Adaptive networks with preferred degree from the mundane to the astonishing

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Thanks to
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Materials Theory

Outline

- Motivation
- Model Specifications
- Simulation & Analytic Results
- Summary and Outlook

Motivation

- Statistical physics: Many *interacting* d.o.f.
- Network of nodes, *linked* together
- Active nodes, static links
  - Ising, Potts, … spin glass, … real spins/glass
  - MD (particle ⇒ node, interaction ⇒ link, in a sense)
  - Models of forest fires, epidemics, opinions…
- <u>Static nodes, active links</u> *(a baseline study)*
- Active nodes, active links
  - annealed random bonds, … real gases/liquids (in a sense)
  - networks in real life: biological, social, infrastructure, …

Motivation

- Static/active nodes, active links

… especially in the setting of…

Social Networks.

- Make new friends, break old ties
- Establish/cut contacts (just joined LinkedIn)
- …according to some ***preference***

(link activity ≠ in growing networks)

- Preferences can be dynamic!

Motivation

- For simplicity, think about epidemics:
  - SIS or SIRS (susceptible, infected, recovered)
  - Many studies of phase transitions
  - but the majority are on ***static*** networks (e.g., square lattice)
- Yet, if you hear an epidemic is raging, you are likely to ***do something***! (as opposed to a tree, in a forest fire)
- Most models "rewire" connections, but…
- …I am more likely to just ***cut ties***!!!

…won't you?!?

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Physics
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Model Specs

- ***N*** nodes have ***preferred*** degree(s): κ
- Links are dynamic, controlled by κ
- *Single* homogeneous (one κ) community
- Dynamics of *two* communities (e.g., two κ's)
- Overlay node variables (health, wealth, opinion, …)
- Feedback & coupling of nodes + links

static nodes
active links

active nodes
active links

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Model Specs

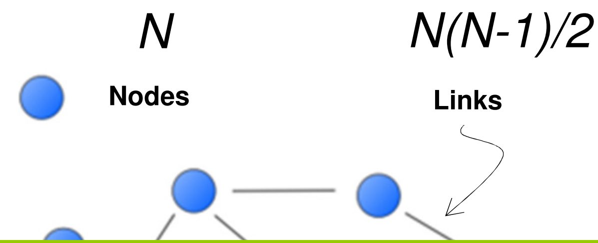
- *N* nodes have ***pre***
- Links are dynamic,
- *Single* homogeneous (one κ) community
- Dynamics of *two* communities (e.g., two κ's)
- Overla)
- Feedba

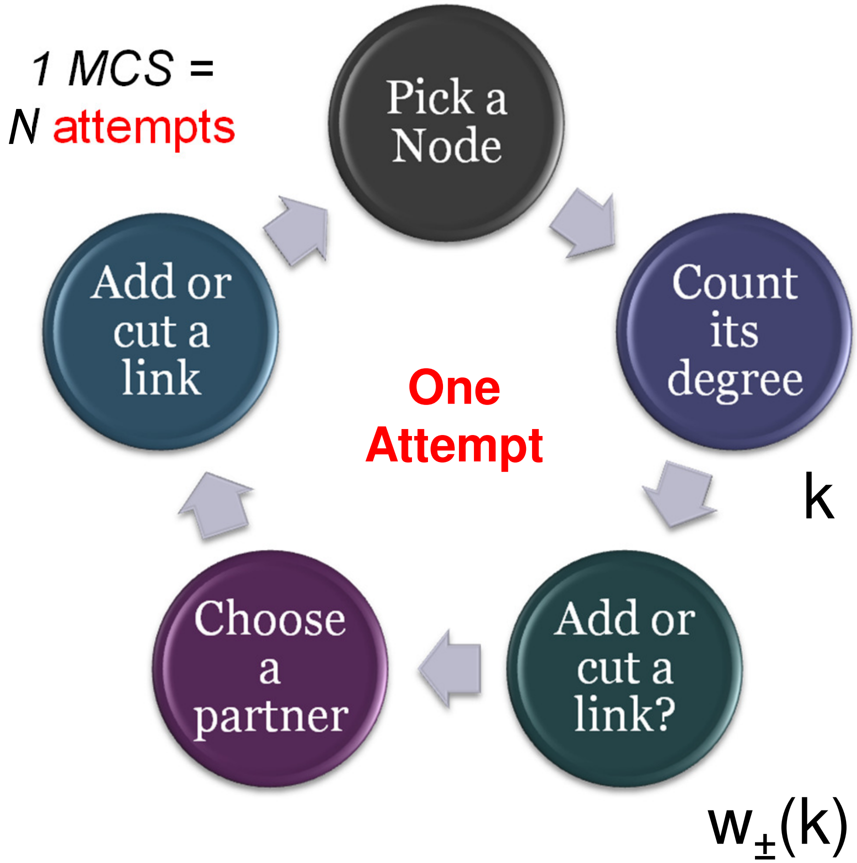
Main focus of *this talk*!

Two communities of
extreme introverts and extroverts

the "*XIE*" model

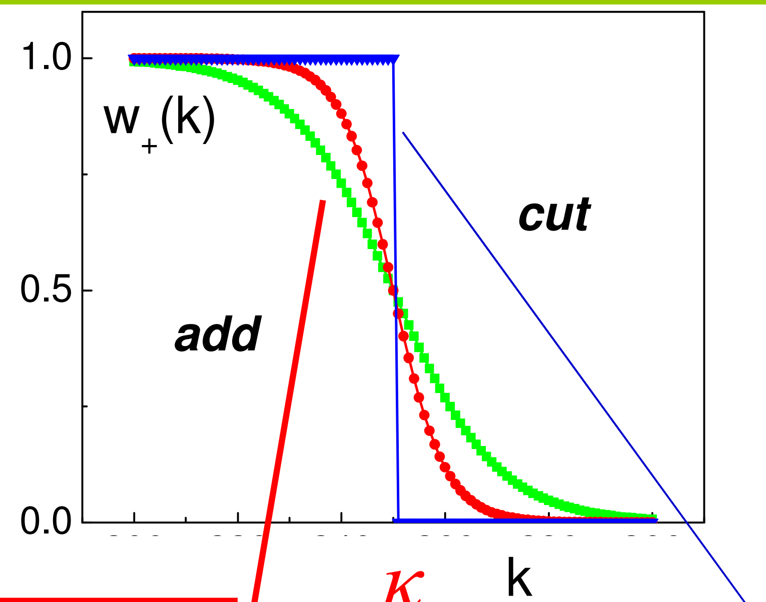
Single homogeneous community





Rate function $\mathbf{w_+(k)}$

- *sets preferred degree* κ
- *interpolates between 0 and 1*



Tolerant
Easy going

Rigid
Inflexible

Choose, for simplicity, the rate for cutting a link:
$w_-(k) = 1-w_+(k)$

N *N(N-1)/2*

Nodes Links

Single homogeneous community

1 MCS =
N attemp

Add c
cut a
link

k:

What quantities are of interest?

…in the steady state…

- Degree distribution $\rho\,(k)$

  < number of nodes with k links >

  surely, around κ ; *Gaussian? or not?*

- Average diameter of network
- Clustering properties
- ⋮

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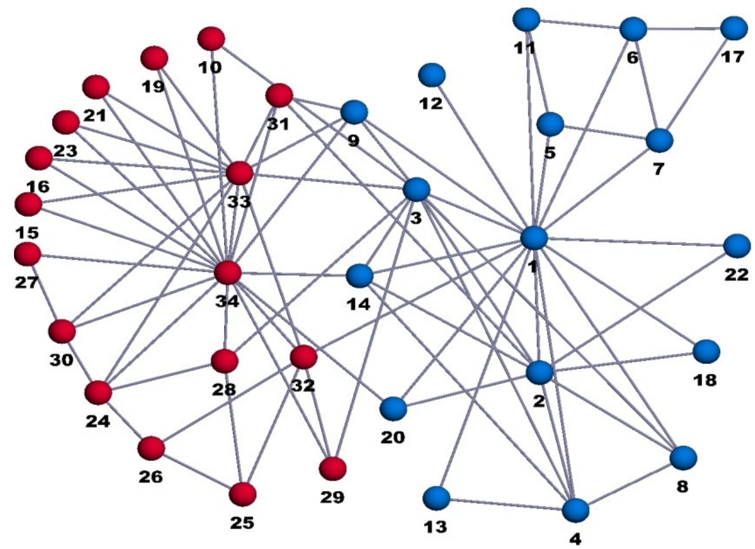
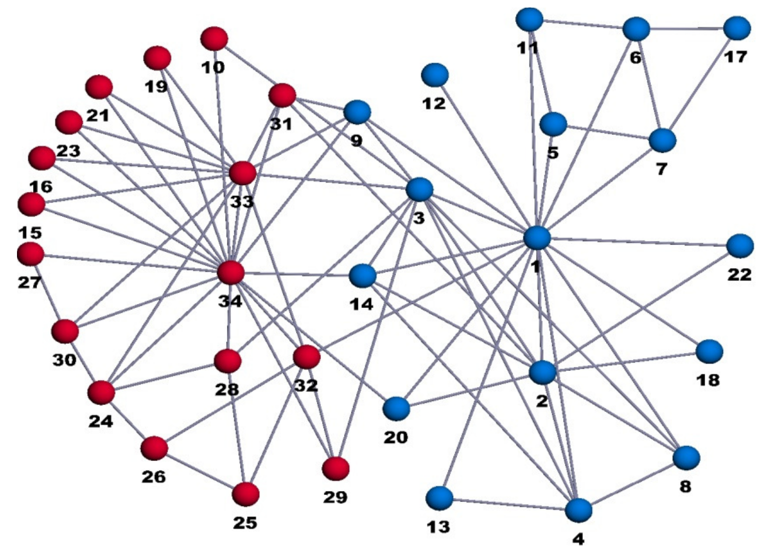
Two communities



Many possible ways…
to have two different groups and
to couple them together !!

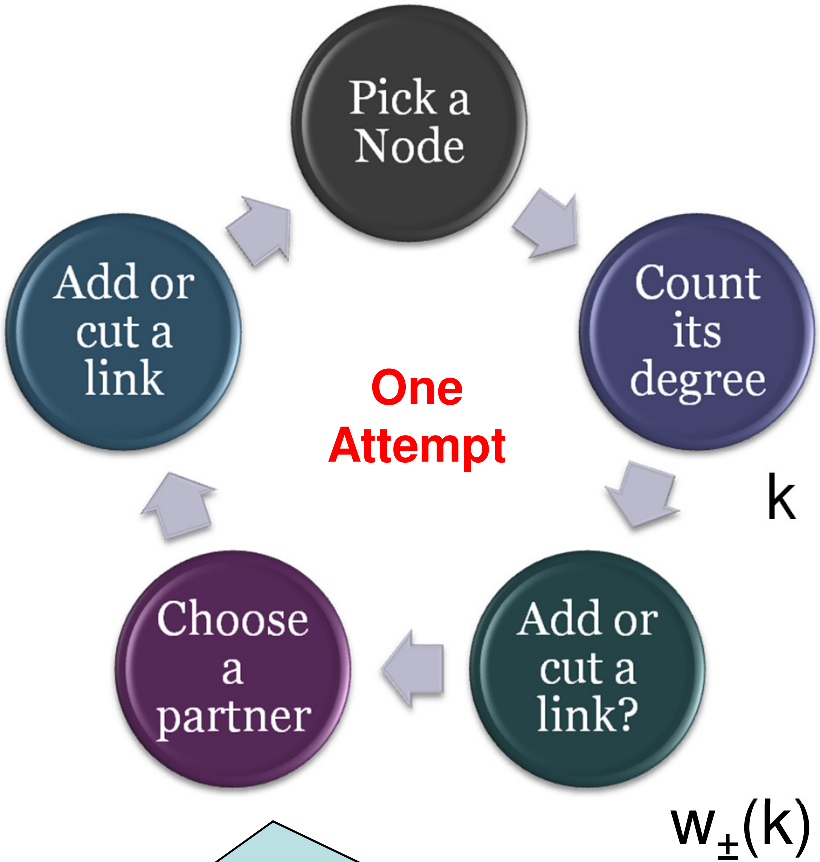
- *different sizes:* $\boldsymbol{N_1 \neq N_2}$
- *different* $\mathbf{W_+}$*'s, e.g., same form, with* $\kappa_1 \neq \kappa_2$
- *various ways to introduce cross-links, e.g., …*

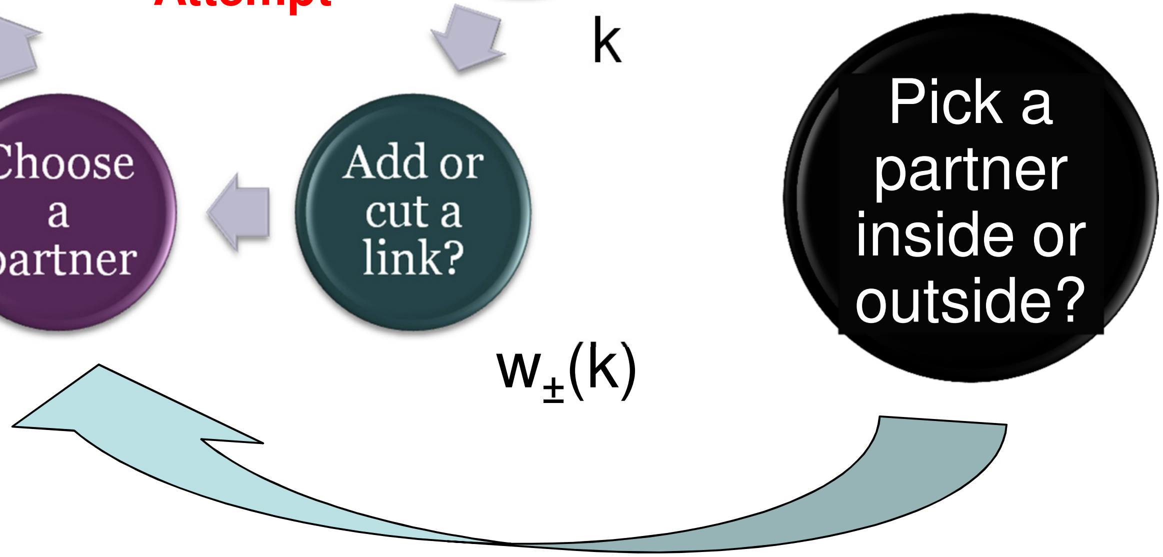
Two communities



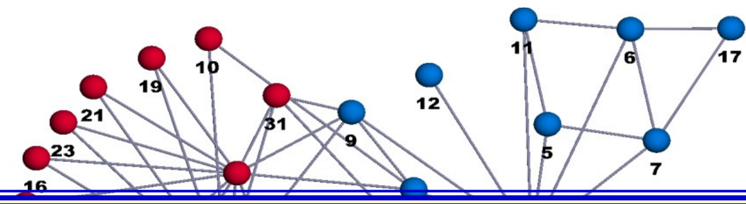
…with probability ***S*** *or* ***1-S***

obviously, can have $S_1 \neq S_2$





Two communities



Add c
cut a
link

What else is interesting?

- Degree distributions *same? or changed?*
- "Internal" *vs.* "external"
  degree distributions
- Total number of cross-links
- How to measure "frustration"?
- ⋮

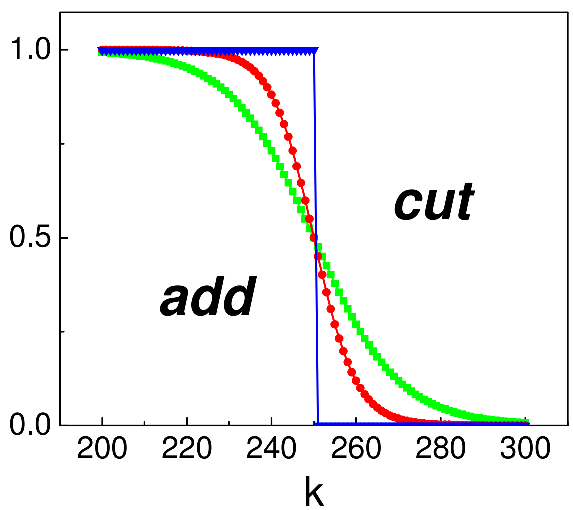
S_2

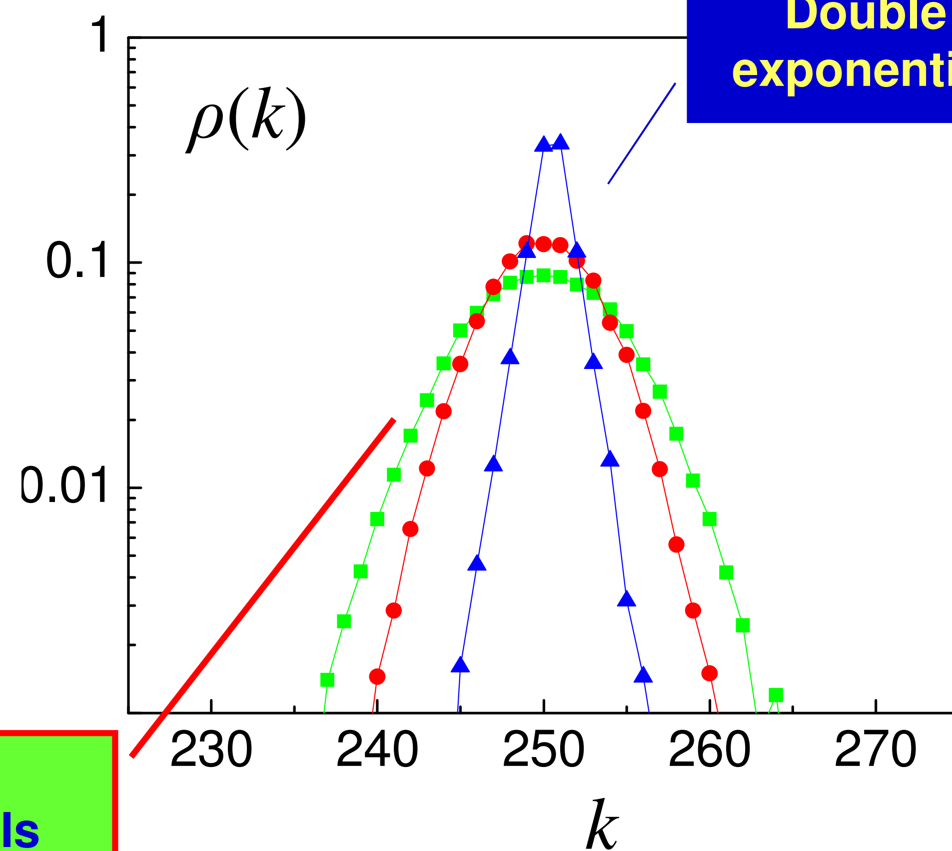
Single homogeneous community

Degree distribution $\rho(k)$ *< number of nodes with k links >*

10^4 MCS

N =1000, $\mathcal{K}$=250_+





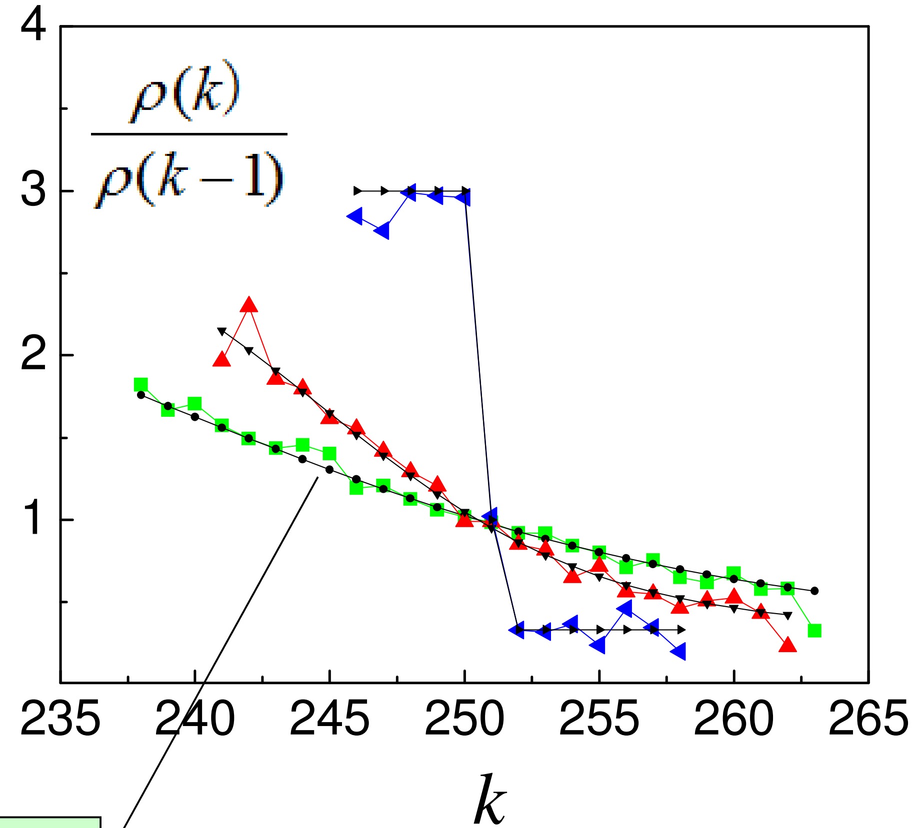
Double exponential

Gaussian → exponential tails

Single homogeneous community

An approximate argument leads to a prediction for the following *stationary* degree distribution:

$$\frac{\rho(k)}{\rho(k-1)} = \frac{1/2 + w_+(k-1)}{1/2 + w_-(k)}$$



Black lines from "theory"

N = 1000, κ = 250

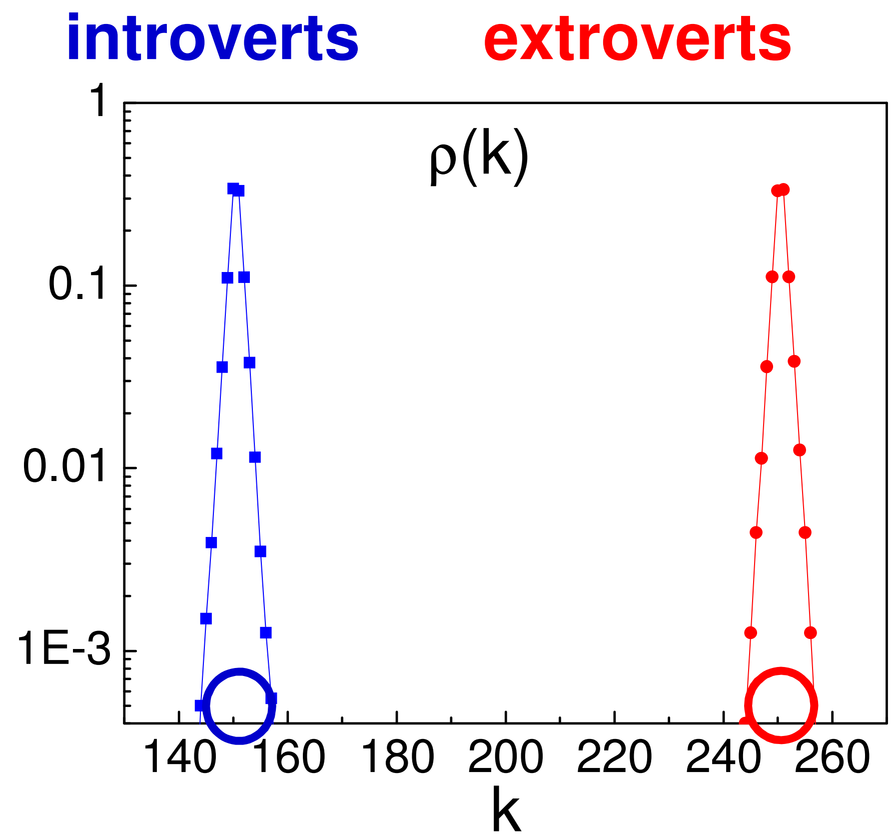
Two communities

$N_1 = N_2 = 1000$

$\kappa_1 = 150$, $\kappa_2 = 250$

Rigid w's

$S_1 = S_2 = 0.5$



Our simple argument for
….degree distributions
in a single network,
generalized to include
S_1 and S_2 :

$$\frac{\rho(k)}{\rho(k-1)} = \frac{\frac{1}{2}\left(1 - S_1 + S_2\right) + w_+(k-1)}{\frac{1}{2}\left(1 - S_1 + S_2\right) + w_-(k)}$$

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Two communities

But, there are puzzles !!

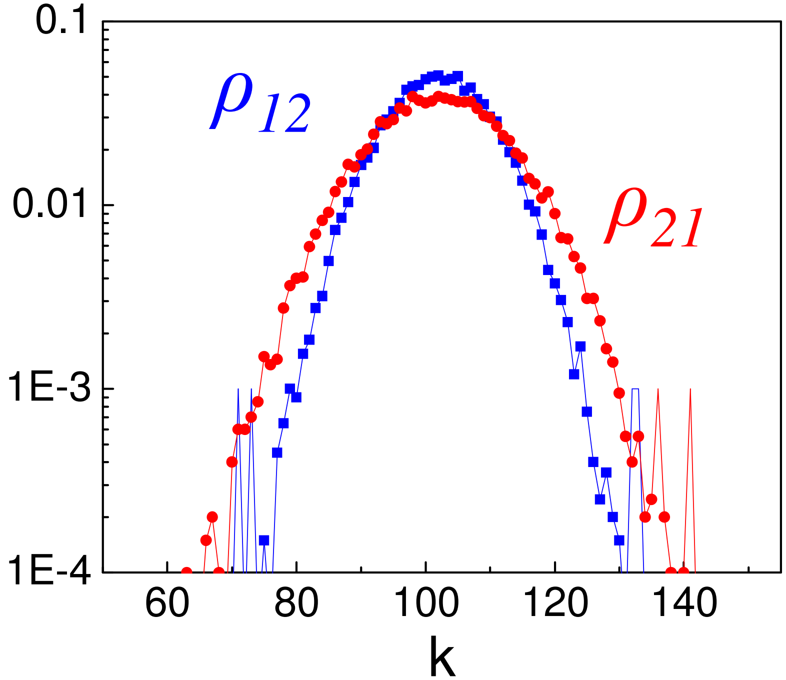
$N_1 = N_2 = 1000$

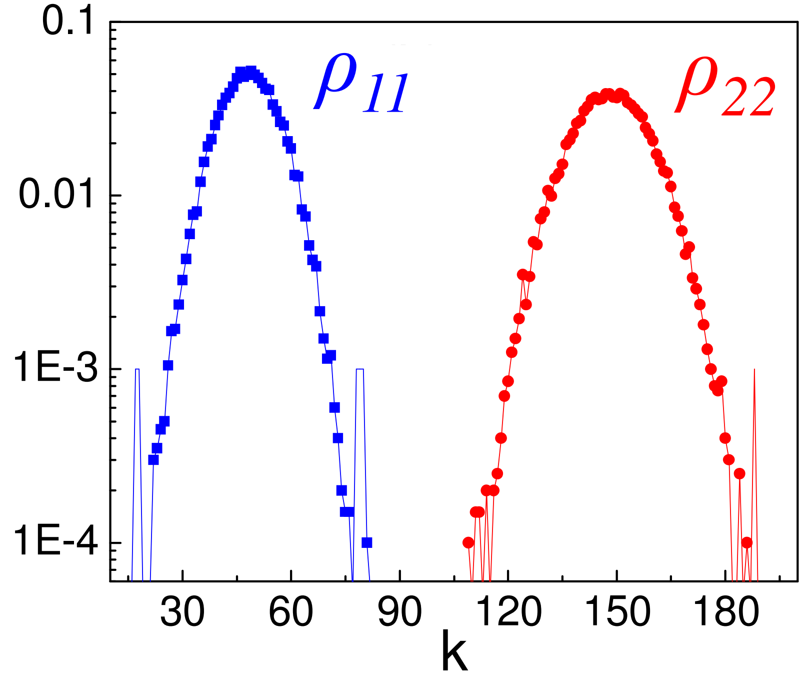
$\kappa_1 = 150$, $\kappa_2 = 250$

Rigid w's

$S_1 = S_2 = 0.5$

Degree distribution of cross links

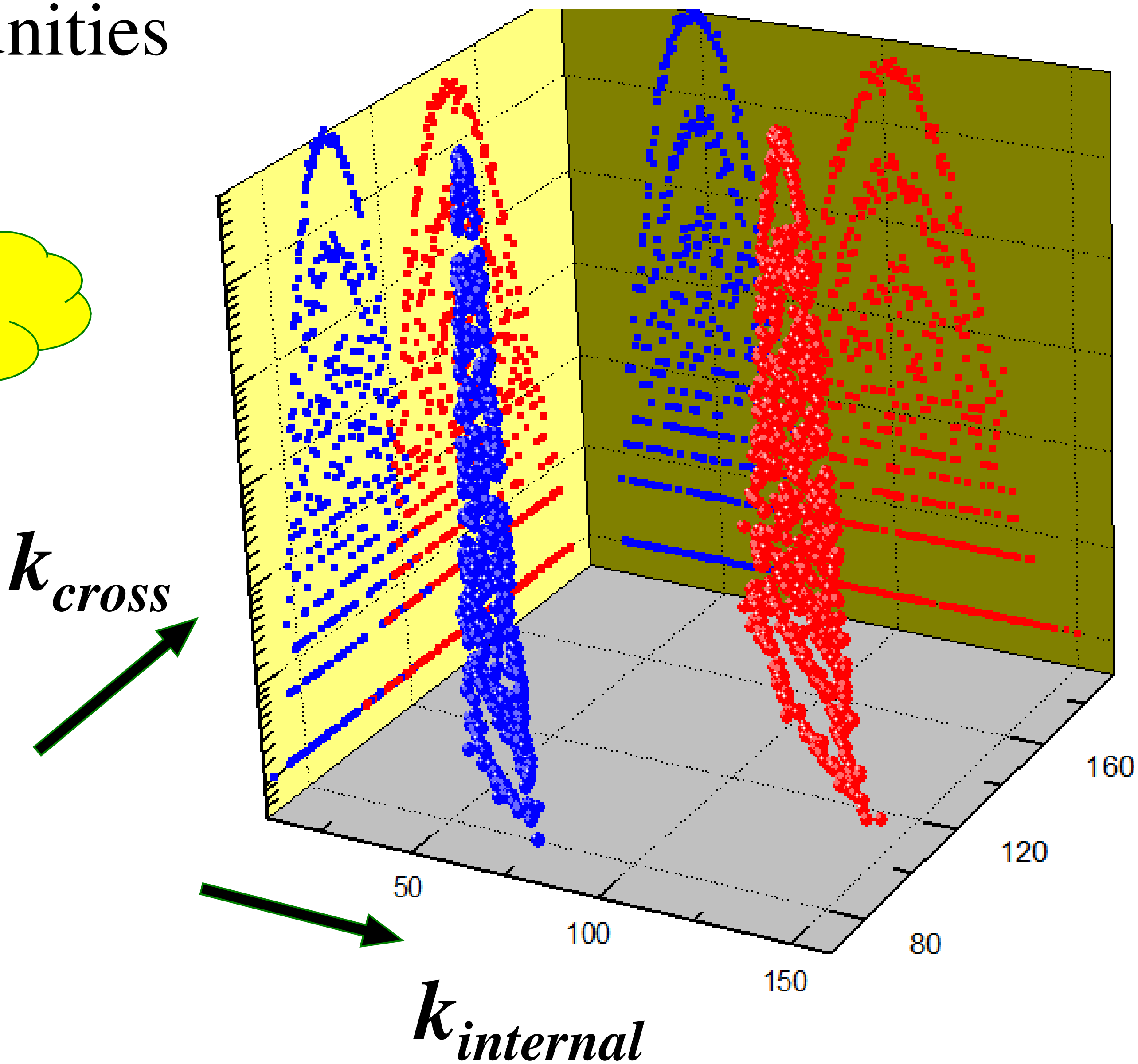




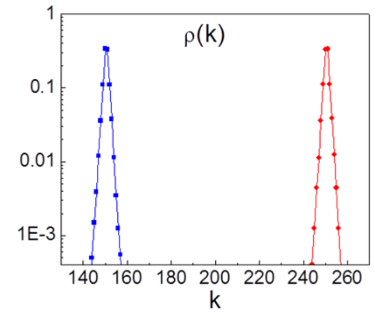
Degree distribution of "internal" links

Two communities

Schematic;
$(N_I \neq N_E)$



Two communities



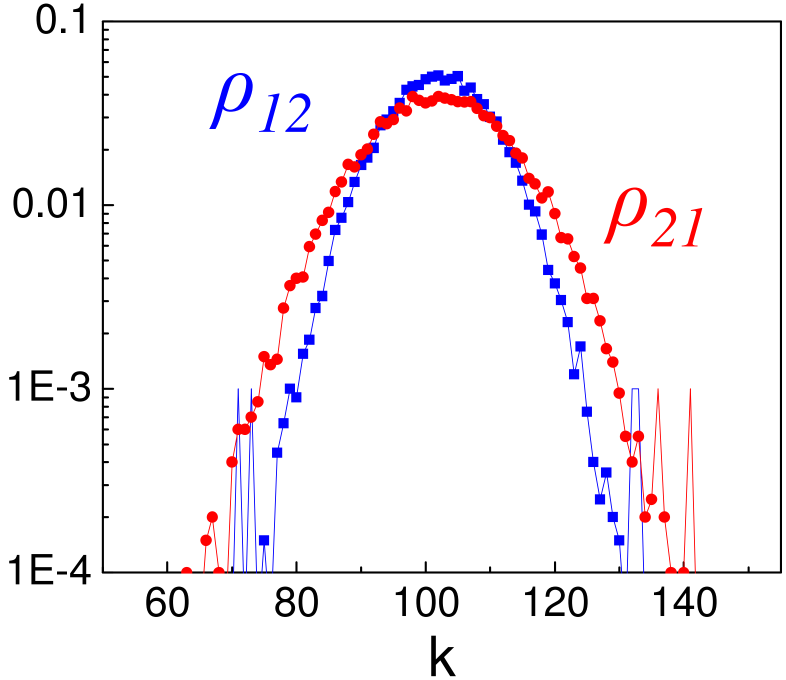
$N_1 = N_2 = 1000$

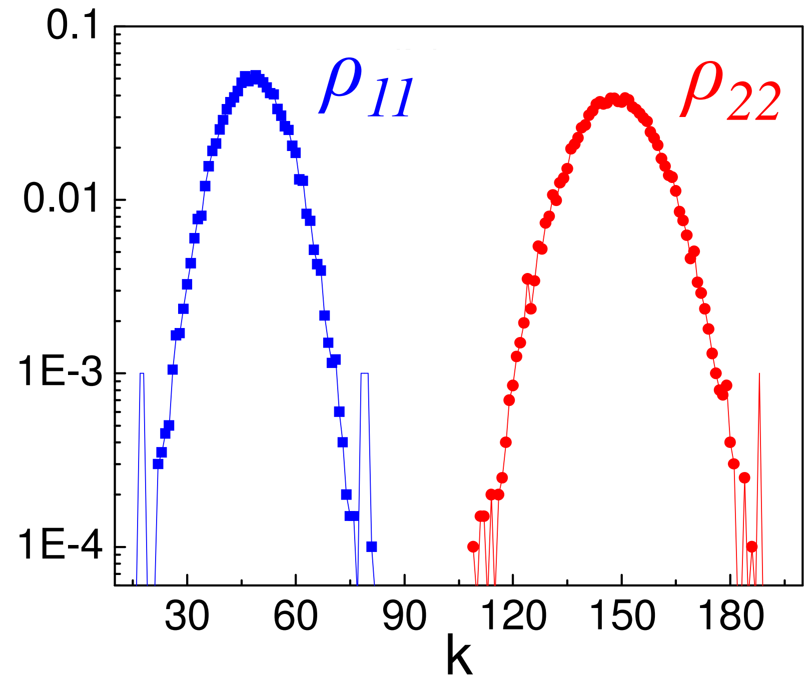
$\kappa_1 = 150$, $\kappa_2 = 250$

Rigid w's

$S_1 = S_2 = 0.5$

Degree distribution of cross links





Degree distribution of "internal" links

Two communities

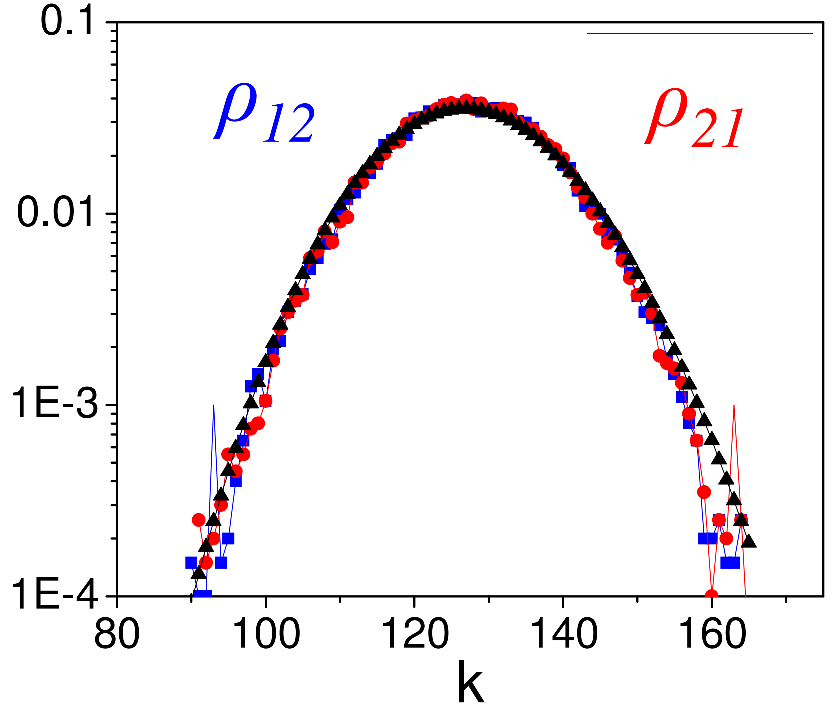
But, there are puzzles, even for the ***symmetric*** case !!

$N_1 = N_2 = 1000$

$\kappa_1 = \kappa_2 = 250$

Rigid w's

$S_1 = S_2 = 0.5$



No surprises here,

e.g., $125 = 0.5 \times 250$,

BUT …

Two communities

But, there are puzzles, even for the ***symmetric*** case !!

$N_1 = N_2 = 1000$

$\kappa_1 = \kappa_2 = 250$

Rigid w's

$S_1 = S_2 = 0.5$

The whole distribution wanders, at very long time scales!

For simplicity, study behavior of $\boldsymbol{X}$, the *total* number of cross-links.

Note: With $N_1 = N_2 = 1000$, if every node has exactly $\kappa = 250$ links, X lies in $[0, 250\mathrm{K}]$.

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Two communities

But, there are puzzles, even for the ***symmetric*** case !!

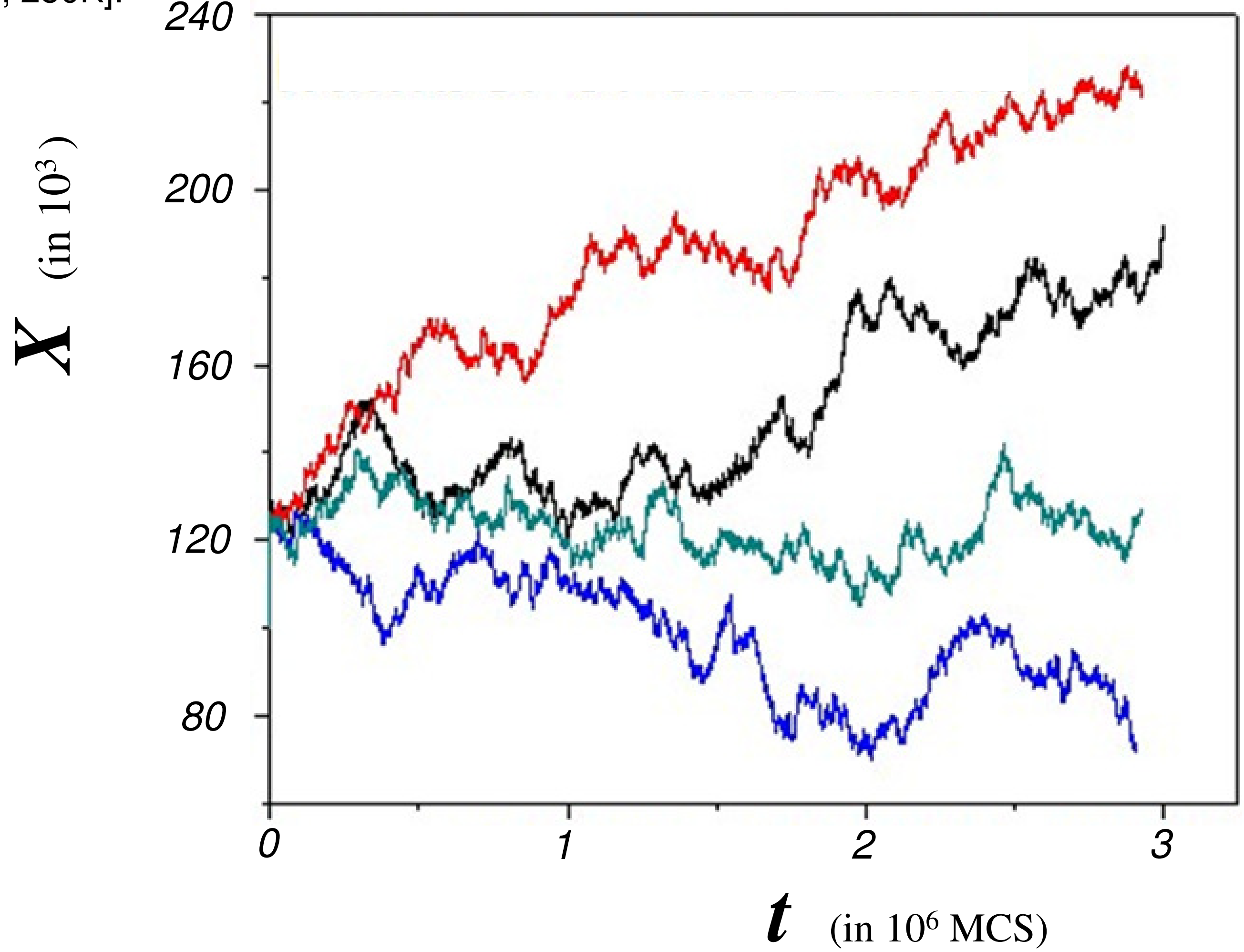
$N_1 = N_2 = 1000$

$\kappa_1 = \kappa_2 = 250$

Rigid w's

$S_1 = S_2 = 0.5$

X lies in [0, 250K].



Two communities

But, there are puzzles, even for the ***symmetric*** case !!

$N_1 = N_2 = 1000$

$\kappa_1 = \kappa_2 = 250$

***Rigid** w's*

$S_1 = S_2 = 0.5$

X lies in [0, 250K].

Many issues poorly understood …

- Dynamics ***violates*** detailed balance
- Stationary distribution is ***not known***
- If *X* does pure RW, $t \sim |X|^2 \sim 10^7$ *MCS*

Hoping to gain some insight, we consider the simplest possible case:

the "*XIE*" model

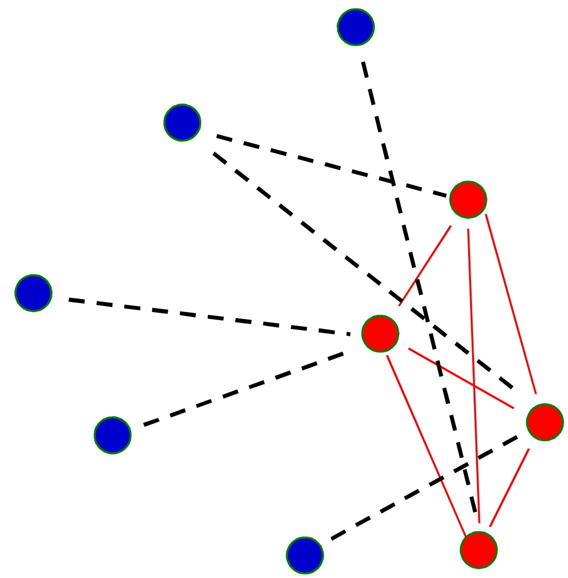
0 1 2 3

t (in 10^6 MCS)

Two communities of *eXtreme* *I*'s & *E*'s

- *I*'s always cut: $\kappa = 0$
- *E*'s always add: $\kappa = \infty$

Only cross links: are active!



An extraordinary transition

in a ***minimal*** adaptive network of introverts and extroverts

Two communities of *eXtreme* *I*'s & *E*'s

- *I*'s always cut: $\kappa = 0$
- *E*'s always add: $\kappa = \infty$
- Adjacency matrix reduces to Incidence: $N_I \times N_E$
- just ***Ising model*** with spin-flip dynamics!
- …with only two control parameters: N_I, N_E
- Unexpected bonuses:
  - Detailed balance restored!!
  - Exact $P^*(\{a_{ij}\})$ obtained analytically.
  - Problem is "***equilibrium***" like…
  - "***Hamiltonian***" is just $-\ln P^*$
- … but so far, nothing can be computed <u>*exactly*</u>.

Two communities of *eXtreme* *I*'s & *E*'s

Extraordinary
phase transition!!

...lence: $N_I \times N_E$

...n-flip dynamics!

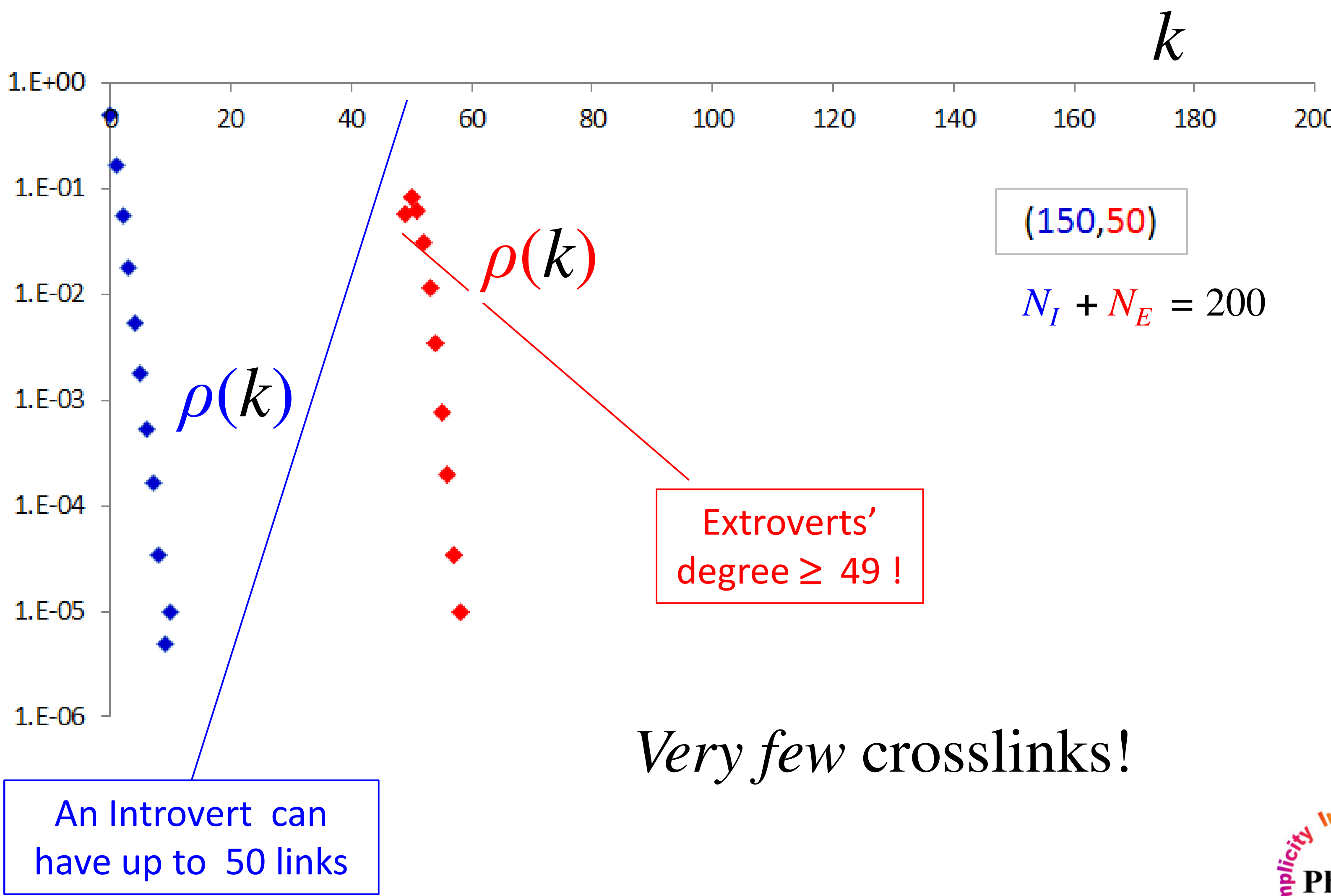
- ...with only **two** cont...
- Unexpected bonu...
  - Detailed balance res...
  - Exact $P^*(\{a_{ij}\})$ obta...
  - Problem is "***equilib...***
  - "***Hamiltonian***" is ju...
- ... but so far, nothing ...

from MC simulations
with

$$N_I + N_E = 200$$

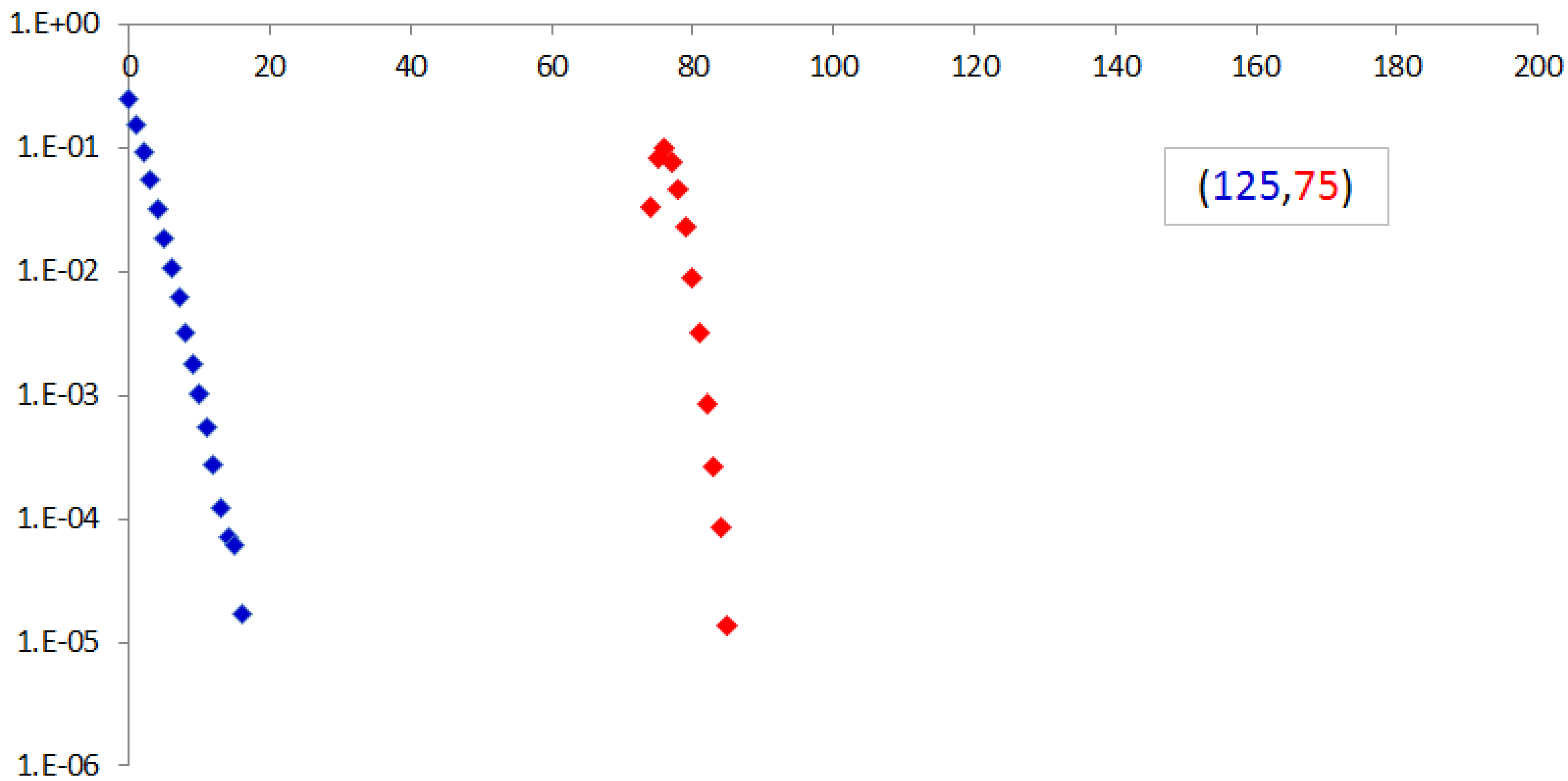
degree: $k \in [0,199]$

Two communities of *eXtreme* *I*'s & *E*'s

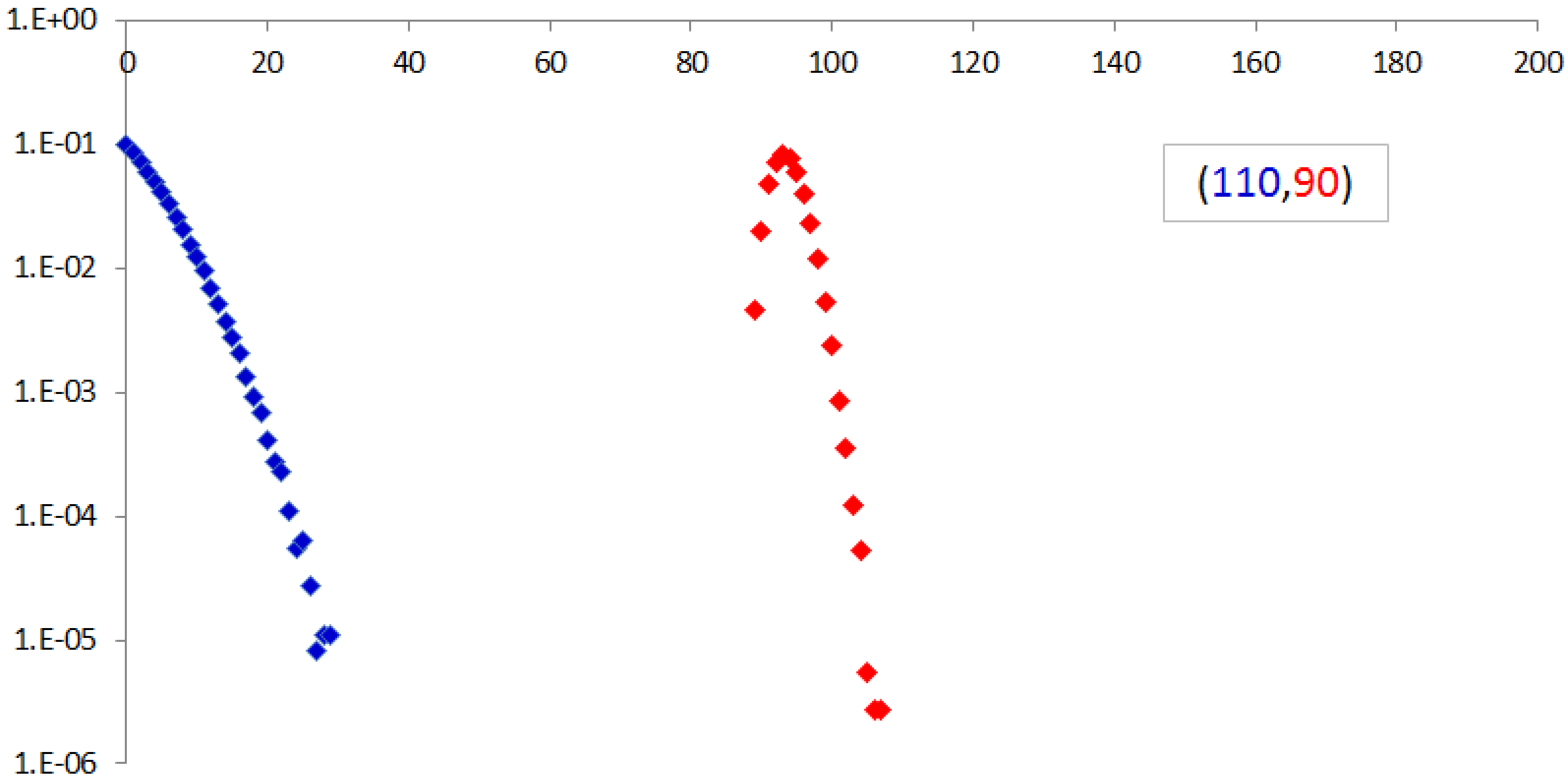


Very few crosslinks!

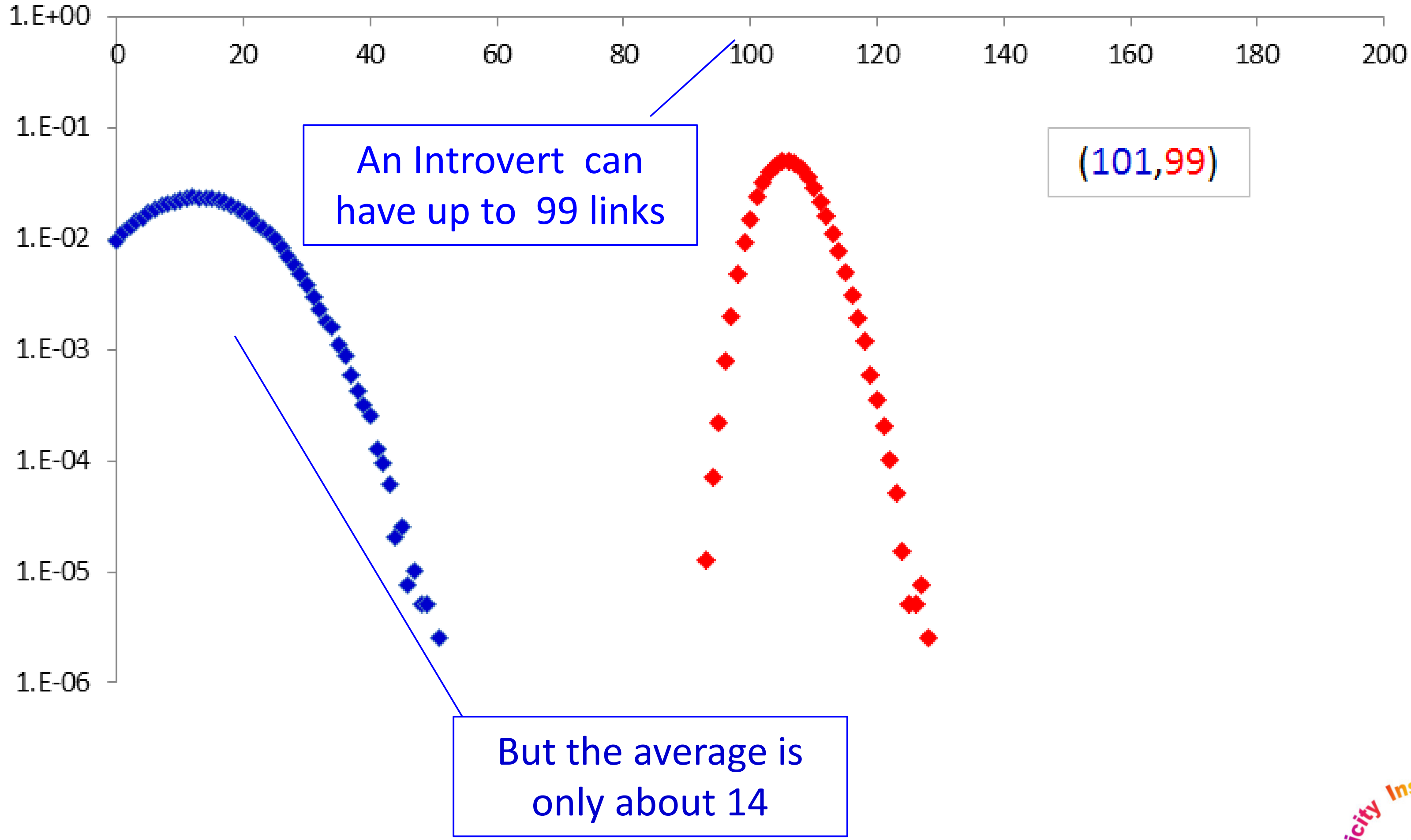
Two communities of *eXtreme I*'s & *E*'s



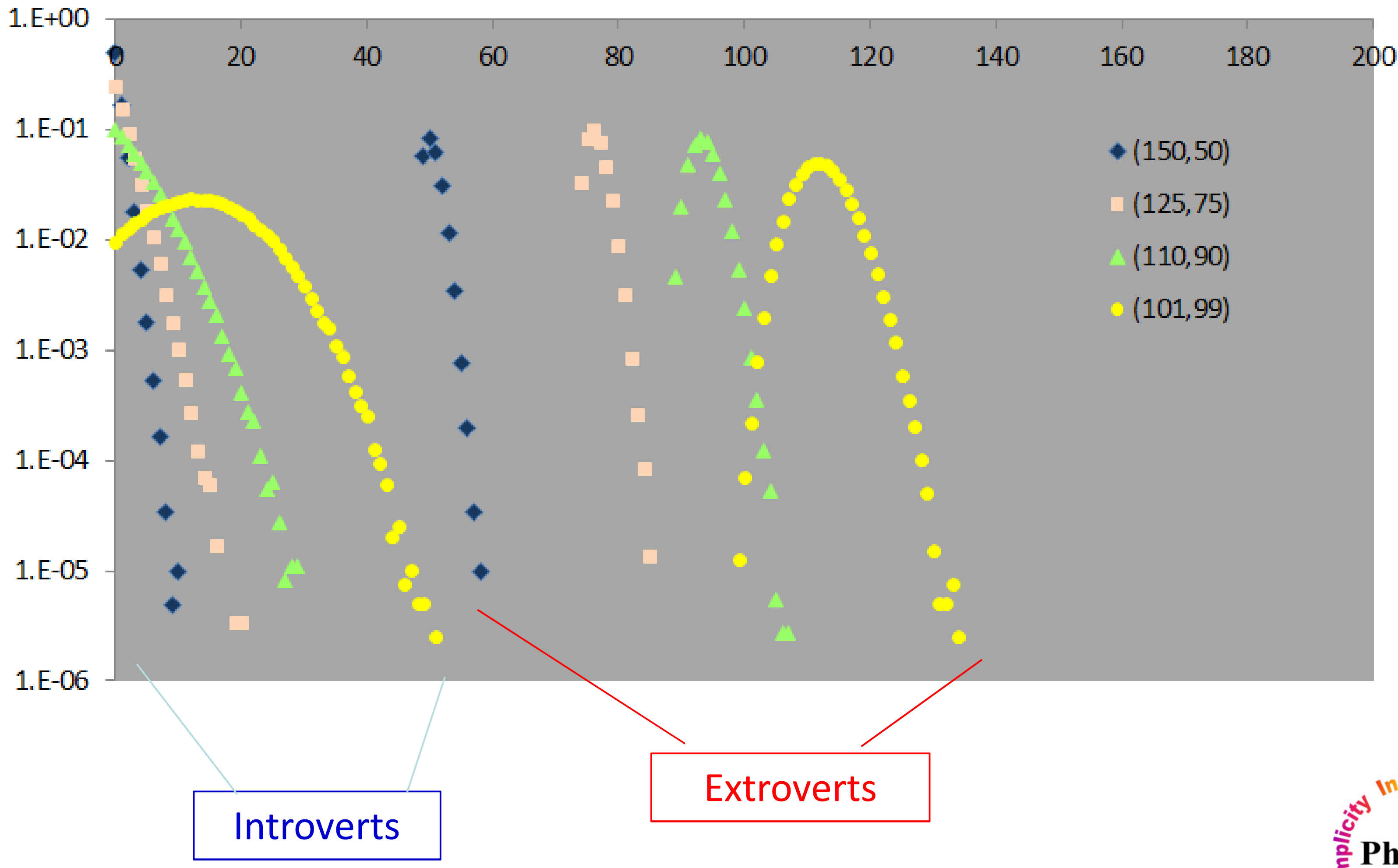
Two communities of *eXtreme* *I*'s & *E*'s



Two communities of *eXtreme* *I*'s & *E*'s

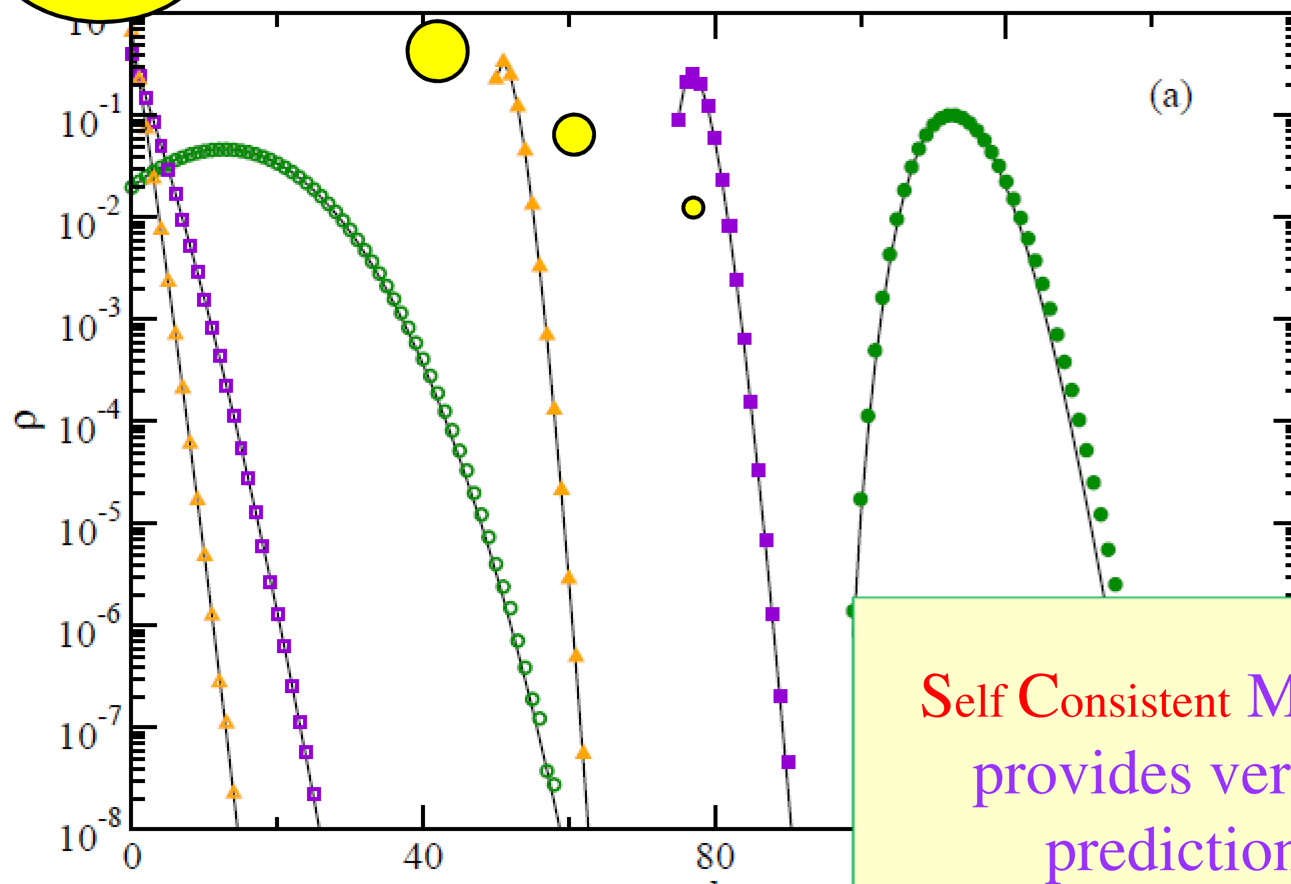


Two communities of *eXtreme I*'s & *E*'s



_NO_ fit parameters!

es of *eXtreme* *I*'s & *E*'s



Self Consistent MF theory provides very good predictions… except for (100,100)!

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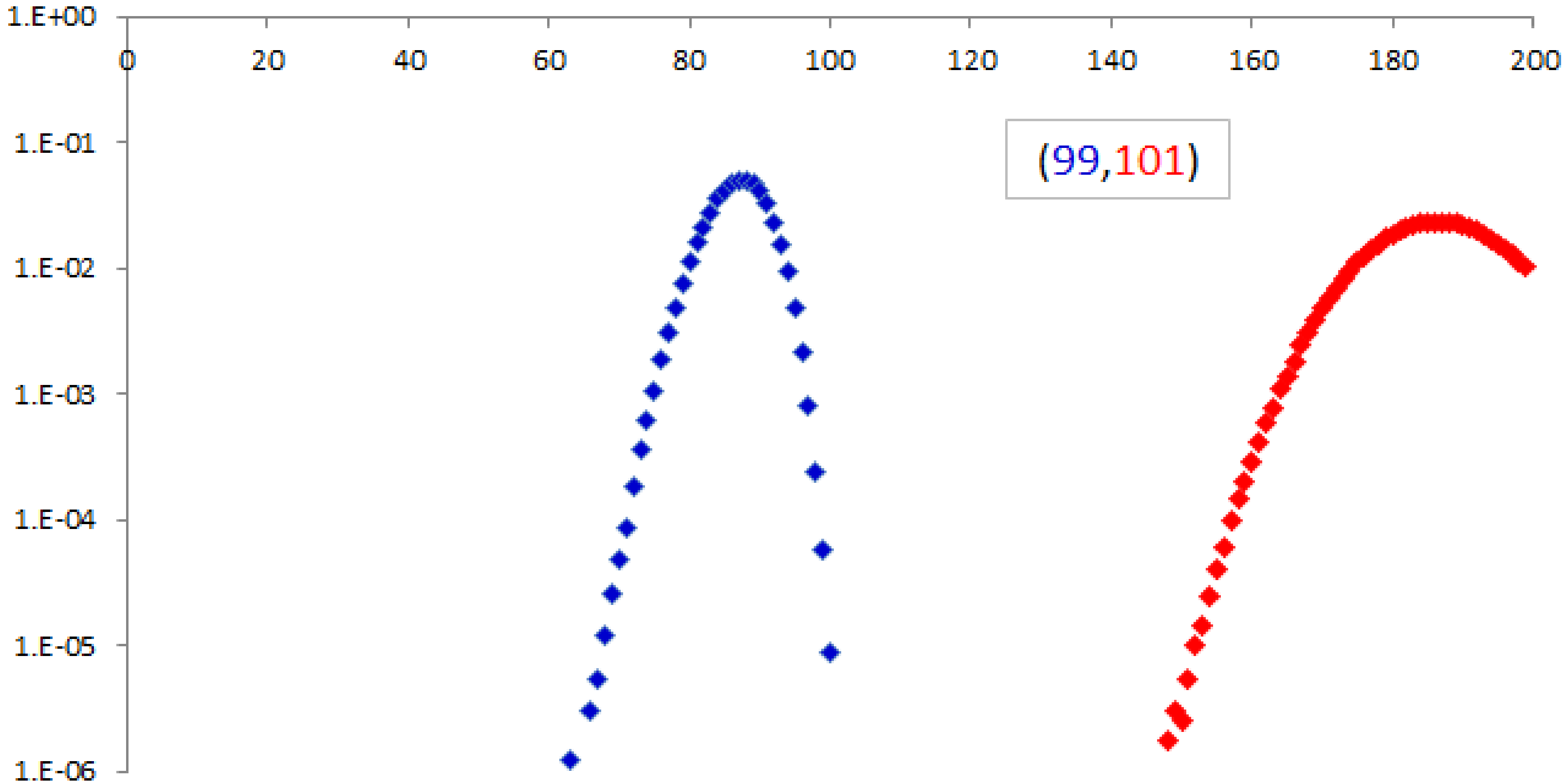
Extraordinary transition

(101,99) → (99,101)

when just 2 *I*'s "change sides"



Particle -*Hole* Symmetry

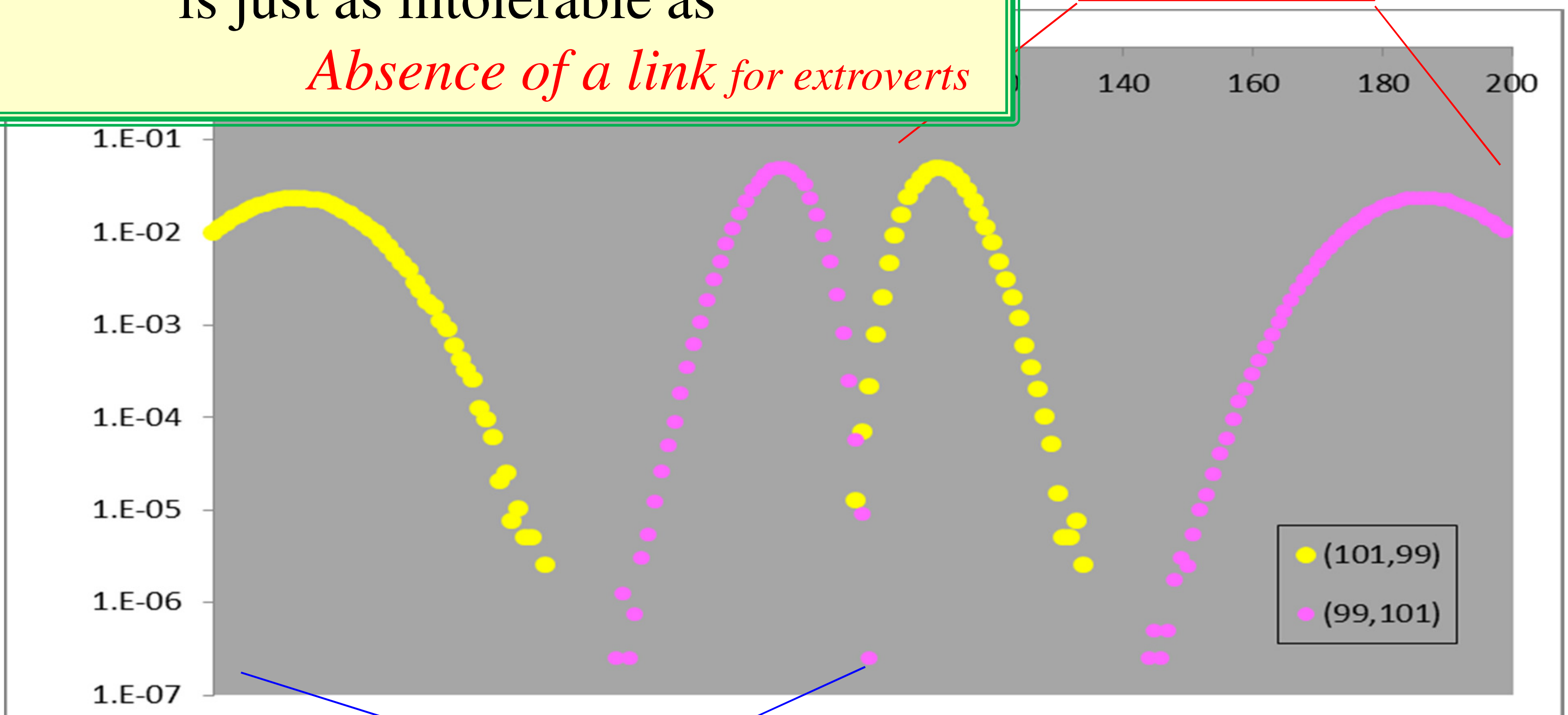
Presence of a link for introverts

is just as intolerable as

Absence of a link for extroverts

particle ⇔ *hole*
$\oplus \quad N_I \Leftrightarrow N_E$

Extroverts



Introverts

Particle -*Hole* Symmetry

Presence of a link for introverts

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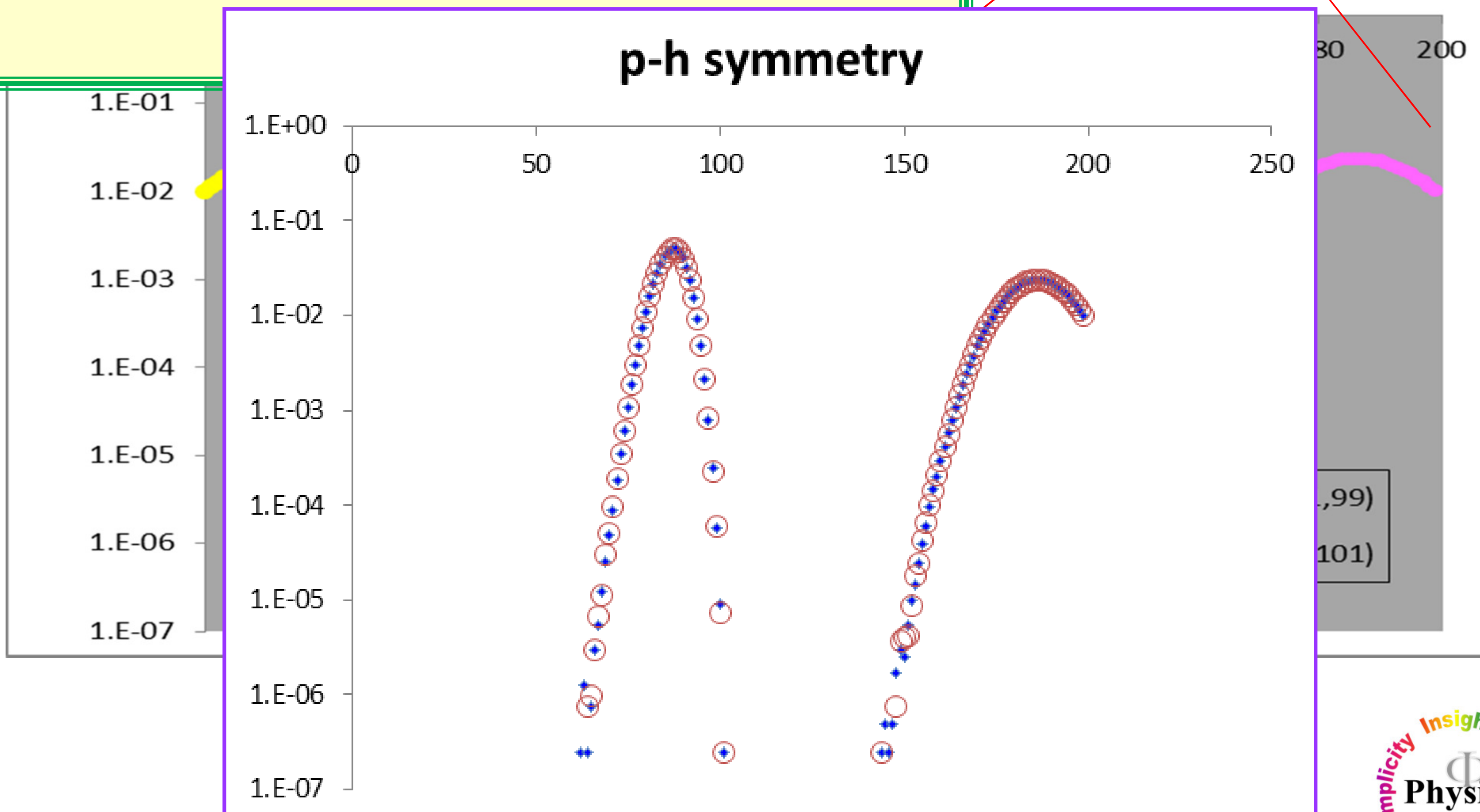
particle $\Leftrightarrow$ *hole*

$\oplus \quad N_I \Leftrightarrow N_E$



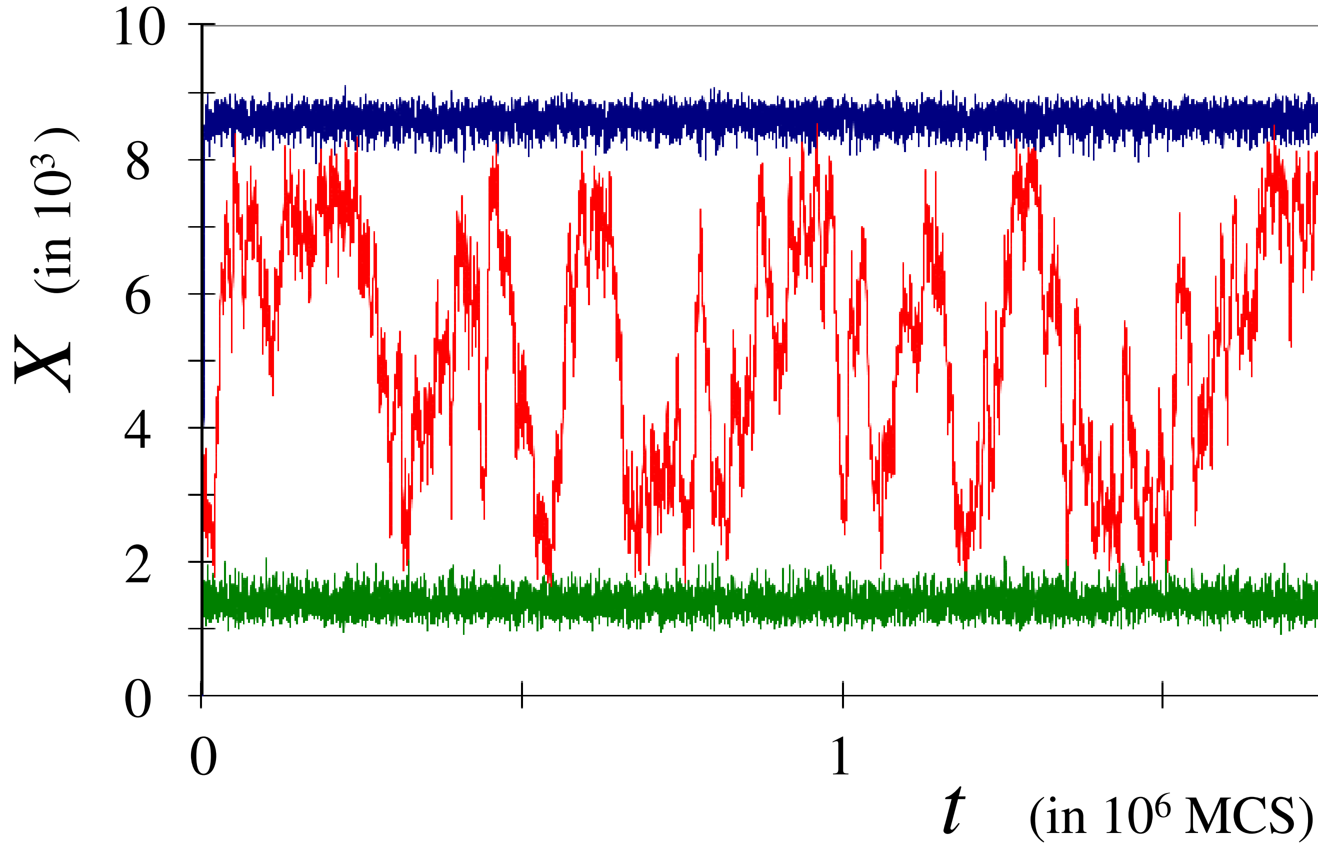
Extraordinary ***critical point:*** (100,100)

- Giant fluctuations, very slow dynamics
- $N \times N$ Ising with spin-flip dynamics and …
- a "Hamiltonian" with long range, multi-spin interactions!
- The degree distributions did not stabilize, even after 10^7 MCS!
- … critical slowing down, with unknown z
- As before, study X instead, but unlike before,
- X *does* reach the boundaries (in 10^6 MCS with N=100)

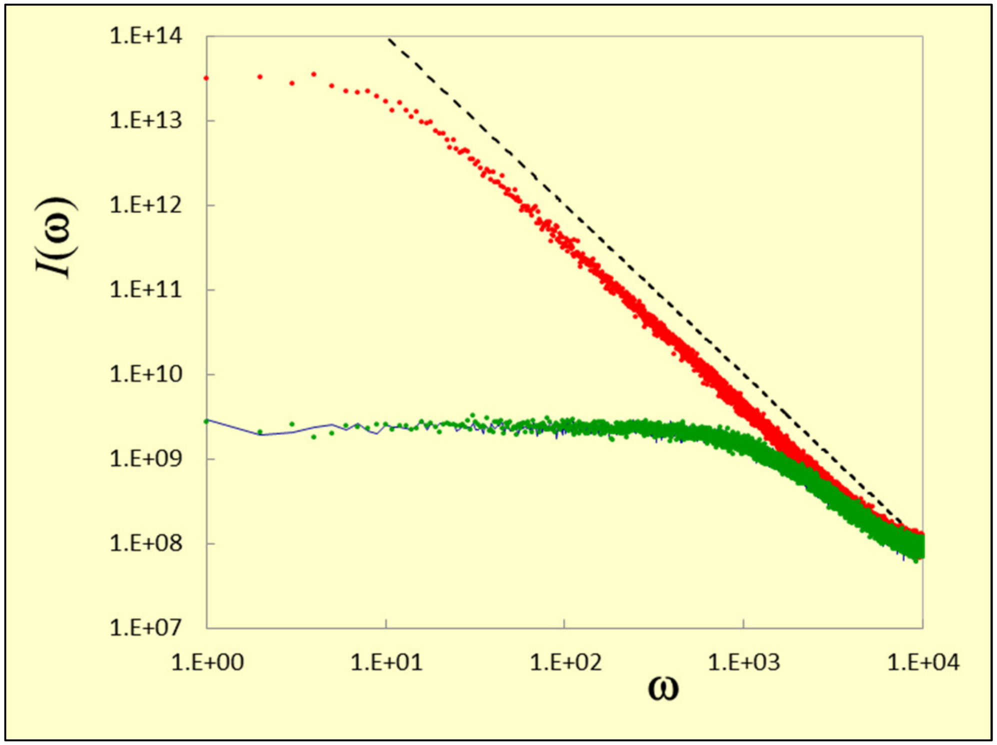
Time traces of X

for (I,E) being

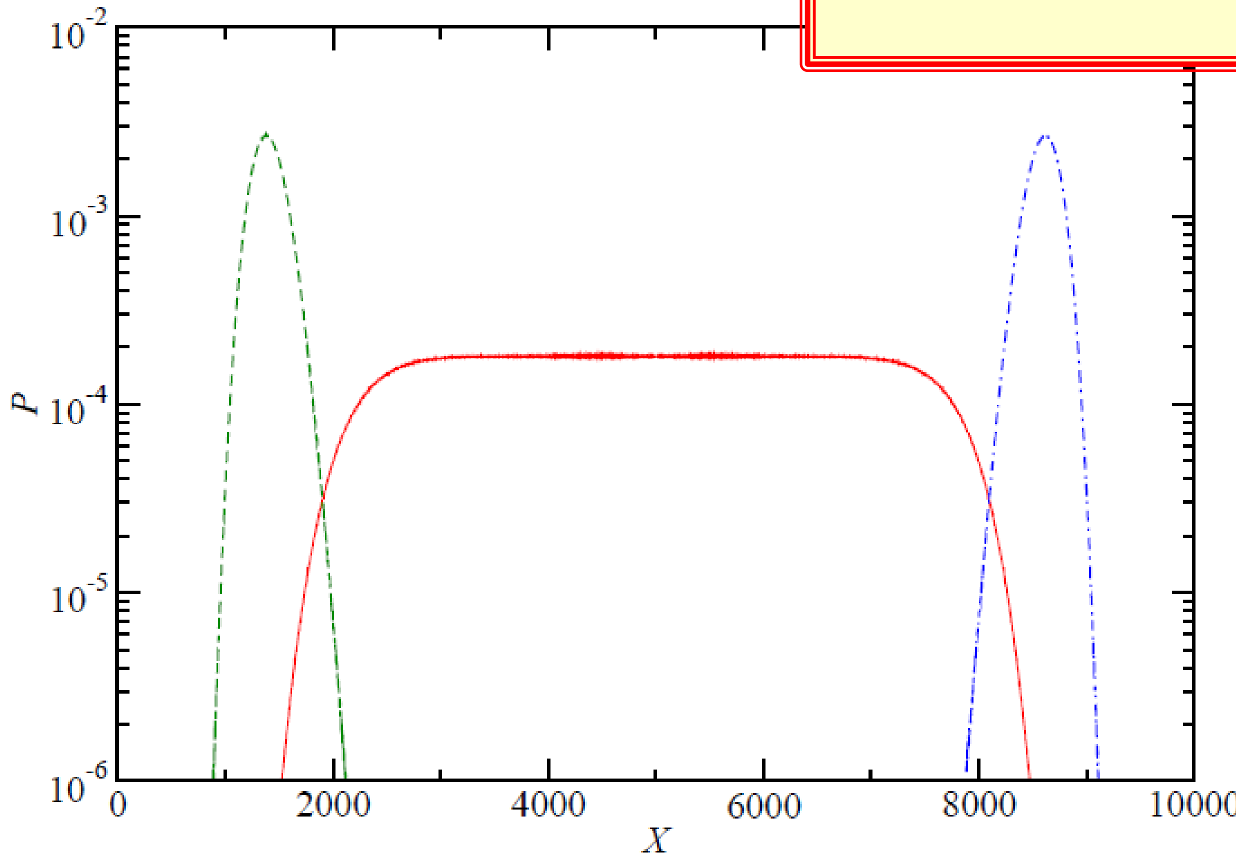
(101,99) (100,100) (99,101)



Power spectrum consistent with pure Random Walker: $\sim 1/f^2$ up to the "walls"



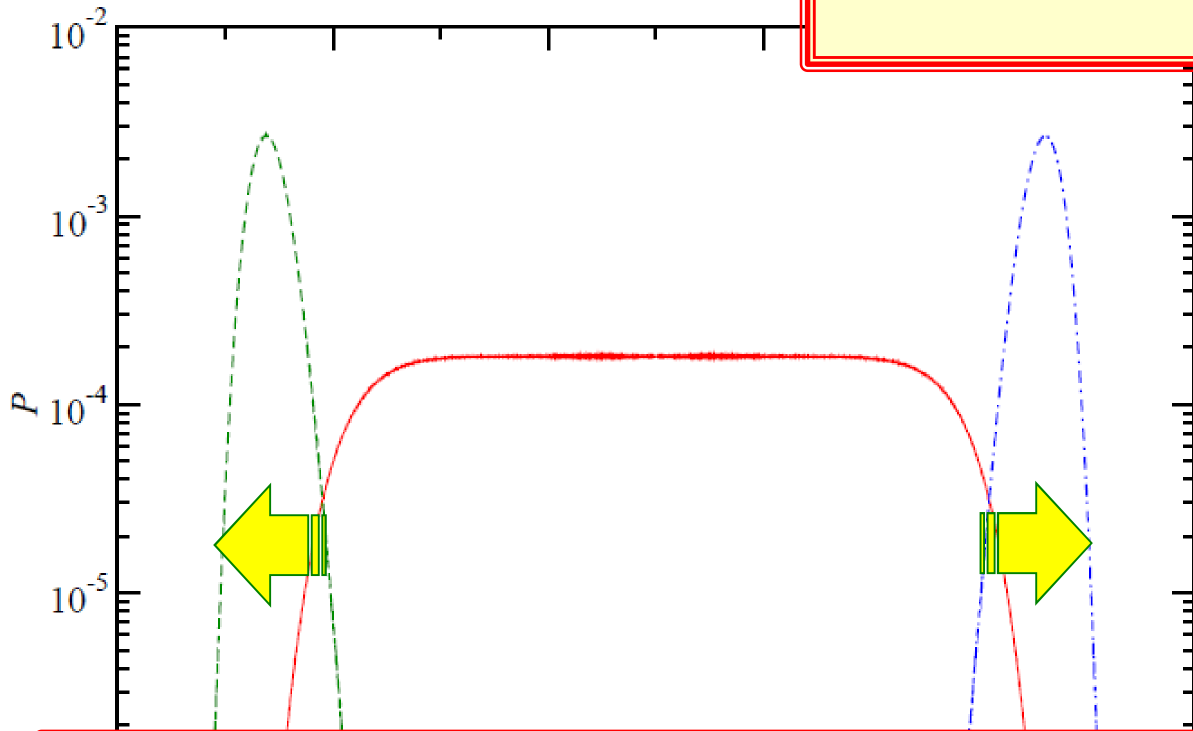
$P(X)$ near & at criticality



How to find location and width of the "soft walls"?

$P(X)$ near & at criticality

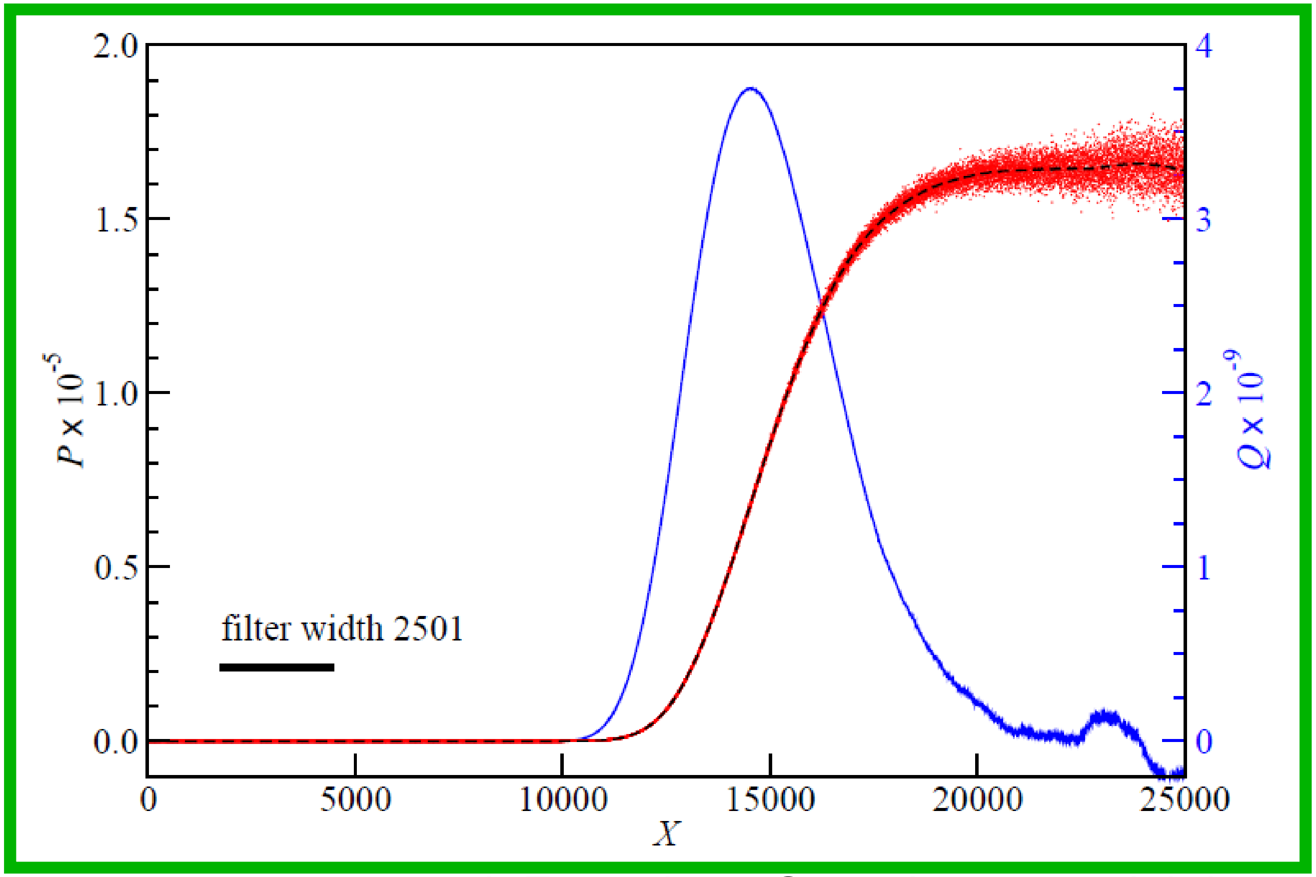
How to find location and width of the "soft walls"?



They move out with larger N !

Steepest "descent" in P $\Rightarrow$

max of $Q(X) \equiv dP/dX$



…exploit Ferrenberg Swendsen re-weighting

$$X_{max} / N^2 \sim N^{-0.38}$$

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An extraordinary transition

in a ***minimal*** adaptive network of introverts and extroverts

- Using the Ising magnetic language,
  - X maps into M:

$$m = 2\bar{\rho} - 1 = 2(\bar{X}/N_I N_E) - 1$$

  - $N_E - N_I$ corresponds to H, an external magnetic field:

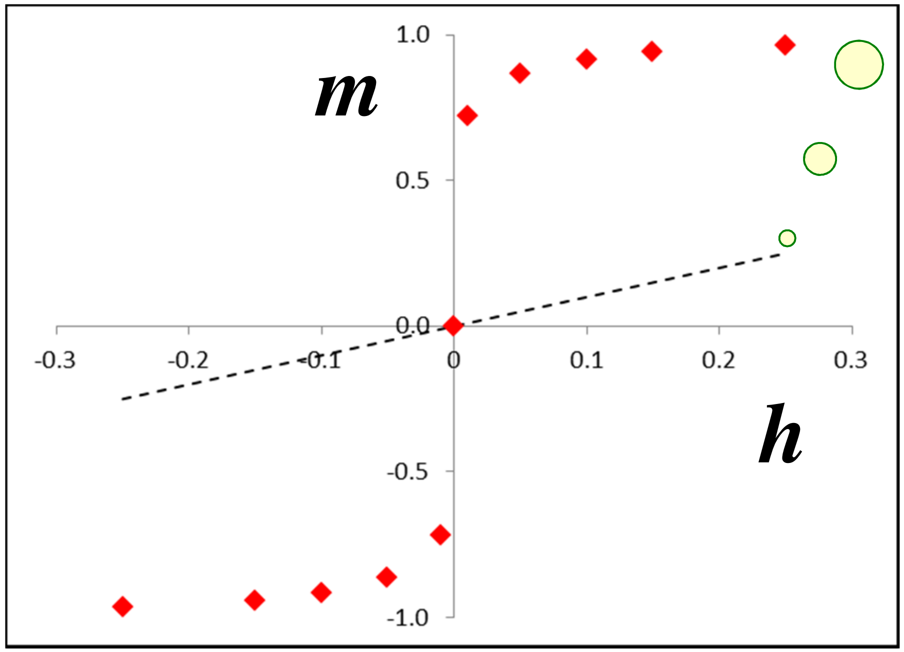
$$h = \frac{N_E - N_I}{N_E + N_I}$$

- Naïve expectation is just $m = h$

$1-\rho \sim$ prob to cut $\propto N_I$

$\rho \sim$ prob to add $\propto N_E$

…like this:

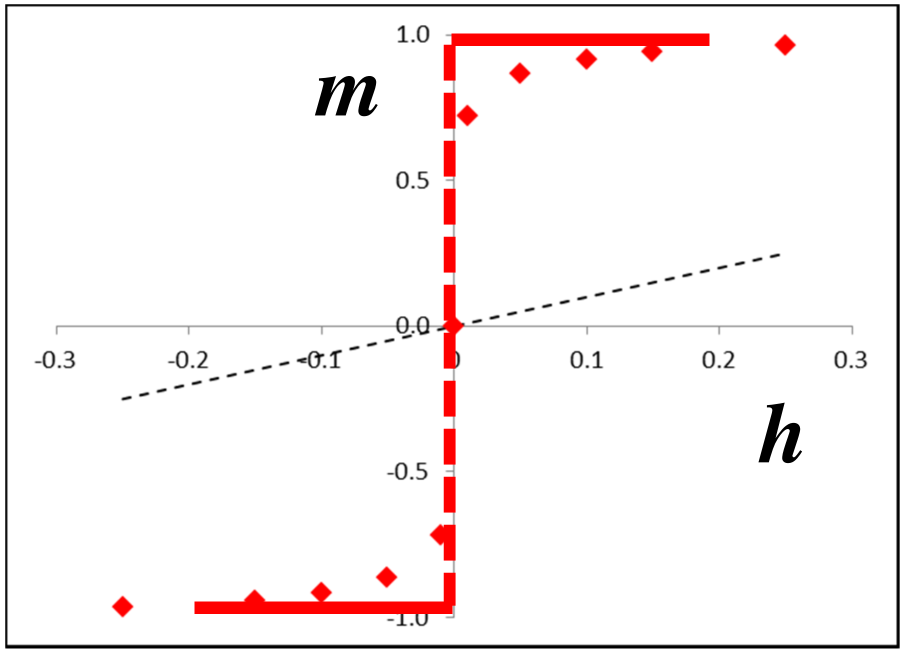


… but the system ***doesn't*** think so!

(125,75) (115,85) (110,90) (105,95) (101,99) (100,100) …

…like this:

Expectation for large N



… but the system ***doesn't*** think so!

(125,75) (115,85) (110,90) (105,95) (101,99) (100,100) …

- Reminds us of $m(h)$ in ferromagnetism below criticality…
- Lots of issues with this picture…
- Mixed order transitions
  - ‘extreme Thousless effect’
  - Bar & Mukamel *PRL* **112**, 015701 (2014)
  - …but there is neither (natural) temperature nor magnetic field
- Symmetry breaking control parameter here is the ***aspect ratio*** of the lattice!

- ⋮

Incidence Matrix for ***XIE*** model

Degrees of nodes i , j & extroverts' "holes"

…exactly like $N_I \times N_E$ Ising

$a_{ij} = 1, 0$
$i = 1, \ldots, N_I$
$j = 1, \ldots N_E$

$k_i \equiv \Sigma_j a_{ij} \in [0, N_E]$
$p_j \equiv \Sigma_i a_{ij} \in [0, N_I]$
$\bar{p}_j \equiv N_I - p_j$

Exact stationary distribution:

$$P^*(\{a_{ij}\}) = \frac{1}{\Omega} \prod_i (k_i!) \prod_j (\bar{p}_j!)$$

"partition function"

particle $\Leftrightarrow$ *hole*
$\oplus \quad N_I \Leftrightarrow N_E$

Physics

“Hamiltonian”

$$\mathcal{H} = -\sum_{i=1}^{N_I} \ln(k_i!) - \sum_{j=1}^{N_E} \ln(\bar{p}_j!)$$

- has long range, multi-spin interactions
- but *__peculiarly anisotropic__*:

…involves all spins within its row and column!!

- surely “much worse” than usual Ising!
- Our $P(X)$ is precisely Ising’s $P(M)$.
- exact, analytic forms not known!
- BTW, analogue of $\rho\,(k)$ in usual Ising model (almost) never studied

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W. Kob: “How about trying Mean Field Theory?”

- Start with

$$P(X) \equiv \sum_{\{\mathbb{A}\}} \delta(X, \Sigma_{ij} a_{ij}) P^*(\{a_{ij}\})$$

- and replace $a_{ij} \rightarrow X/(N_I N_E)$
- so that

$$k_i = \Sigma_j a_{ij} \rightarrow X/N_I$$

$$\bar{p}_j = N_I - \Sigma_i a_{ij} \rightarrow N_I - X/N_E$$

- Meanwhile,

$$\sum_{\{\mathbb{A}\}} \delta(X, \Sigma_{ij} a_{ij}) = \binom{N_I N_E}{X}$$

- so that

$$P(X) \to \binom{N_I N_E}{X} P^*\begin{pmatrix} k_i \to X/N_I \;; \\ \bar{p}_j \to N_I - X/N_E \end{pmatrix}$$

- A better perspective is to define a "Landau free energy"

$$F(\rho) \equiv \frac{-\ln P(X)}{N_I N_E}$$

$$\rho \equiv \frac{X}{N_I N_E}$$

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$$F(\rho) \propto \rho \ln N_I + [1-\rho] \ln N_E - \frac{1}{2}\left(\frac{\ln \rho}{N_E} + \frac{\ln[1-\rho]}{N_I} \right) + O\left(\frac{1}{N}\right)$$

Leading order is *linear !!*

"Restoring forces" down by $O(1/N\ln N)$ *!!*

So, typical ("off critical") minima are
very close to the boundaries!

Mostly flat
for the "critical" case!

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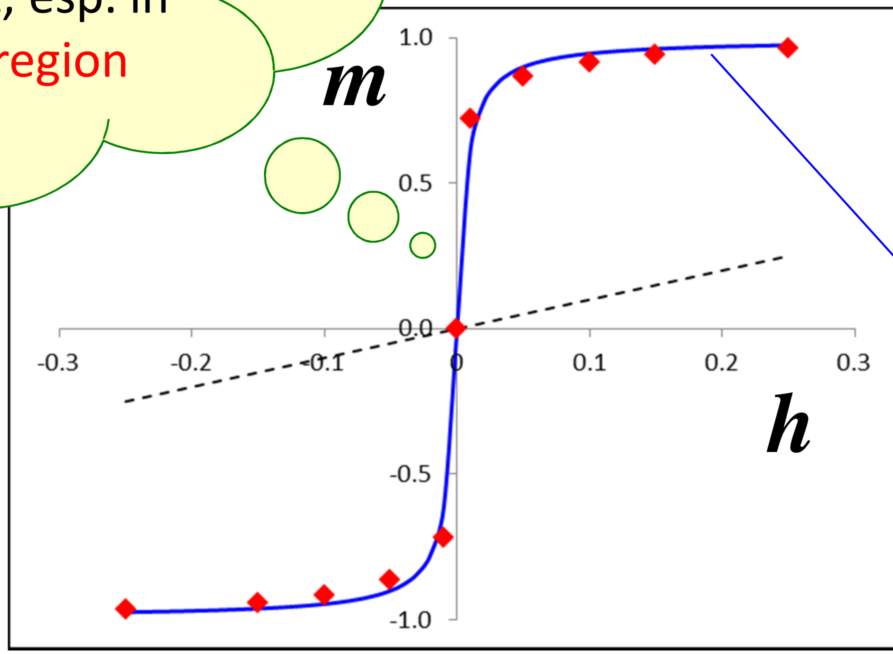
$$F(\rho) \propto \rho \ln N_I + [1-\rho]\ln N_E - \frac{1}{2}\left(\frac{\ln\rho}{N_E} + \frac{\ln[1-\rho]}{N_I}\right) + O\left(\frac{1}{N}\right)$$

- Meets qualitative expectations.
- Provides insight into this "extraordinary transition."
- Need FSS analysis for details!

… yet a <u>surprising</u> fit, with ***NO*** adjustable parameters:

Not bad, for a first try…
Obvious room for
improvement, esp. in
the critical region



Mean Field Approach
from min of F

$$\boxed{N_I + N_E = 200}$$

$$\boxed{m = 2X/(N_I N_E) - 1}$$

$$\boxed{h = (N_I - N_E)/200}$$

(125,75) (115,85) (110,90) (105,95) (101,99) (100,100) …

Summary and Outlook

- Many systems in real life involve networks with ***active*** links
- Dynamics from intrinsic preferences, adaptation, etc.
- Remarkable behavior, even in a ***minimal*** model
- Some aspects understood, many puzzles remain
- Exact $P^*(\{a_{ij}\})$ found!
- Didn't talk about other aspects, e.g., SIS on these networks
- Obvious questions, about *XIE* as well as more typical two communities interacting.
- Generalizations to more ***realistic*** systems.
  - Populations with many κ's, not just two distinct groups
  - Links can be stronger or weaker (close friend vs. acquaintance)
  - Interaction of networks with very different characteristics (e.g., social, internet, power-grid, transportation…)
  - How does failure of one affect another?

    $\vdots$

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