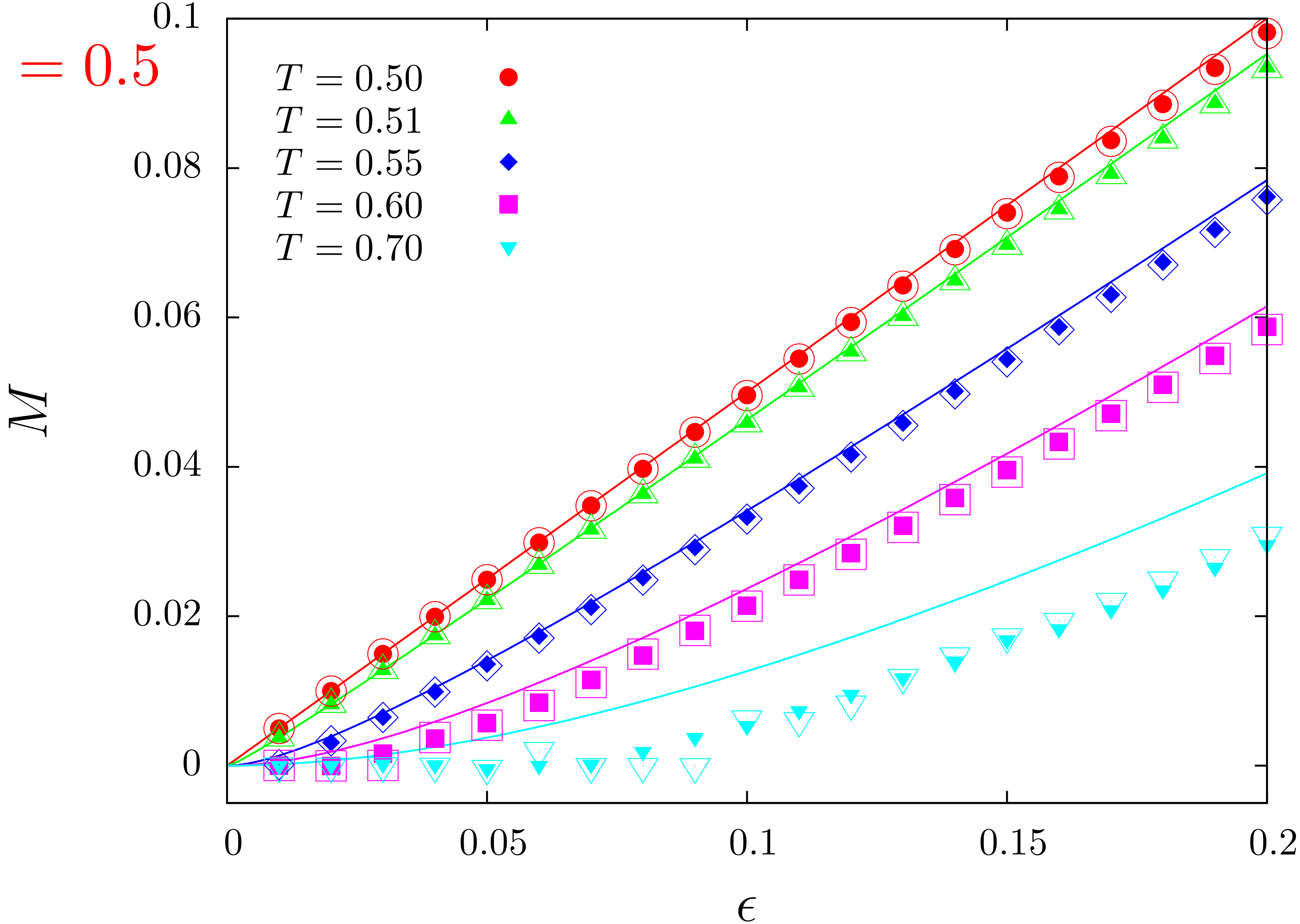
$h = 0:$

$T > T_{\mathrm{c}}$: $$M = \left(\frac{-b\Delta T + \sqrt{(b\Delta T)^2 + 4\epsilon a c}}{2c}\right)^2$$

$T = T_{\mathrm{c}}$: $$M = \frac{a}{c}\epsilon$$

Response to perturbation (numerical test)

$T_c = 0.5$



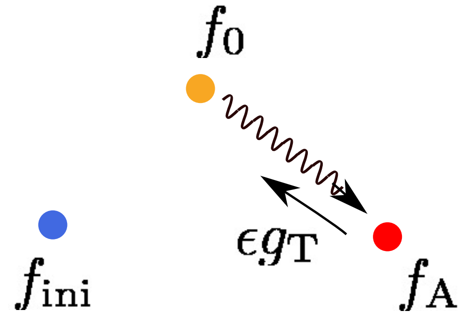
Ogawa-YYY, PRE **89**, 052114 (2014)

Discrepancy ?

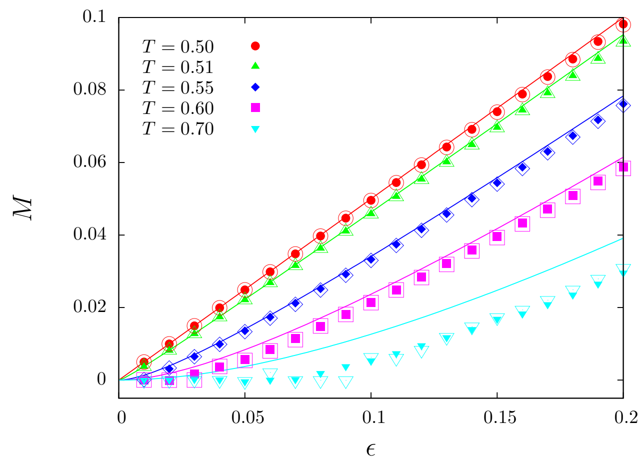
- We omitted $O(\epsilon^2)$ term.

$$\Downarrow$$

The transient part $\epsilon g_{\rm T}$ can be omitted.



- Omitting transient part $\epsilon g_{\rm T}$ implies omitting the Landau damping.

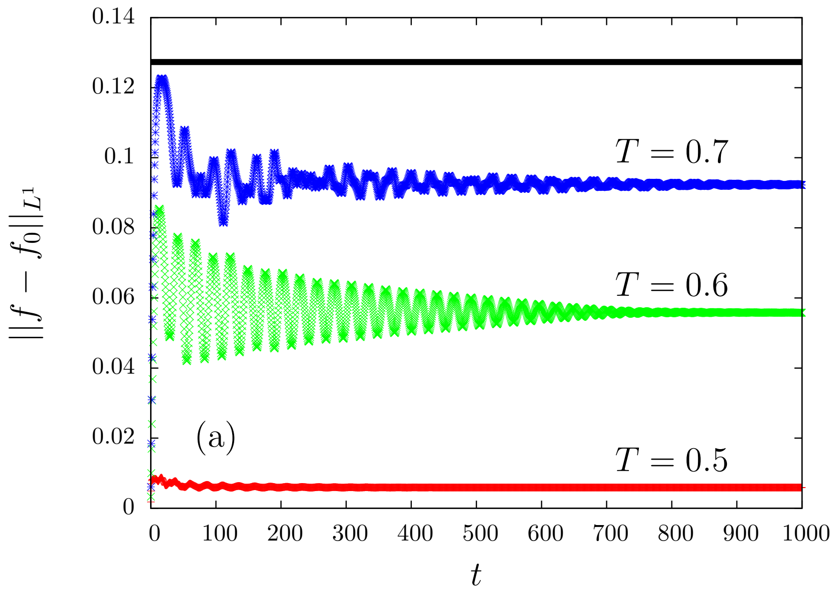


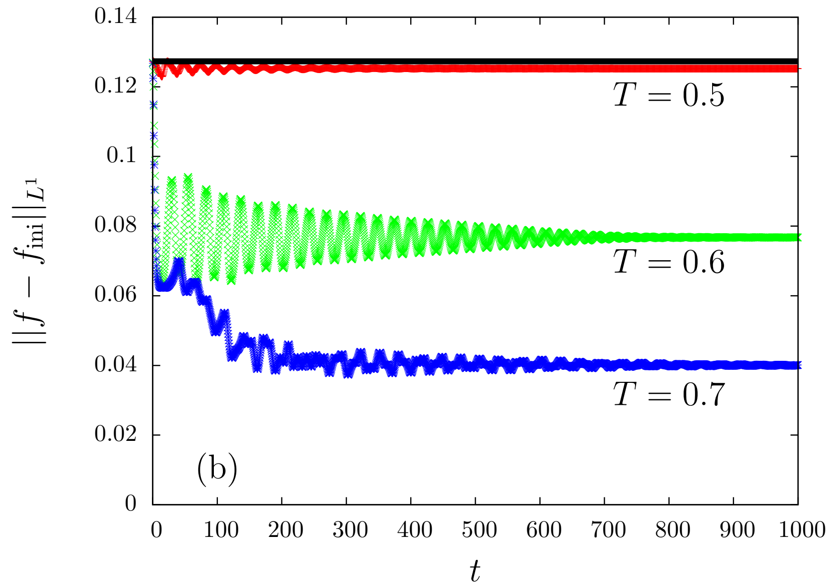
- $T = T_{\rm c}$: Damping rate is zero, and the theory works well.

  $T \nearrow$: Damping rate grows, and the theory gets worse.

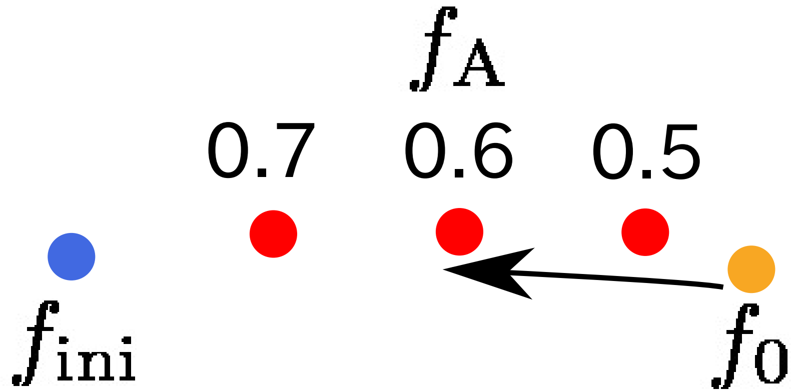
Numerical evidences

$$\|f - f_0\|_{L^1} \qquad\qquad \|f - f_{\mathrm{ini}}\|_{L^1}$$





Ogawa-YYY, PRE **89**, 052114 (2014)



Summary [Ogawa-YYY, PRE 89, 052114 (2014)]

- We proposed a nonlinear response theory for long-range Hamiltonian systems.
- It works not only for thermal eq. but also for QSSs.

Response to external field: $\gamma_- = \beta(\delta - 1)$

	Landau theory	Response theory
$M \propto (T_c - T)^\beta$	$\beta = 1/2$	$\beta = 1/2$
$dM/dh \propto (T - T_c)^{-\gamma_+}$	$\gamma_+ = 1$	$\gamma_+ = 1$
$dM/dh \propto (T_c - T)^{-\gamma_-}$	$\gamma_- = 1$	$\gamma_- = 1/4$
$M \propto h^{1/\delta}$	$\delta = 3$	$\delta = 3/2$

Response to perturbation:

The theory works well at the critical point (no damping).

Thank you for your attention.

Appendix A

T-linearization and omitting the transient part ϵg_{T}

T-linearization

Vlasov equation:

$$\frac{\partial f}{\partial t} = \{\mathcal{H}[f], f\}$$

Asymptotic-Transitent decomposition of f :

$$f(q,p) = f_{\mathrm{A}}(q,p) + \epsilon g_{\mathrm{T}}(q,p,t)$$

A-T decomposition of $\mathcal{H}[f]$:

$$\mathcal{H}[f](q,p) = \mathcal{H}[f_{\mathrm{A}}](q,p) + \epsilon \mathcal{V}[g_{\mathrm{T}}](q,p,t)$$

Substituting into the Vlasov equation:

$$\begin{aligned}\frac{\partial f}{\partial t} &= \{\mathcal{H}[f_{\mathrm{A}}], f\} + \epsilon\{\mathcal{V}[g_{\mathrm{T}}], f\} \\ &= \{\mathcal{H}[f_{\mathrm{A}}], f\} + \epsilon\{\mathcal{V}[g_{\mathrm{T}}], f_{\mathrm{A}}\} + \epsilon^2\{\mathcal{V}[g_{\mathrm{T}}], g_{\mathrm{T}}\}\end{aligned}$$

Formal solution

$$\frac{\partial f}{\partial t} = \{\mathcal{H}[f_{\mathrm{A}}], f\} + \epsilon\{\mathcal{V}[g_{\mathrm{T}}], f_{\mathrm{A}}\}$$
$$= \mathcal{L}_{\mathrm{A}} f + \epsilon\{\mathcal{V}[g_{\mathrm{T}}], f_{\mathrm{A}}\}$$

Formal solution:

$$f(t) = e^{t\mathcal{L}_{\mathrm{A}}} f_0 + \epsilon \int_0^t e^{(t-s)\mathcal{L}_{\mathrm{A}}} \left(\frac{\partial \mathcal{V}[g_{\mathrm{T}}]}{\partial q} \frac{\partial f_{\mathrm{A}}}{\partial p} \right) ds$$

Definition of asymptotic state f_{A}:

$$f_{\mathrm{A}} = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau f(t) dt = \langle f \rangle_t \quad \left(= \lim_{t \to \infty} f(t) \right)$$

Definition of asymptotic state

Formal solution:

$$f(t) = e^{t\mathcal{L}_{\mathrm{A}}} f_0 + \epsilon \int_0^t e^{(t-s)\mathcal{L}_{\mathrm{A}}} \left(\frac{\partial \mathcal{V}[g_{\mathrm{T}}]}{\partial q} \frac{\partial f_{\mathrm{A}}}{\partial p} \right) ds$$

Ergodic like formula:

$$\left\langle e^{t\mathcal{L}_{\mathrm{A}}} f_0 \right\rangle_t = \langle f_0 \rangle_J$$

Thus, we have

$$f_{\mathrm{A}} = \langle f_0 \rangle_J + \epsilon \left\langle \int_0^t e^{(t-s)\mathcal{L}_{\mathrm{A}}} \left(\frac{\partial \mathcal{V}[g_{\mathrm{T}}]}{\partial q} \frac{\partial f_{\mathrm{A}}}{\partial p} \right) ds \right\rangle_t$$

Lemma

$$\mathcal{V}\left[\left\langle\int_0^t e^{(t-s)\mathcal{L}_\mathrm{A}}\left(\frac{\partial\mathcal{V}[g_\mathrm{T}]}{\partial q}\frac{\partial f_\mathrm{A}}{\partial p}\right)ds\right\rangle_t\right]=0$$

(Proof)

$$\frac{\partial\mathcal{V}[g_\mathrm{T}]}{\partial q}=\sum_k T_k(t)e^{ikq}$$

$$\begin{aligned}
\langle\cdot\rangle_t &= \sum_k \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt\int_0^t e^{(t-s)\mathcal{L}_\mathrm{A}}\left(T_k(s)e^{ikq}\frac{\partial f_\mathrm{A}}{\partial p}\right)ds \\
&= \sum_k\int_0^\infty dsT_k(s)\lim_{\tau\to\infty}\frac{1}{\tau}\int_{-s}^{\tau-s}e^{u\mathcal{L}_\mathrm{A}}\left(e^{ikq}\frac{\partial f_\mathrm{A}}{\partial p}\right)du \\
&= \sum_k\left\langle e^{ikq}\frac{\partial f_\mathrm{A}}{\partial p}\right\rangle_J\int_0^\infty T_k(s)ds
\end{aligned}$$

The p-odd function $\left\langle e^{ikq}\partial_p f_\mathrm{A}\right\rangle_J$ does not contribute to potential $\mathcal{V}$. ■

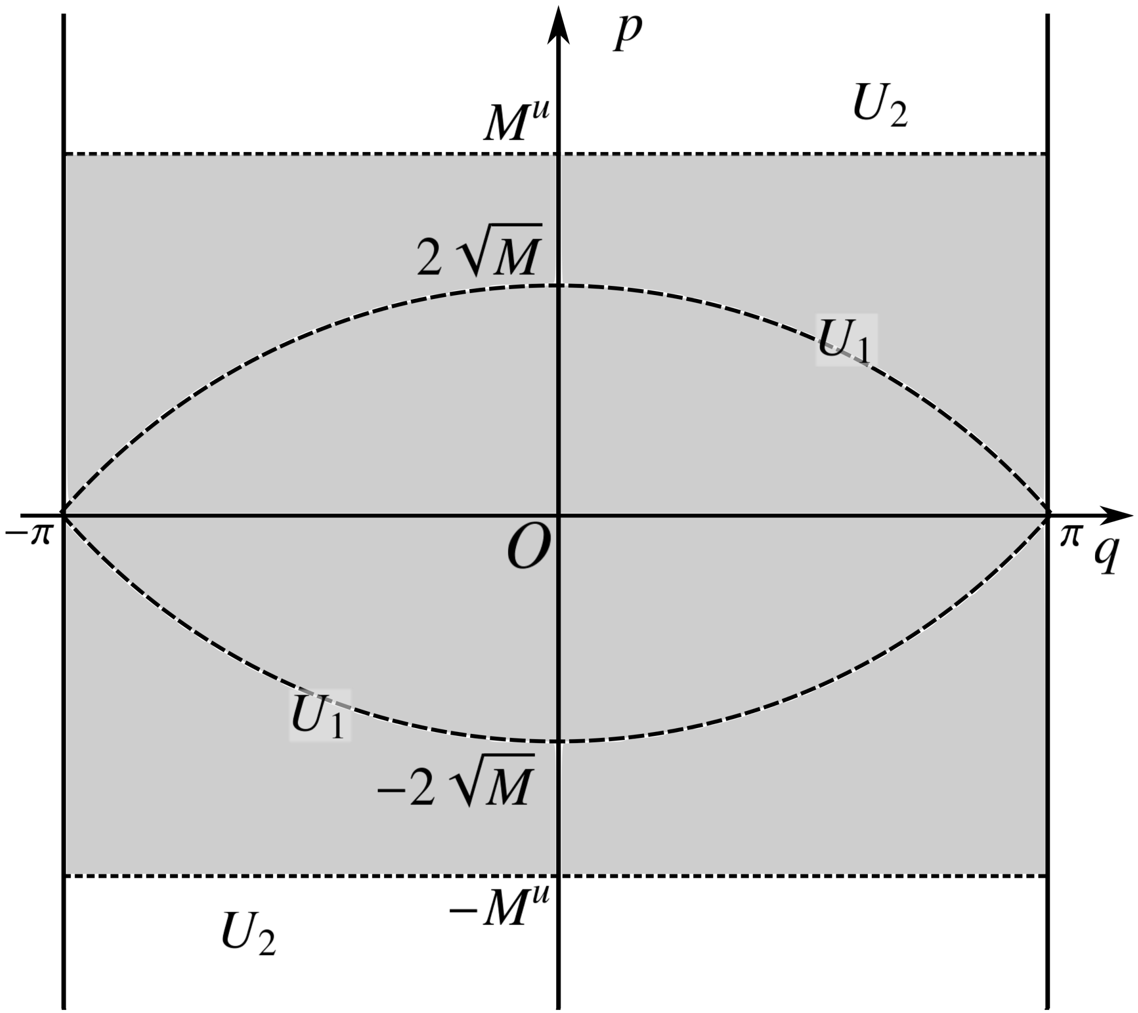
Appendix B

Expansion of the sel-consistent equation

$$M = \int \cos q \left\langle f_0 \right\rangle_J dqdp = \int \left\langle \cos q \right\rangle_J f_0 dqdp$$

Division of phase space

We divide the phase space into U_1 and U_2:



$$U_2 = \{|p| > M^u\}$$

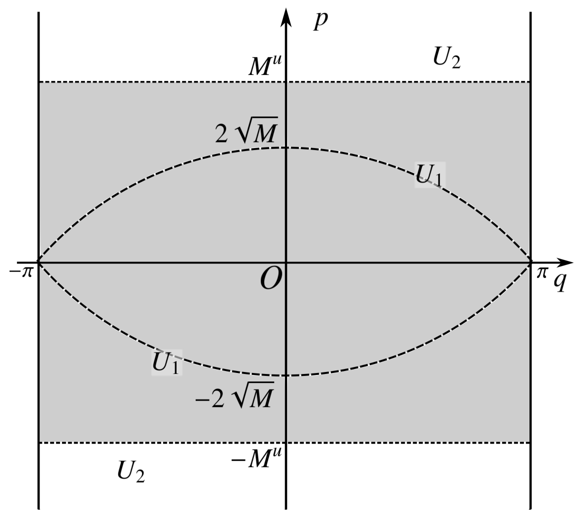
$$U_1 = \{|p| < M^u\}$$

$$0 < u < 1/2$$

Ogawa-YYY, PRE **89**, 052114 (2014)

Expansin in each region

- In U_1:
  $|p|$ is small, and we expand
  $f_0(q,p)$ into the Taylor series wrt p



- In U_2:
  Action J is written by $k = \sqrt{(\mathcal{H} + M + h)/2(M+h)}$.
  k is large, and we expand
  $\langle \cos q \rangle_J$ into the power series of $1/k$

- The scaling $p \simeq k\sqrt{M+h}$ provides
  the power series of $\sqrt{M+h}$