High-Energy Tail of the Velocity Distribution of Driven Inelastic Maxwell Gases

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Inelastic gas

Inelastic collision: $(v_1 - v_2) = -r(v_1^* - v_2^*)$

coefficient of restitution $\quad r \in [0, 1)$

Momentum conservation: $v_1 + v_2 = v_1^* + v_2^* \qquad (m = 1)$

Collision rules: $\begin{cases} v_1 = \epsilon v_1^* + (1-\epsilon) v_2^* \\ v_2 = (1-\epsilon) v_1^* + \epsilon v_2^* \end{cases} \qquad \epsilon = \dfrac{1-r}{2}$

Change of energy during collision:

$$\Delta E = -\epsilon(1-\epsilon)\left(v_1^* - v_2^*\right)^2$$

Energy decay with time

$$\Delta E \propto -\epsilon(\Delta v)^2, \qquad \Delta t \sim \ell/(\Delta v), \qquad \Delta v \sim v \sim \sqrt{E}$$

$$\Rightarrow \frac{dE}{dt} \sim -\epsilon\, E^{3/2} \qquad \Rightarrow \qquad E(t) \sim (1 + A\epsilon\, t)^{-2}$$

(Haff's law)

Late time behavior: (sticky gas like)

$$E(t) \sim t^{-2/3}$$

(in 1D)

clustering: $M(t) \sim t^{2/3}$

Carnevale, Pomeau & Young (1990).

$MV =$ constant $\Rightarrow$ $M \sim E^{-1}$

Clustering in 1D

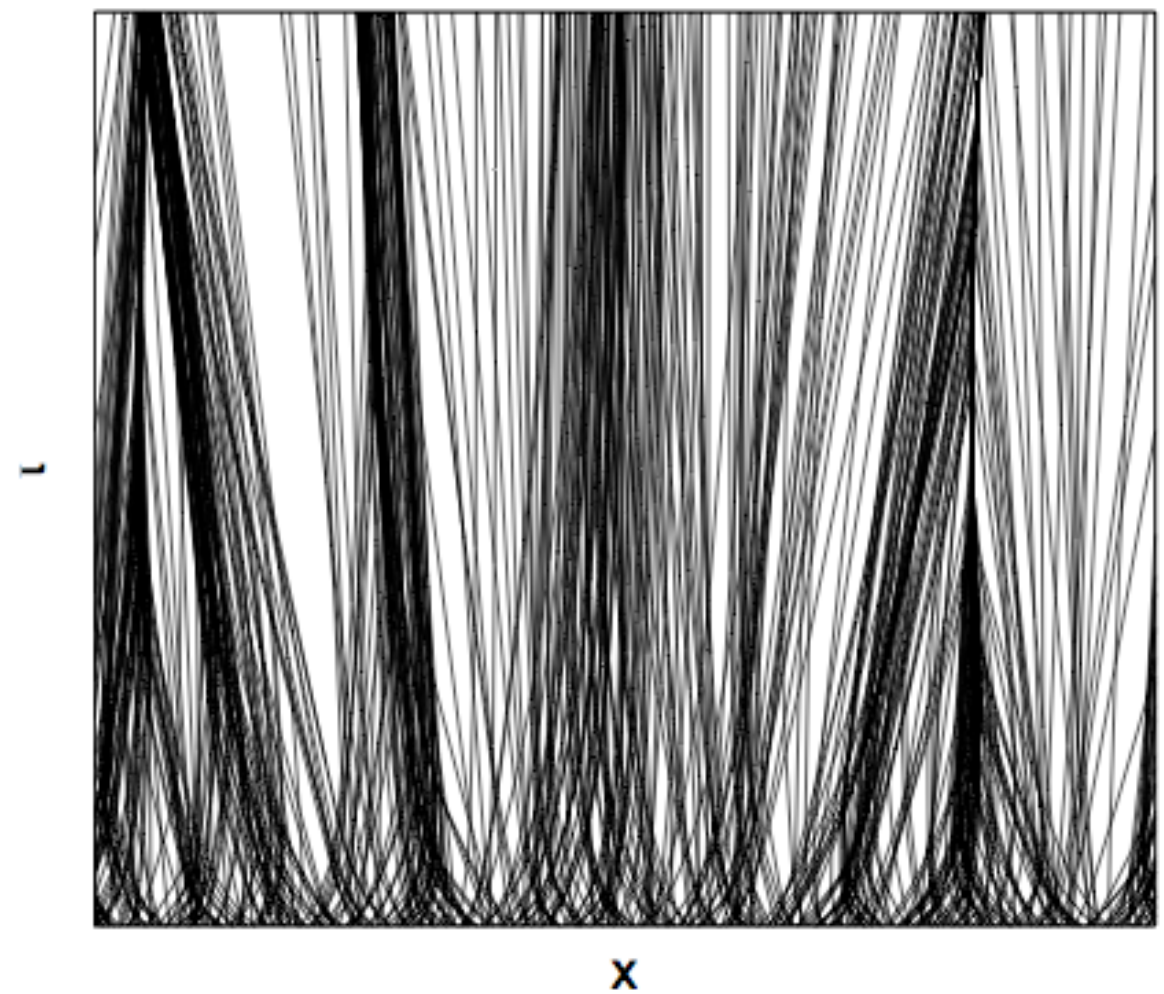

FIG. 3. Space-time evolution of a 500 particle system with $r = 0.9$ and $\delta = 10^{-2}$, up to $t = 600$.

Ben-Naim, Chen, Doolen & Redner (1999)

Connection with Burgers equation.

Clustering Instability in Dissipative Gases

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Applied & Computational Mathematics, Princeton University, Princeton, New Jersey 08544
(Received 3 September 1991)

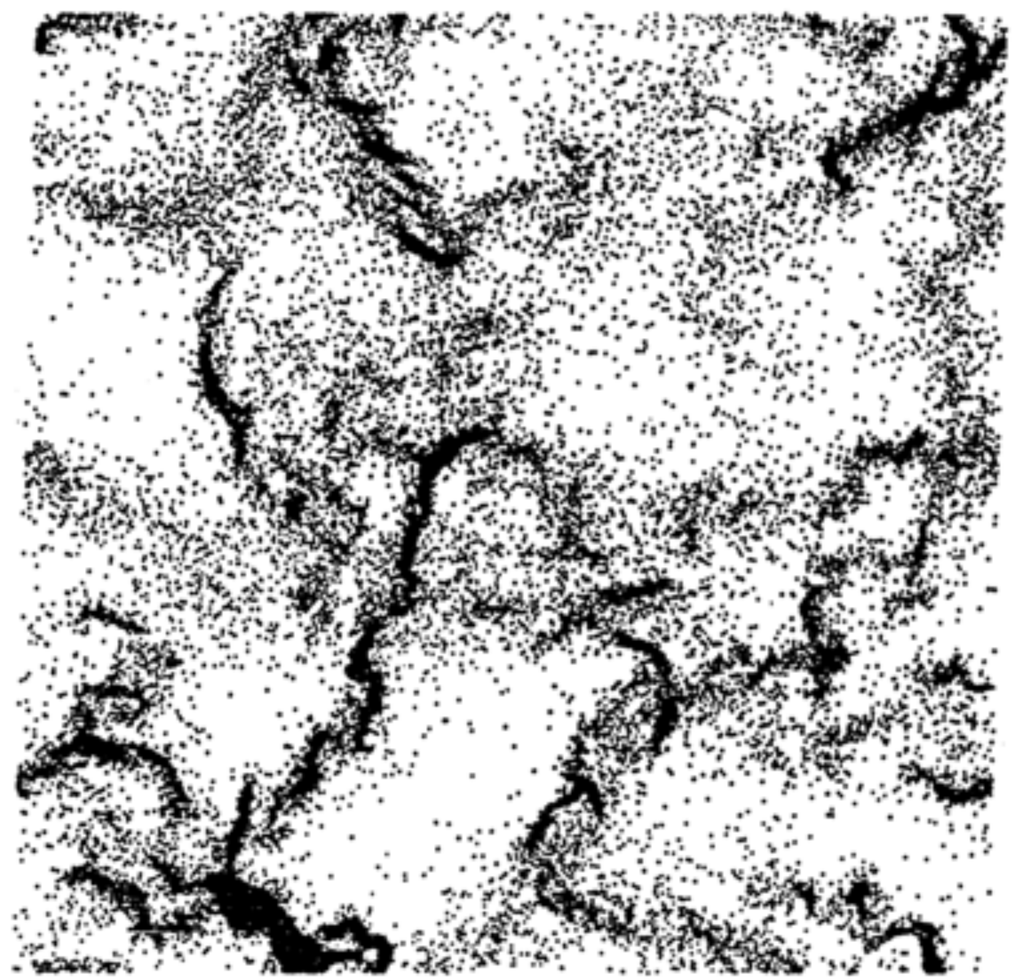
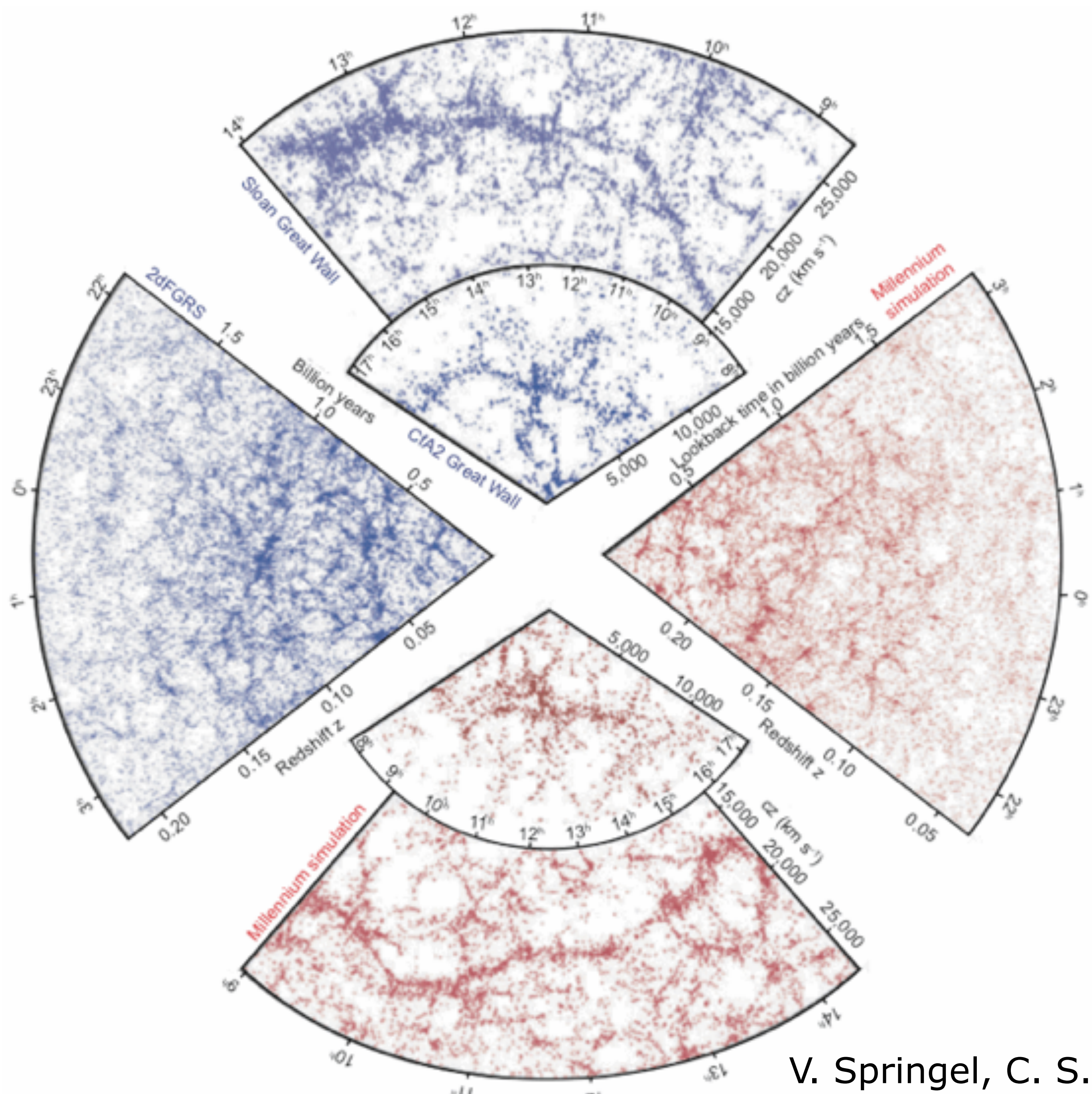


FIG. 3. A typical configuration of particles exhibiting clusters. Here the coefficient of restitution is 0.6, the time corresponds to 500 collisions per particle, and the area fraction is 0.05. The number of particles is 40000.

Large-scale structure of the universe



V. Springel, C. S. Frenk & S. D. M. White, *Nature* **440**, 1137 (2006)

Driven Granular Gases

What is the velocity distribution?

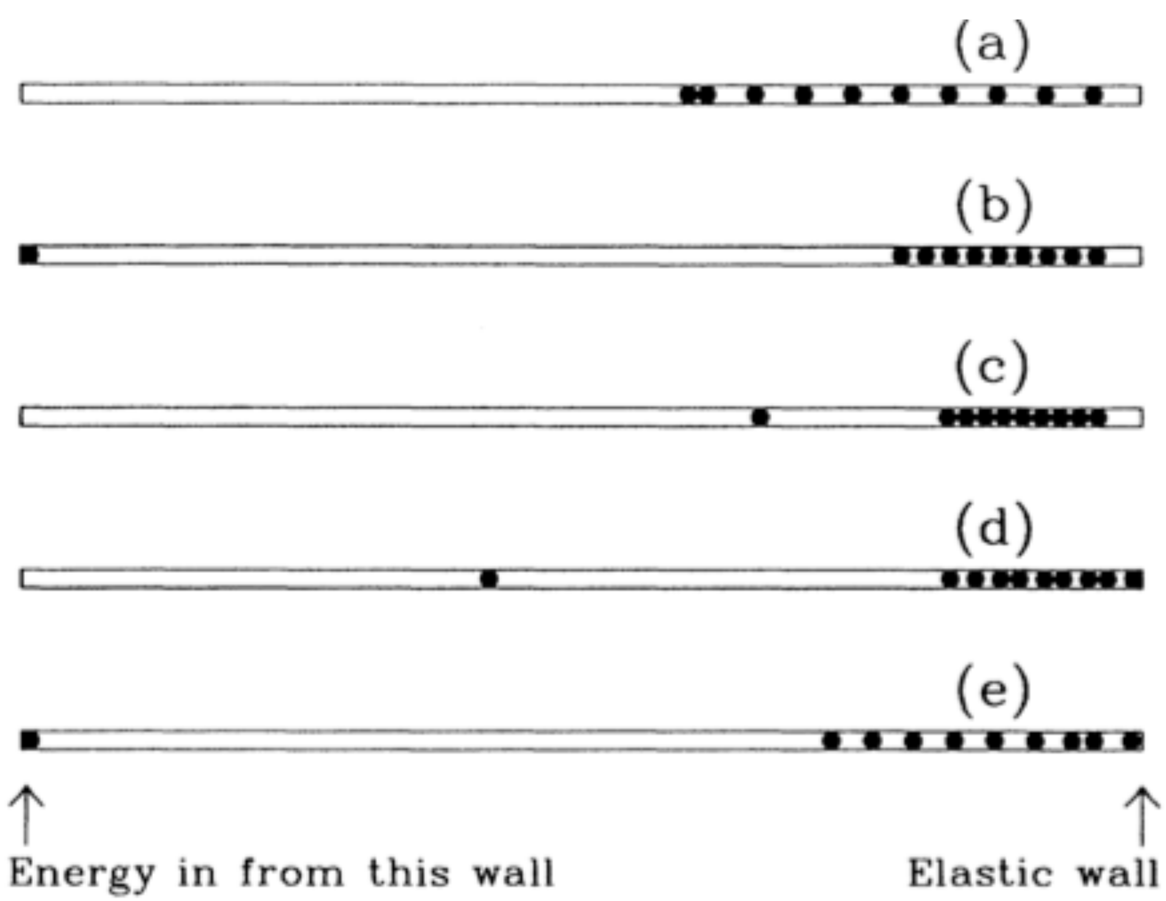
Is the steady state equilibrium like?

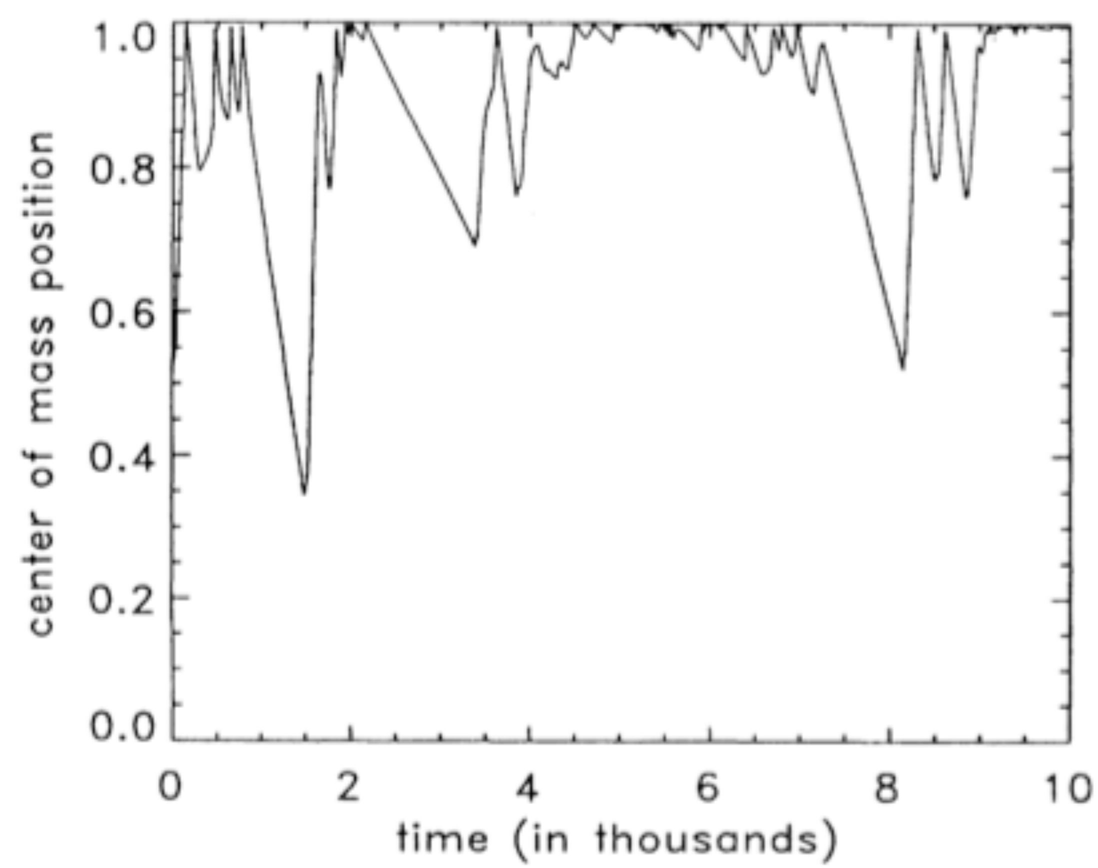
VOLUME 74, NUMBER 8 PHYSICAL REVIEW LETTERS 20 FEBRUARY 1995

Breakdown of Hydrodynamics in a One-Dimensional System of Inelastic Particles

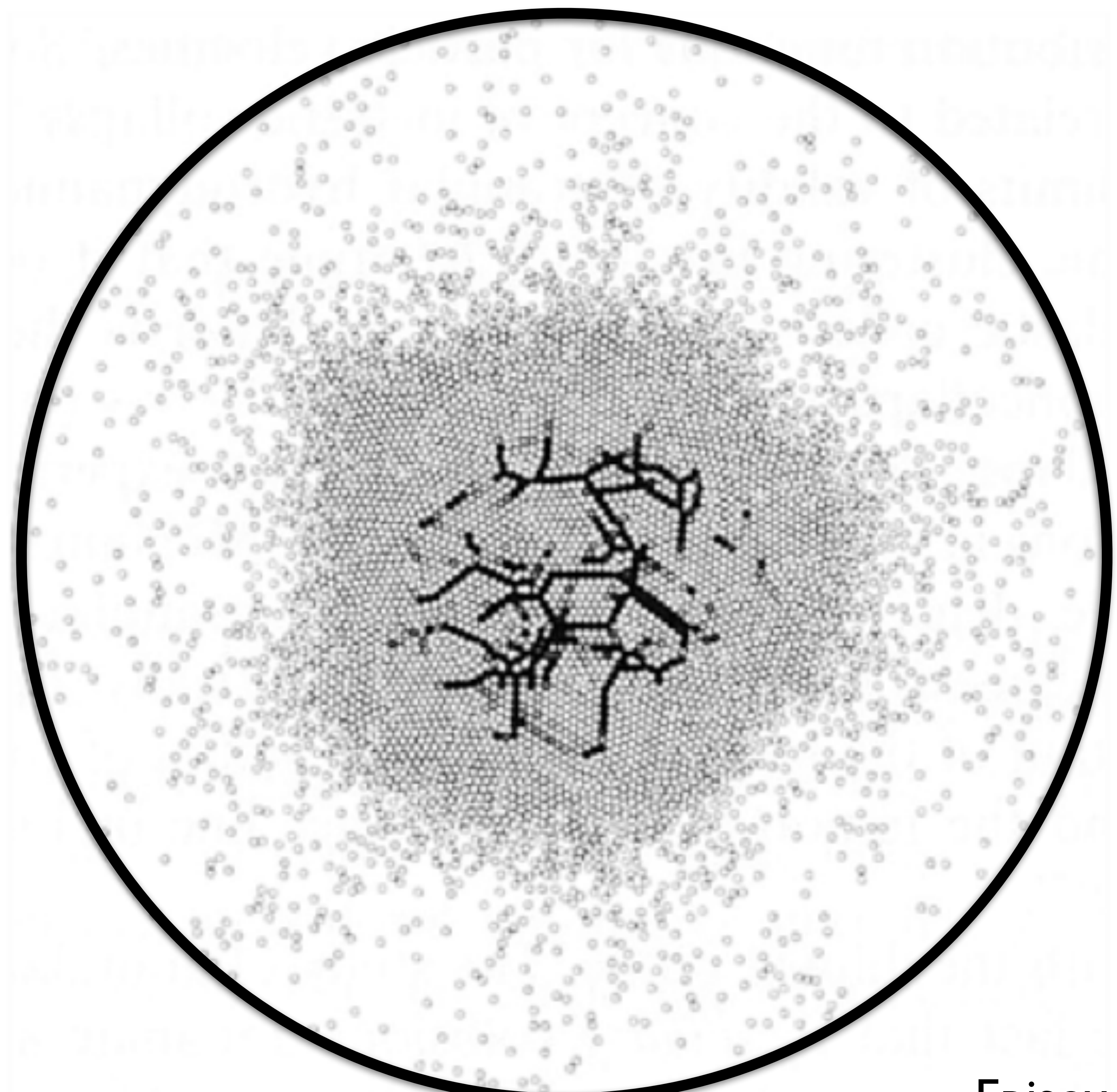
Yunson Du, Hao Li, and Leo P. Kadanoff
The James Franck Institute, The University of Chicago, Chicago, Illinois 60637
(Received 15 August 1994)

We study dynamics of nearly elastic particles constrained to move on a line with energy input from the boundaries. We find that for typical initial conditions, the system evolves to an "extraordinary" state with particles separated to two groups: The majority of the particles get clamped into a small region of space and move with very slow velocities; a few remaining particles travel between the boundaries at much higher speeds. Such a state clearly violates equipartition of energy. The simplest hydrodynamic approach fails to give a correct description of the system.





Clustering in the boundary driven granular gas in 2D



Episov & Pöschel (1997)

Uniformly driven gas

RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 60, NUMBER 3 SEPTEMBER 1999

Velocity distributions and density fluctuations in a granular gas

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(Received 21 April 1999)

Velocity distributions in a vibrated granular monolayer are investigated experimentally. Non-Gaussian velocity distributions are observed at low vibration amplitudes but cross over smoothly to Gaussian distributions as the amplitude is increased. Cross-correlations between fluctuations in density and temperature are present only when the velocity distributions are strongly non-Gaussian. Confining the expansion of the granular layer results in non-Gaussian velocity distributions that persist to high vibration amplitudes.
[S1063-651X(99)50409-1]

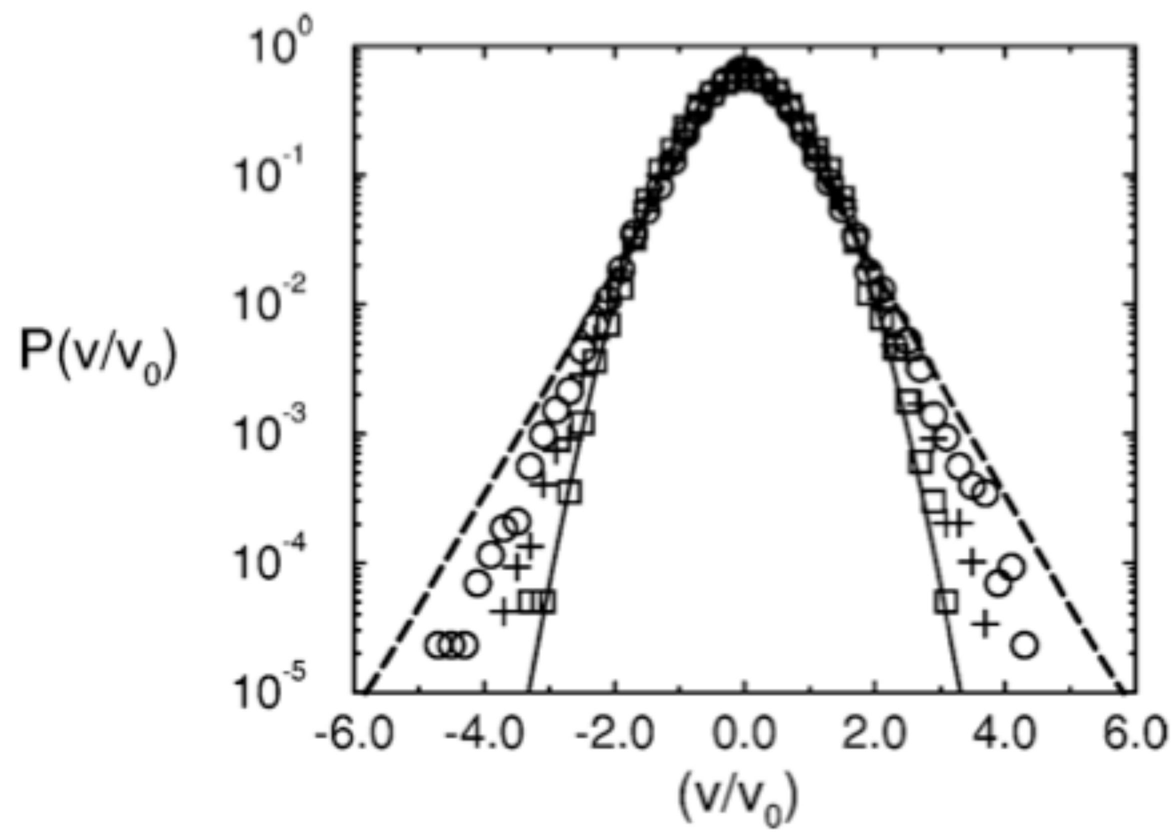

FIG. 1. Log-linear plot of velocity distribution functions for increasing Γ at constant frequency. As the acceleration is increased, the distributions go from having nearly exponential to Gaussian tails. (○) $\Gamma = 0.93$, (+) $\Gamma = 1.5$, (□) $\Gamma = 3.0$.

RAPID COMMUNICATIONS

PHYSICAL REVIEW E VOLUME 62, NUMBER 2 AUGUST 2000

Non-Gaussian velocity distributions in excited granular matter in the absence of clustering

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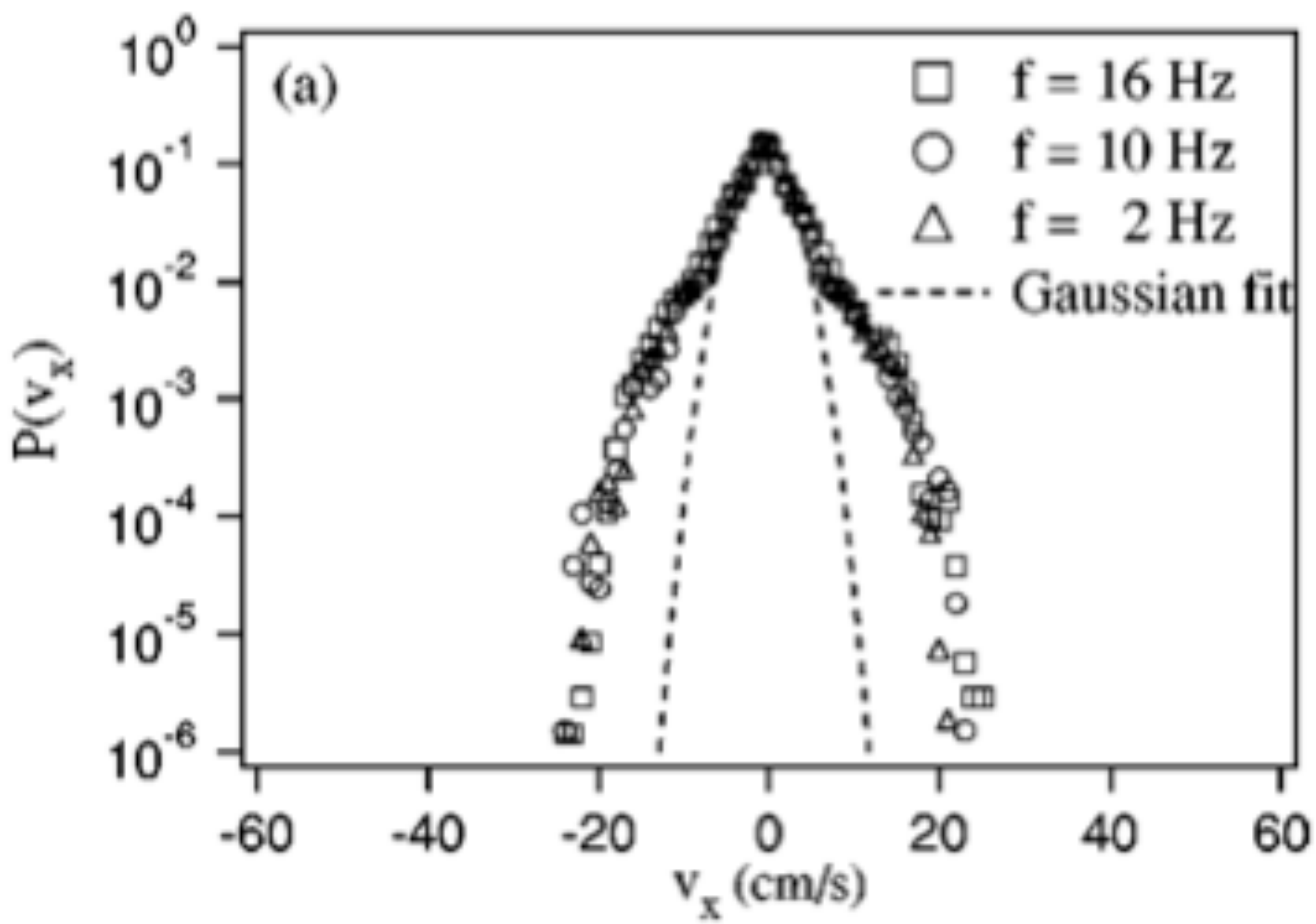
(Received 18 January 2000)

The velocity distribution of spheres rolling on a slightly tilted rectangular two-dimensional surface is obtained by high speed imaging. The particles are excited by periodic forcing of one of the side walls. Our data suggests that strongly non-Gaussian velocity distributions can occur in dilute granular materials even in the absence of significant density correlations or clustering. When the surface on which the particles roll is tilted further to introduce stronger gravitation, the collision frequency with the driving wall increases and the velocity component distributions approach Gaussian distributions of different widths.



Velocity Fluctuations in a Homogeneous 2D Granular Gas in Steady State

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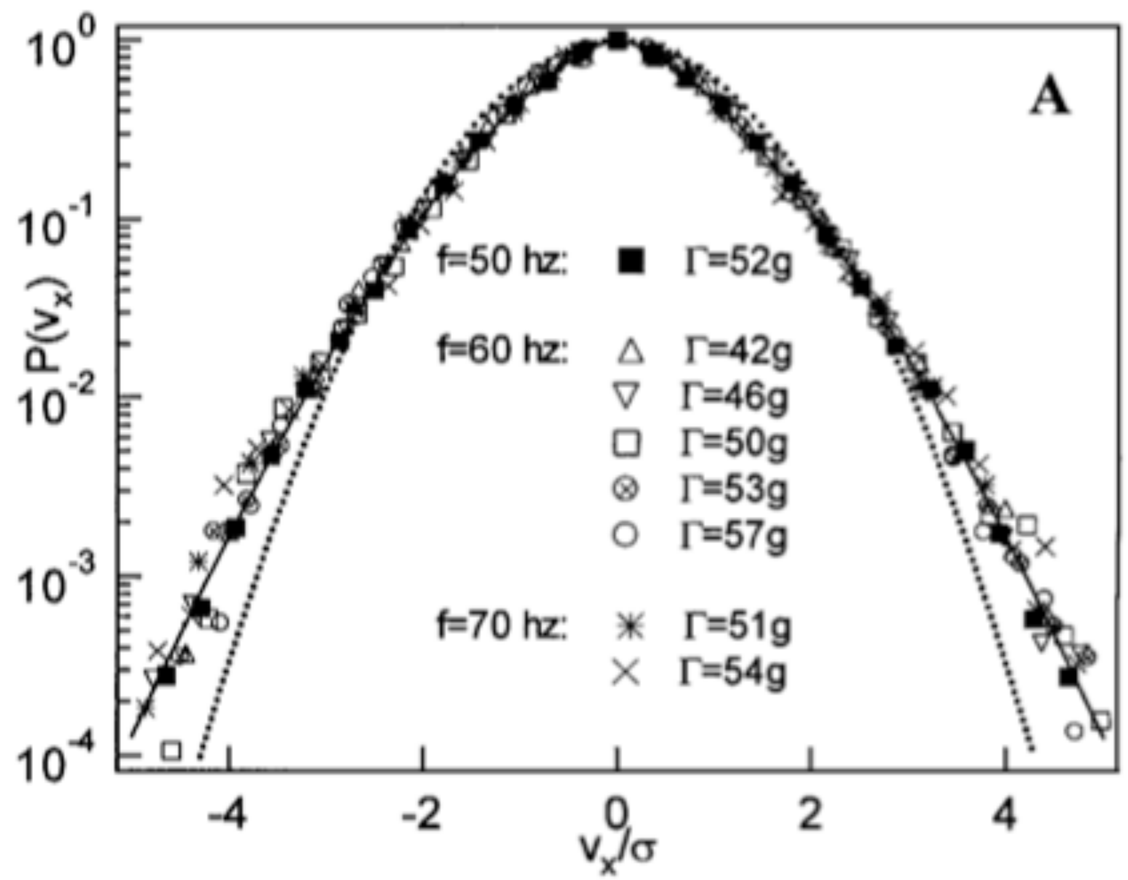
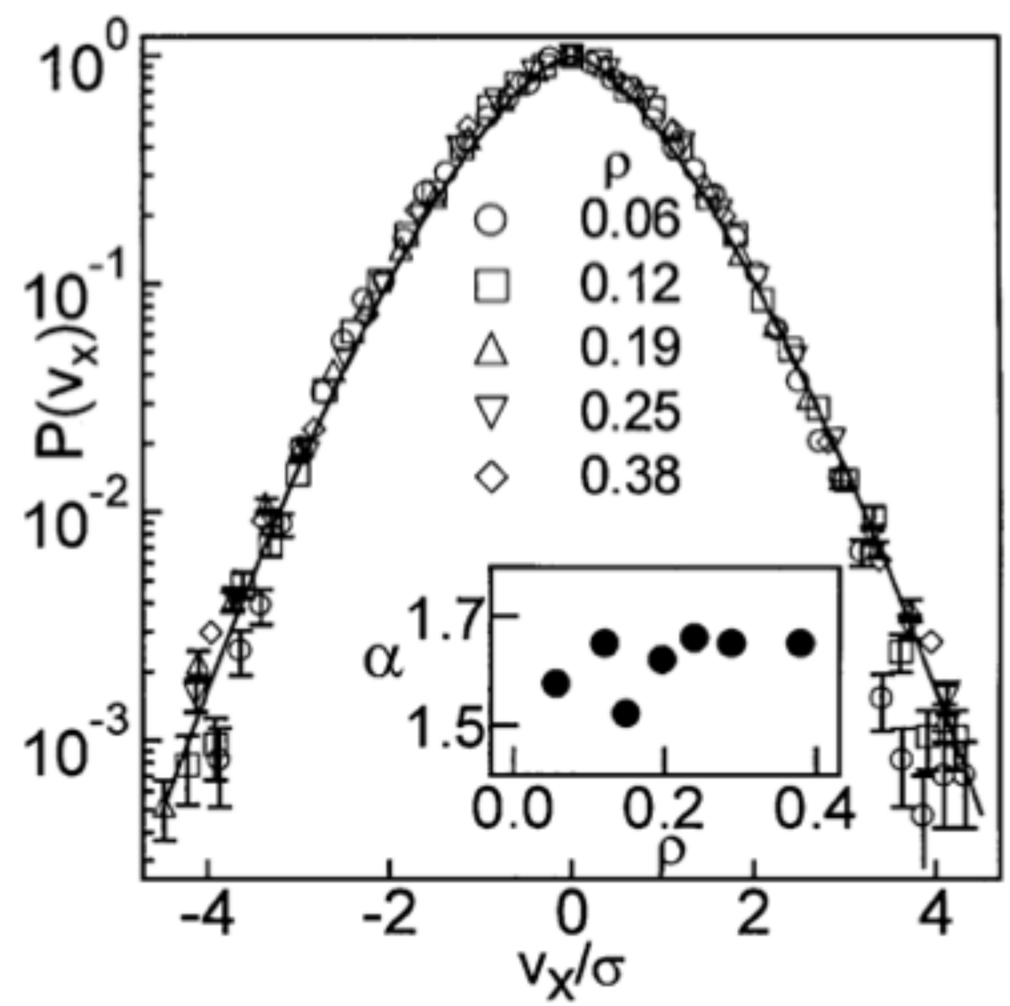
(Received 25 April 2000)

We have measured the spectrum of velocity fluctuations in a granular system confined to a vertical plane and driven into a homogeneous, steady state by strong vertical vibration. The distribution of horizontal velocities is not Maxwell-Boltzmann and is given by $P(v) = C\exp[-\beta(|v|/\sigma)^{\alpha}]$ where $\alpha = 1.55 \pm 0.1$ at all frequencies and amplitudes investigated, and also for varying boundary conditions. The deviation from Maxwell-Boltzmann statistics occurs in the absence of spatial clustering and does not result from an inhomogeneous average over regions of varying local density. Surprisingly, $P(v)$ has the same shape over a wide range of densities.





PHYSICAL REVIEW E **66**, 061302 (2002)

Velocity fluctuations in electrostatically driven granular media

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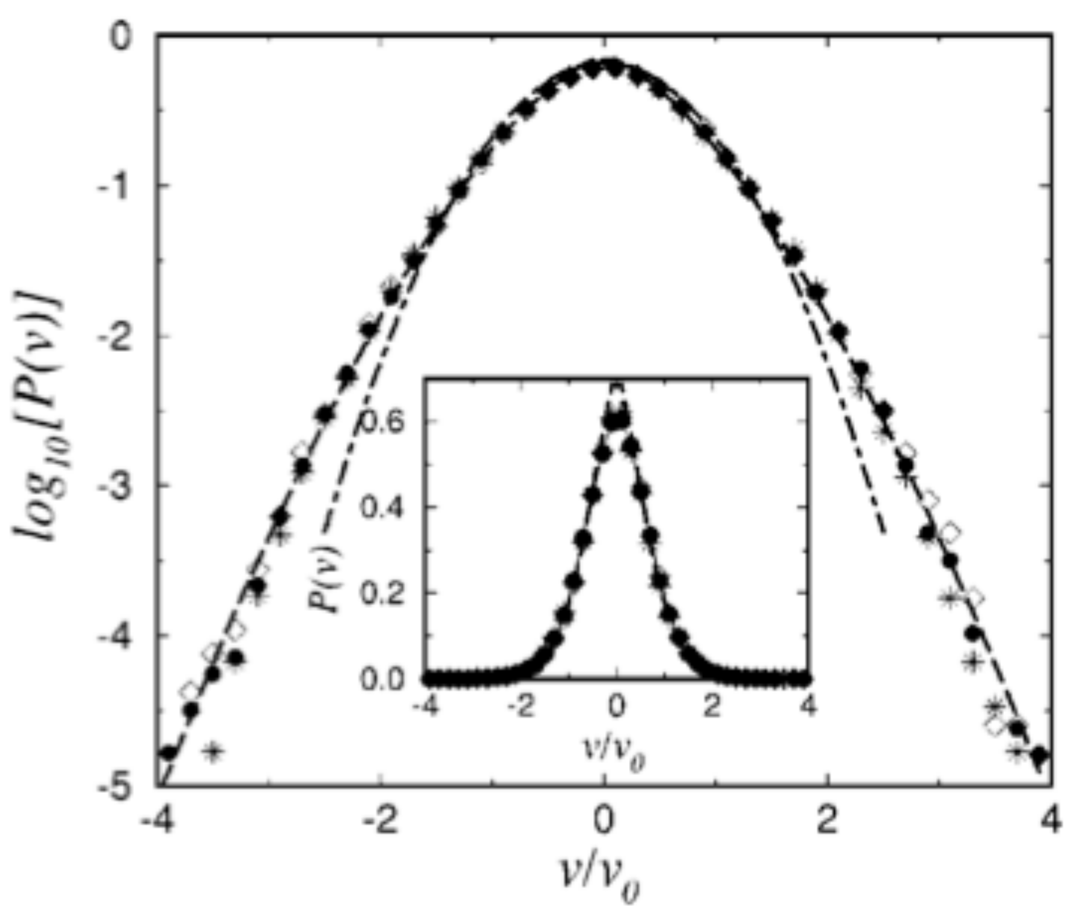
We study experimentally the particle velocity fluctuations in an electrostatically driven dilute granular gas. The velocity distributions have strong deviations from a Maxwellian form over a wide range of parameters. We have found that the tails of the distribution functions are consistent with a stretched exponential law with typical exponents of the order 3/2. Molecular dynamic simulations shows qualitative agreement with experimental data. Our results suggest that this non-Gaussian behavior is typical of most inelastic gases with both short- and long-range interactions.



Kinetic theory approach (homogeneous gas)

- Set up Boltzmann equation for single-particle velocity distribution, under molecular chaos hypothesis,

$$P(v_1, v_2) = P(v_1)P(v_2)$$

- Add a diffusive term to model uniform heating:

$$D\partial_v P(v) \qquad [dv/dt = \eta(t)]$$

- Collision rate $\propto |v_1 - v_2|^\delta$ with $\delta = 1$ (hard sphere)

$$\Rightarrow \quad P(v) \sim \exp\left(-A|v|^{3/2}\right)$$

Noije & Ernst (1998)

VOLUME 93, NUMBER 3 PHYSICAL REVIEW LETTERS week ending 16 JULY 2004

Velocity Distributions in Dissipative Granular Gases

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Motivated by recent experiments reporting non-Gaussian velocity distributions in driven dilute granular materials, we study by numerical simulation the properties of 2D inelastic gases.

Here we show that, rather than a universal distribution with $\alpha = 1.5$, a family of distributions with apparent exponents covering a wide range of values $\alpha < 2$ is expected, depending on both material and experimental conditions.

PHYSICAL REVIEW E **72**, 051301 (2005)

Velocity distributions in dilute granular systems

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(Received 30 October 2003; revised manuscript received 6 August 2004; published 11 November 2005)

We investigate the idea that velocity distributions in granular gases are determined mainly by η, the coefficient of restitution and q, which measures the relative importance of heating (or energy input) to collisions. To this end, we study by numerical simulation the properties of inelastic gases as functions of η, concentration ϕ, and particle number N with various heating mechanisms. For a wide range of parameters, we find Gaussian velocity distributions for uniform heating and non-Gaussian velocity distributions for boundary heating. Comparison between these results and velocity distributions obtained by other heating mechanisms and for a simple model of a granular gas without spatial degrees of freedom, shows that uniform and boundary heating can be understood as different limits of q, with $q \gg 1$ and $q \lesssim 1$ respectively. We review the literature for evidence of the role of q in the recent experiments.

We want to understand starting with the simplest model

The Maxwell model: $\delta = 0$ (collision rate is independent of the velocities of the colliding particles)

$$\partial_t P(v,t) = \sqrt{T} \int_{-\infty}^{\infty} du \left[\frac{1}{r} P(v^*,t) P(u^*,t) - P(v,t) P(u,t) \right]$$

$$+ D \partial_v P(v,t)$$

Ben-Naim & Krapivsky (2000)

$$\Rightarrow \quad P(v, t \to \infty) \sim \exp\big(-A|v|\big)$$

Antal, Droz & Lipowski (2002)
Santos & Ernst (2003)

Is this true?

The model

- Consider a collection of N particles.
- Each particle is characterized by a velocity v_i .
- At each time step, pick two particles at random.
- With probability p , collide them inelastically:

$$v_i = \epsilon v_i^* + (1-\epsilon) v_j^* \quad \text{and} \quad v_j = (1-\epsilon) v_i^* + \epsilon v_j^*$$

- With probability $(1-p)$, apply external force:

$$v_i = v_i^* + \eta_i \quad \text{and} \quad v_j = v_j^* + \eta_j$$

Hierarchy of equations

- Equation for $P_1(v_1)$ involves $P_2(v_1, v_2)$
- Equation for $P_2(v_1, v_2)$ involves $P_3(v_1, v_2, v_3)$
- ... and so on.

Fortunately, we find that, the equations for variance per particle and the velocity correlation per pair close, and satisfy a linear recursion relation.

$$\mathrm{e}(n) = \frac{1}{2N} \sum_{i=1}^{N} \langle v_i^2(n) \rangle \,, \qquad \Sigma(n) = \frac{1}{N(N-1)} \sum_{i \neq j} \langle v_i(n) v_j(n) \rangle$$

$$[\mathrm{e}(n), \Sigma(n)]^T \equiv X_n = R X_{n-1} + C$$

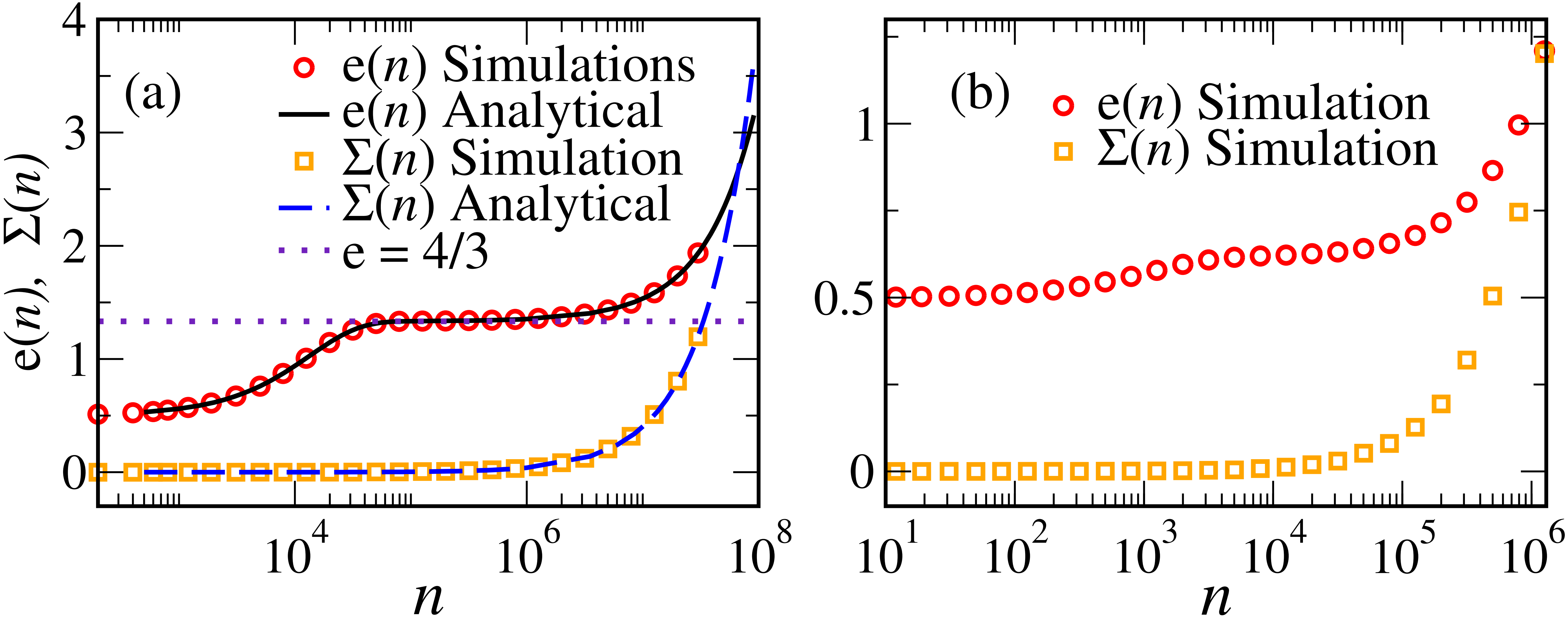
$$R = \begin{bmatrix} 1 - \frac{4p\epsilon(1-\epsilon)}{N} & \frac{2p\epsilon(1-\epsilon)}{N} \\ \frac{8p\epsilon(1-\epsilon)}{N(N-1)} & 1 - \frac{4p\epsilon(1-\epsilon)}{N(N-1)} \end{bmatrix} \qquad C = \begin{bmatrix} (1-p)\frac{\sigma^2}{N} \\ 0 \end{bmatrix}$$

Eigenvalues are: 1 and $1 - \dfrac{p(1-r^2)}{(N-1)}$

$\Downarrow$

Linear dependence with n

The mean energy and the correlation



What we find:

- Eventually, both variance and the correlation increase linearly with time.
- No steady state.
- The assumption $P_2(v_1, v_2) = P_1(v_1)P_1(v_2)$

  is not valid!

  (which was used in the Boltzmann equation)

Modeling the external forcing

The particles collide with a "vibrating wall"

$$v - W = -r_w(v^* - W^*)$$

The wall is massive: $W = W^*$

$$v = -r_w v^* + (1 + r_w)W$$

The velocity of the wall is random: $(1 + r_w)W = \eta$

$$\boxed{v = -r_w v + \eta}$$

$$\boxed{v = -r_w v + \eta} \qquad r_w \in [-1, 1]$$

$$R = \begin{bmatrix} 1 - \frac{\left[4p\epsilon(1-\epsilon)+2(1-p)(1-r_w^2)\right]}{N} & \frac{2p\epsilon(1-\epsilon)}{N} \\ \frac{8p\epsilon(1-\epsilon)}{N(N-1)} & 1 - \frac{\left[4p\epsilon(1-\epsilon)-2(1-p)(1+r_w)^2+4(N-1)(1-p)(1+r_w)\right]}{N(N-1)} \end{bmatrix}$$

No steady state

$r_w = -1$ $\qquad$ $r_w = +1$

For $r_w \neq -1$ both eigenvalues have absolute values less than unity. Thus, the system reaches a steady state.

Mean energy and correlation in the steady state

$$\mathrm{e} = \frac{(\sigma^2/2)\left[2\epsilon(1-\epsilon)+\gamma(1-r_w^2)+2(N-2)\gamma(1+r_w)\right]}{4\epsilon(1-\epsilon)(1-r_w^2)+\gamma(1-r_w^2)^2+(N-2)(1+r_w)[4\epsilon(1-\epsilon)+2\gamma(1-r_w^2)]}\,, \qquad \gamma = (1-p)/p$$

$$\Sigma = \frac{2\sigma^2\epsilon(1-\epsilon)}{4\epsilon(1-\epsilon)(1-r_w^2)+\gamma(1-r_w^2)^2+(N-2)(1+r_w)[4\epsilon(1-\epsilon)+2\gamma(1-r_w^2)]}\,.$$

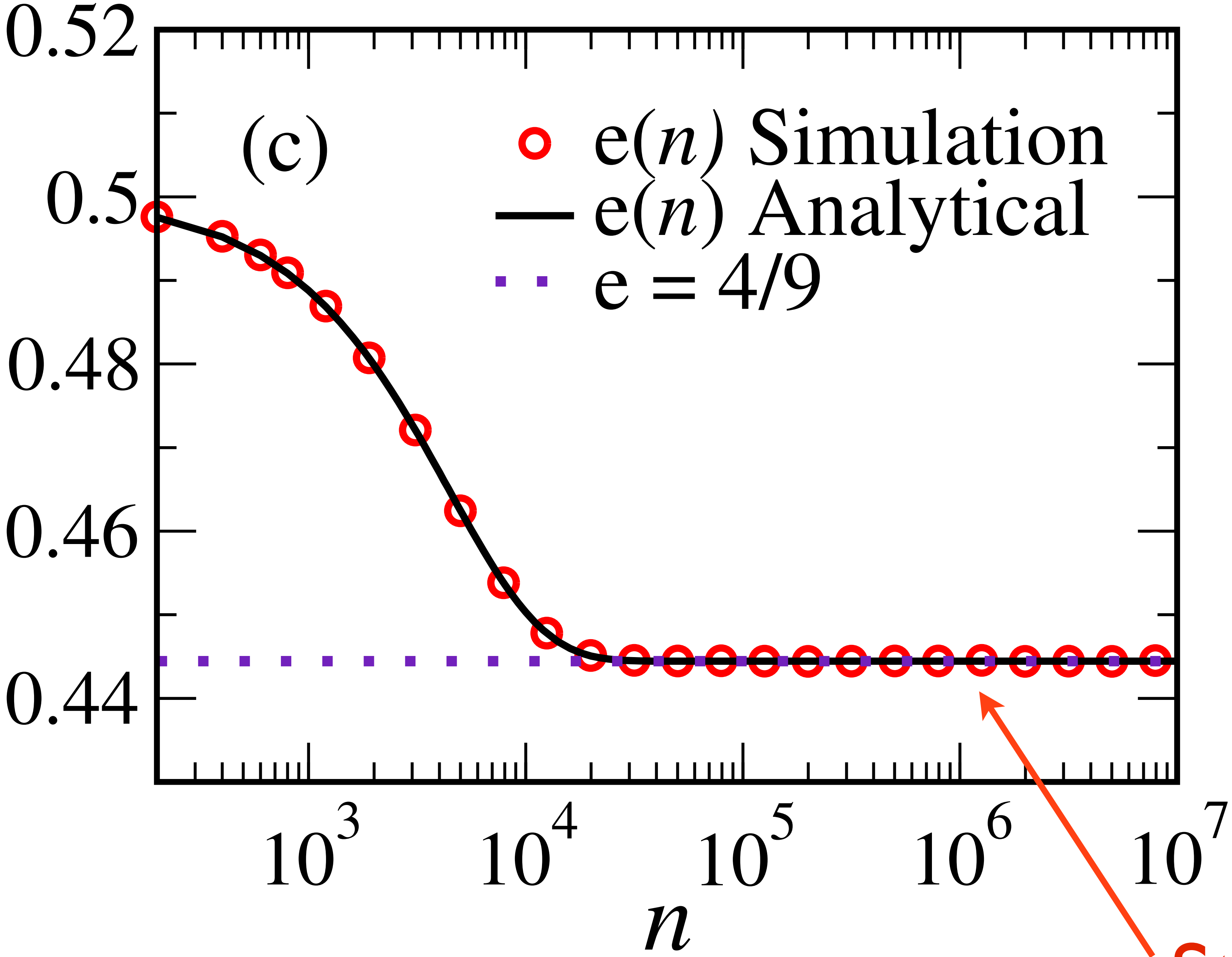
Large N limit:

$$\mathrm{e} = \frac{\gamma\sigma^2}{4\epsilon(1-\epsilon)+2\gamma(1-r_w^2)} + O(N^{-1})$$

$$\Sigma = O(N^{-1})$$

Mean energy in the steady state

$r_w \neq -1$



Steady state

- The correlation is $O(N^{-1})$ $\rightarrow 0$ as $N \rightarrow \infty$
- The assumption $P_2(v_1, v_2) = P_1(v_1)P(v_2)$ ✓

Generating function $Z(\lambda) = \langle e^{-\lambda v} \rangle$ in the steady state ($r_w \neq -1$)

$$Z(\lambda) = p\, Z(\epsilon\lambda)\, Z([1-\epsilon]\lambda) + (1-p)\, Z(r_w\lambda)\, \underbrace{f(\lambda)}_{\exp(\sigma^2\lambda^2/2)}$$

for Gaussian noise

with $Z(\lambda) = Z(-\lambda)$

and $Z(\lambda) = 1 + \mathrm{e}\lambda^2 + \cdots$ as $\lambda \to 0$

Nonlinear & nonlocal equation!

We can't find the exact solution.

Near-elastic, weak energy injection limit

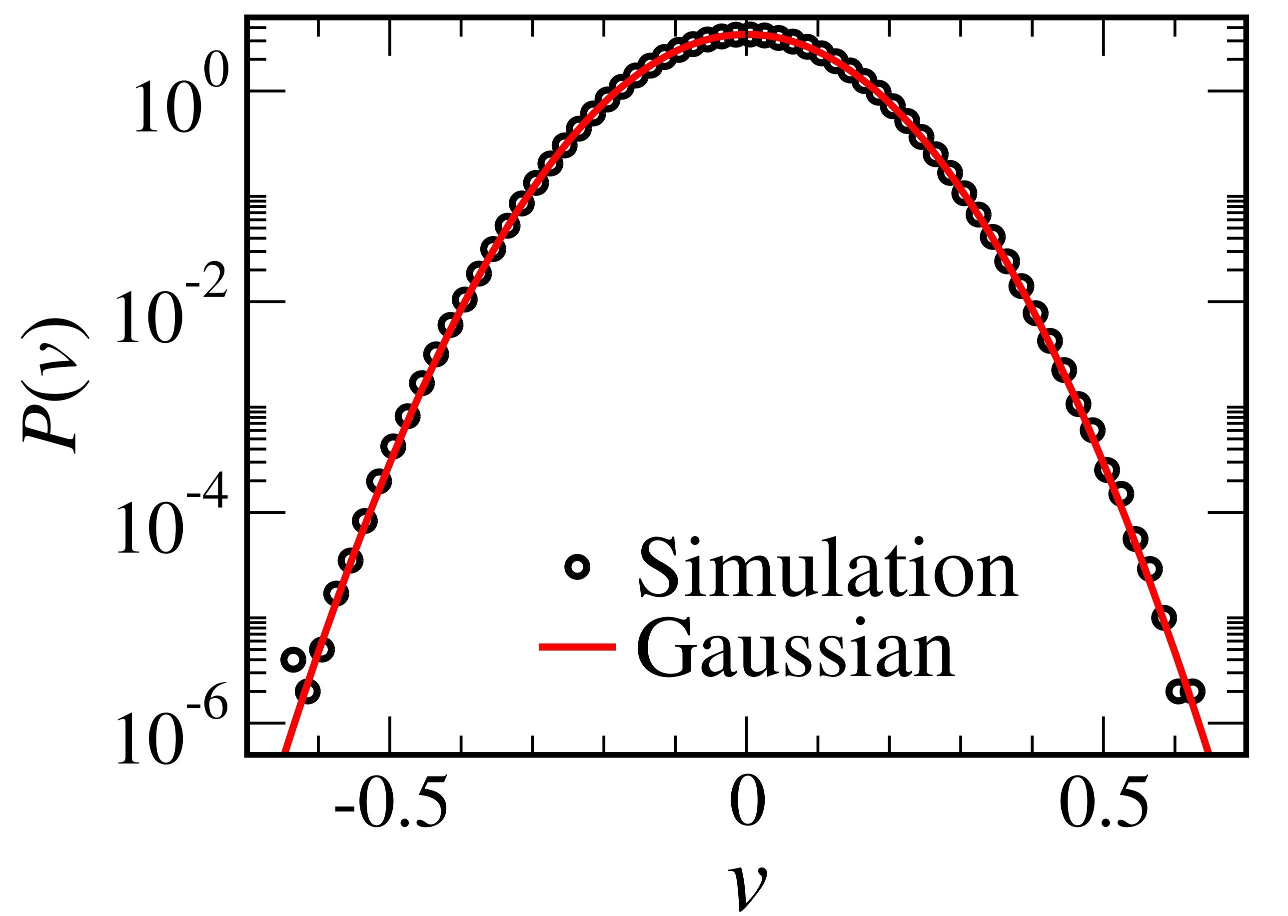
$$\epsilon \to 0, \quad r_w = (1-\theta) \to 1, \quad \sigma \to 0$$

while keeping σ^2/ϵ and θ/ϵ fixed.

$$\frac{dZ}{d\lambda} = \lambda \Delta^2 Z(\lambda)$$

where $\Delta^2 = \dfrac{\gamma\sigma^2/\epsilon}{2\left[1+\gamma\theta/\epsilon\right]}$

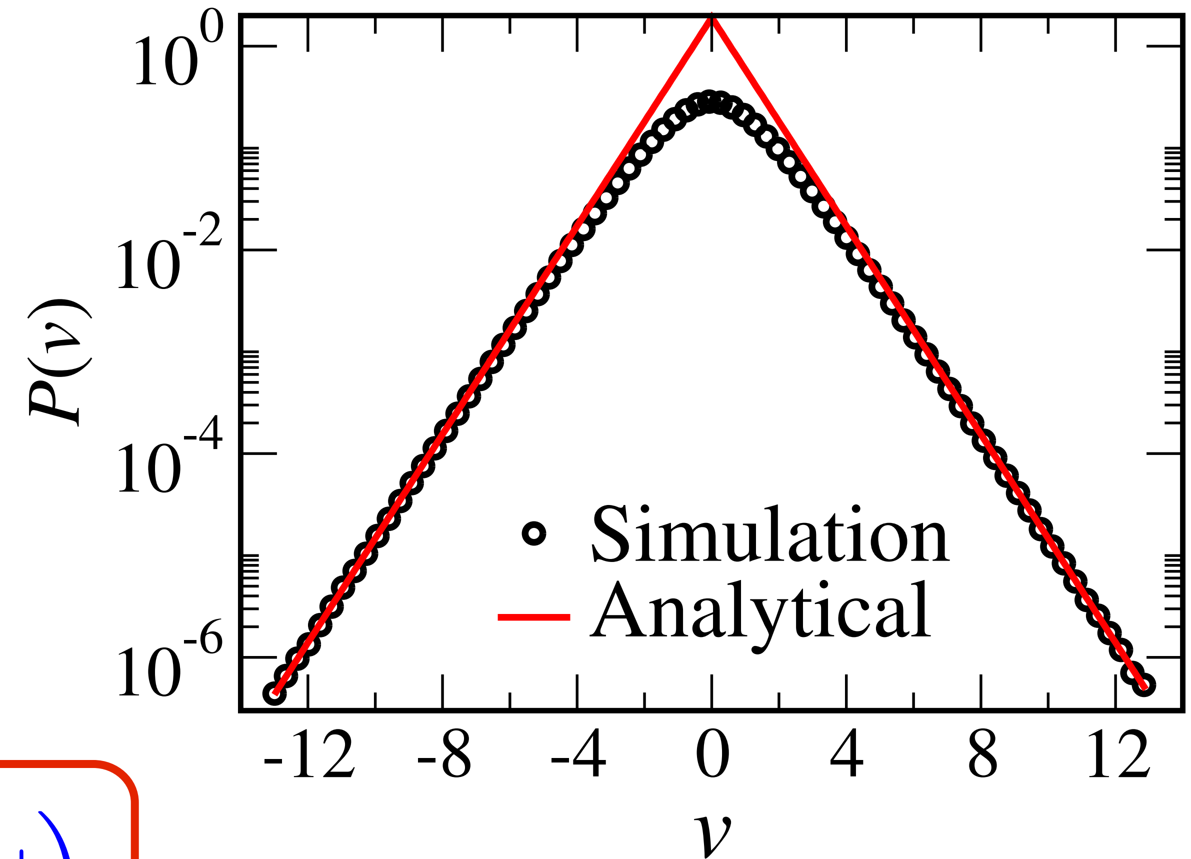
Gaussian with $\langle v^2 \rangle = \Delta^2$



Elastic collision with the "wall" $(r_w = 1)$

$$Z(\lambda) = \left[1 - (1-p)\, f(\lambda)\right]^{-1} p\, Z(\epsilon\lambda) Z([1-\epsilon]\lambda)$$

This can be solved by iteration.



Exponential tail:

$$\boxed{P(v) \sim A(\epsilon) \exp\Big(-|\lambda_0||v|\Big)}$$

$$\boxed{v = -r_w v + \eta}$$

Inelastic collision with the "wall" ($|r_w| < 1$)

- Difficult to guess the tail from numerical simulation, as it requires large number of realizations.
- Convenient to numerically solve the generating function $Z(ik)$ for the special case $\epsilon = r_w = 1/2$.
- Numerically computing the inverse Fourier transform gives the velocity distribution.
- The numerical results were not conclusive!

$$P(v) \sim \exp\bigl(-A|v|^{\alpha}\bigr)$$

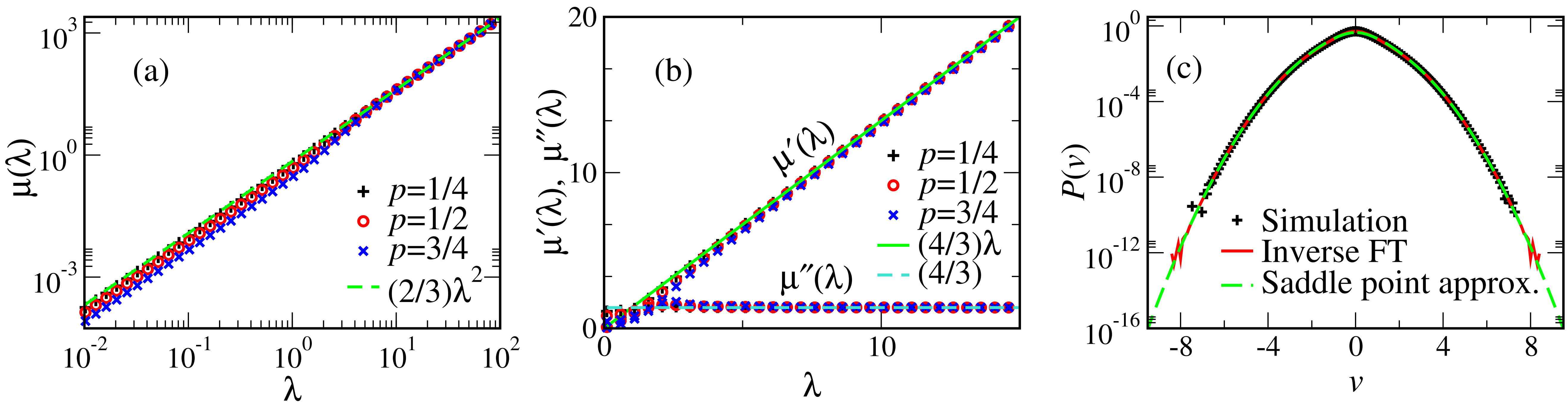
with α gradually increasing (but $\alpha < 2$) as we go towards higher and higher velocities.

Saddle-point analysis

$$P(v) \approx \frac{\exp\left[\mu(\lambda^*) + \lambda^* v\right]}{\sqrt{2\pi|\mu''(\lambda^*)|}}$$ where $\mu(\lambda) = \ln Z(\lambda)$

and $\mu'(\lambda^*) = -v$

Gives Gaussian: $P(v) \approx \sqrt{\frac{1-r_w^2}{2\pi\sigma^2}} \exp\left[-\frac{v^2}{2\sigma^2}(1-r_w^2)\right]$



Even moments

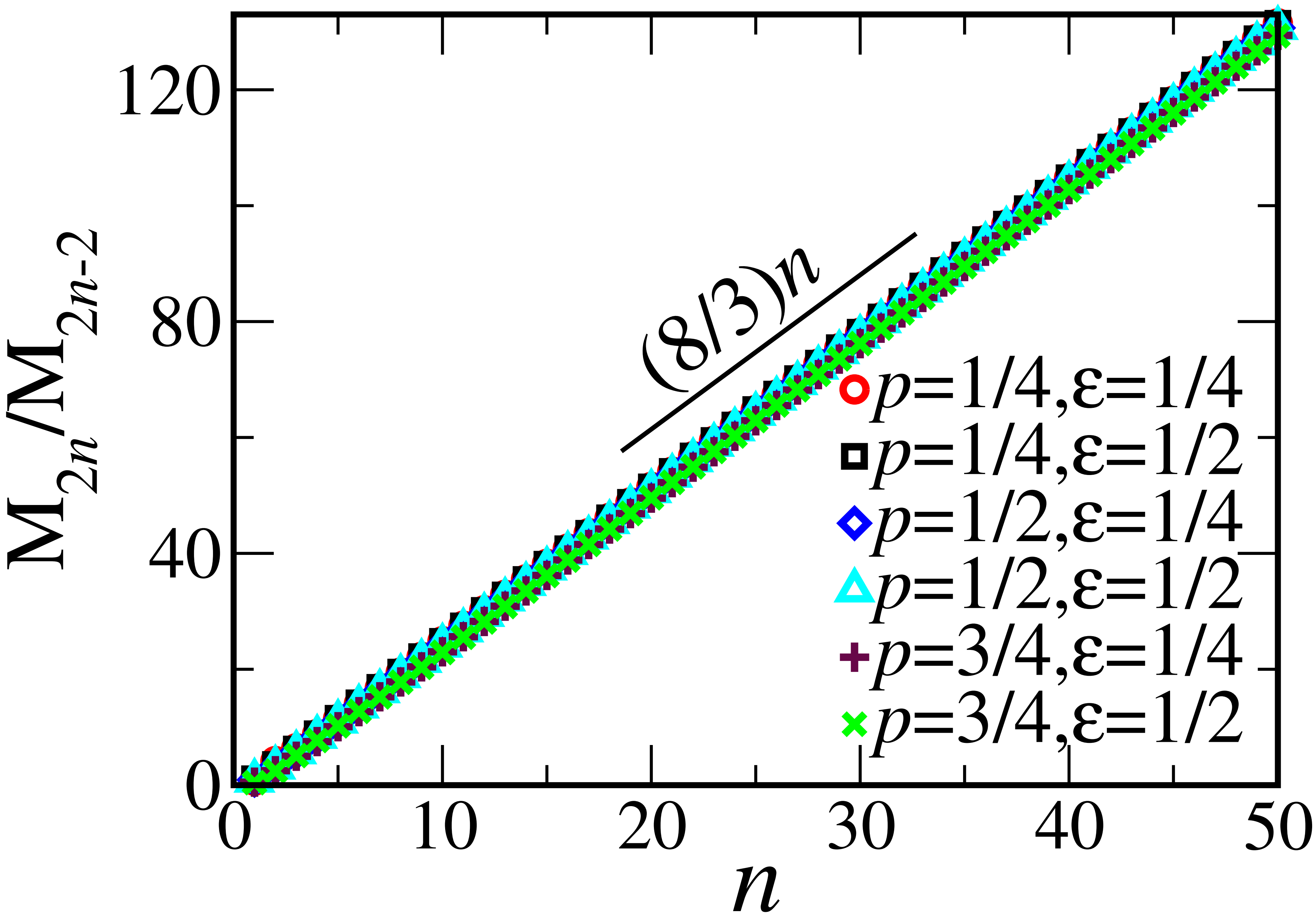
$M_{2n} = \langle v^{2n} \rangle$ satisfies the recursion relation

$$\Big[1-\epsilon^{2n} - (1-\epsilon)^{2n} + \gamma\big(1 - r_w^{2n}\big)\Big] M_{2n} =$$

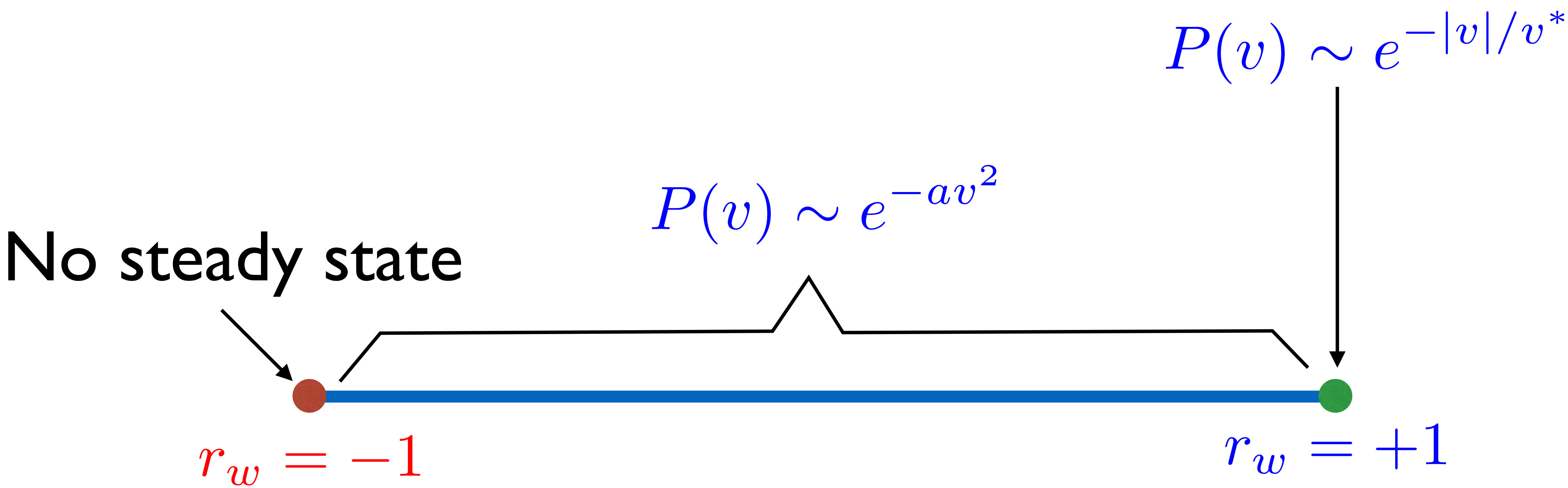
$$\sum_{m=1}^{n-1} \binom{2n}{2m} \epsilon^{2m} (1-\epsilon)^{2n-2m} M_{2m} M_{2n-2m}$$

$$+ \gamma \sum_{m=0}^{n-1} \binom{2n}{2m} r_w^{2m} M_{2m} \frac{(2n-2m)!}{(n-m)!} \left(\frac{\sigma^2}{2}\right)^{n-m}$$

Ratio of moments



$$v = -r_w v + \eta$$



Conclusion

- It is possible to write exact recursion relation for the time evolution of the second moment and velocity correlation together for the driven inelastic Maxwell model.
- This enables us to obtain the form of the high-energy tails and all moments (recursively) of the velocity distribution in the steady state in the thermodynamic limit.
- We do not require any approximations to break the BBGKY hierarchy, which is usually assumed in earlier works but is unnecessary.
- Results are also valid in the continuum time dynamics.