Distinct sites, common sites and maximal displacement of *N* random walkers

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GGI workshop
Florence

Joint work with

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- Gregory Schehr, LPTMS

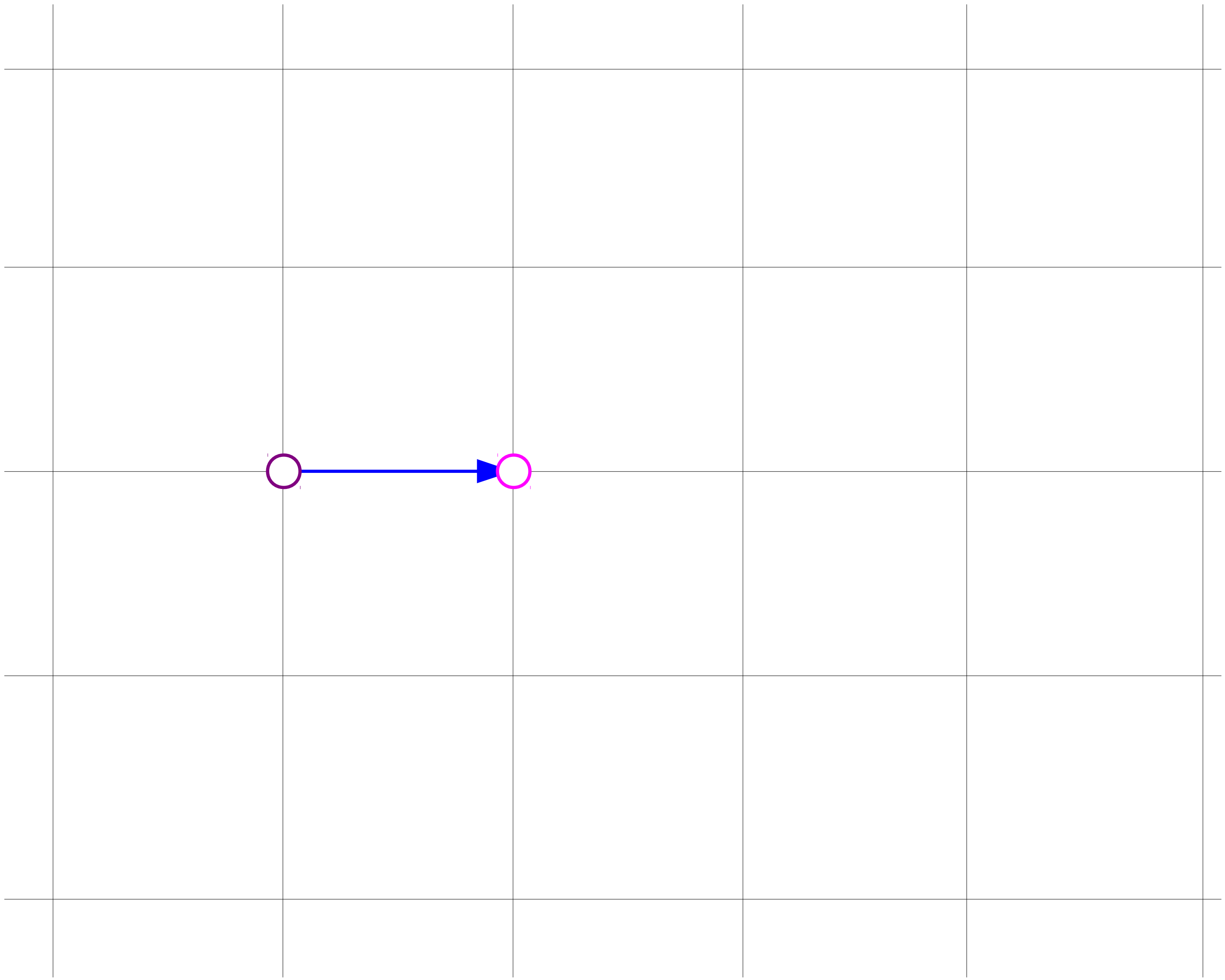
Outline

1. ***N*** **independent walkers**

2. ***N*** **vicious walkers**

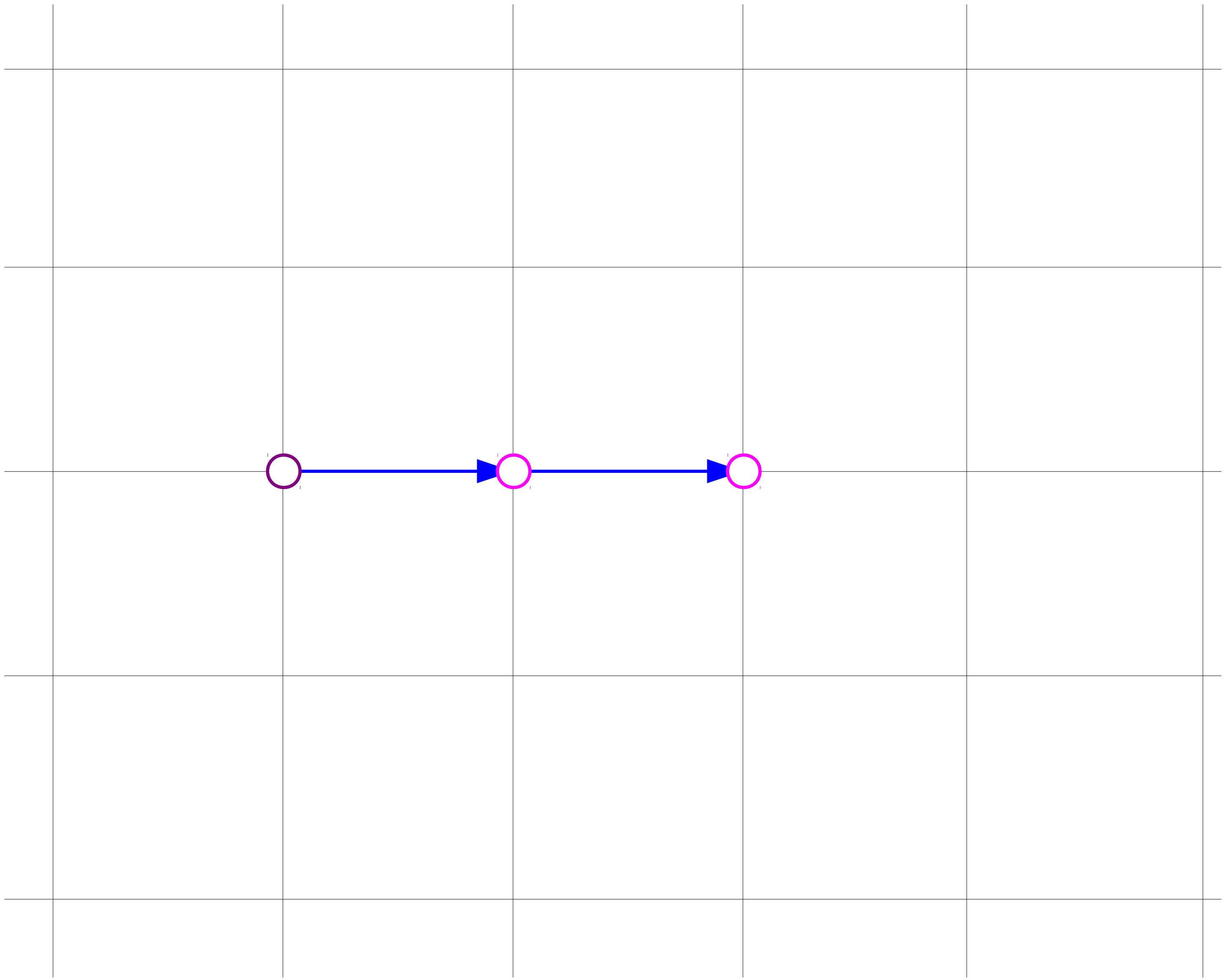
Distinct visited site

Distinct site :- Site visited by the walker



Distinct visited site

Distinct site :- Site visited by the walker



Distinct visited site

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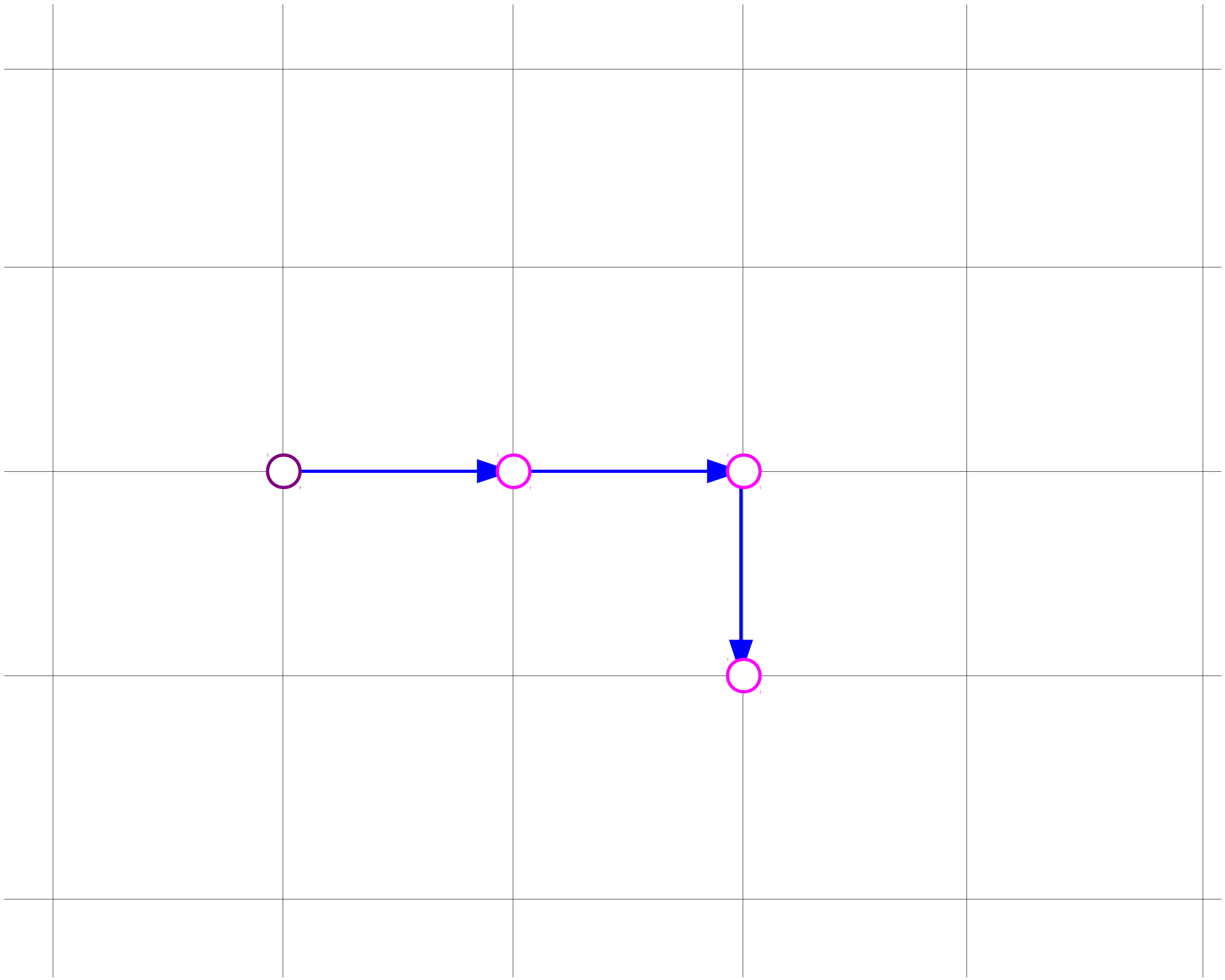
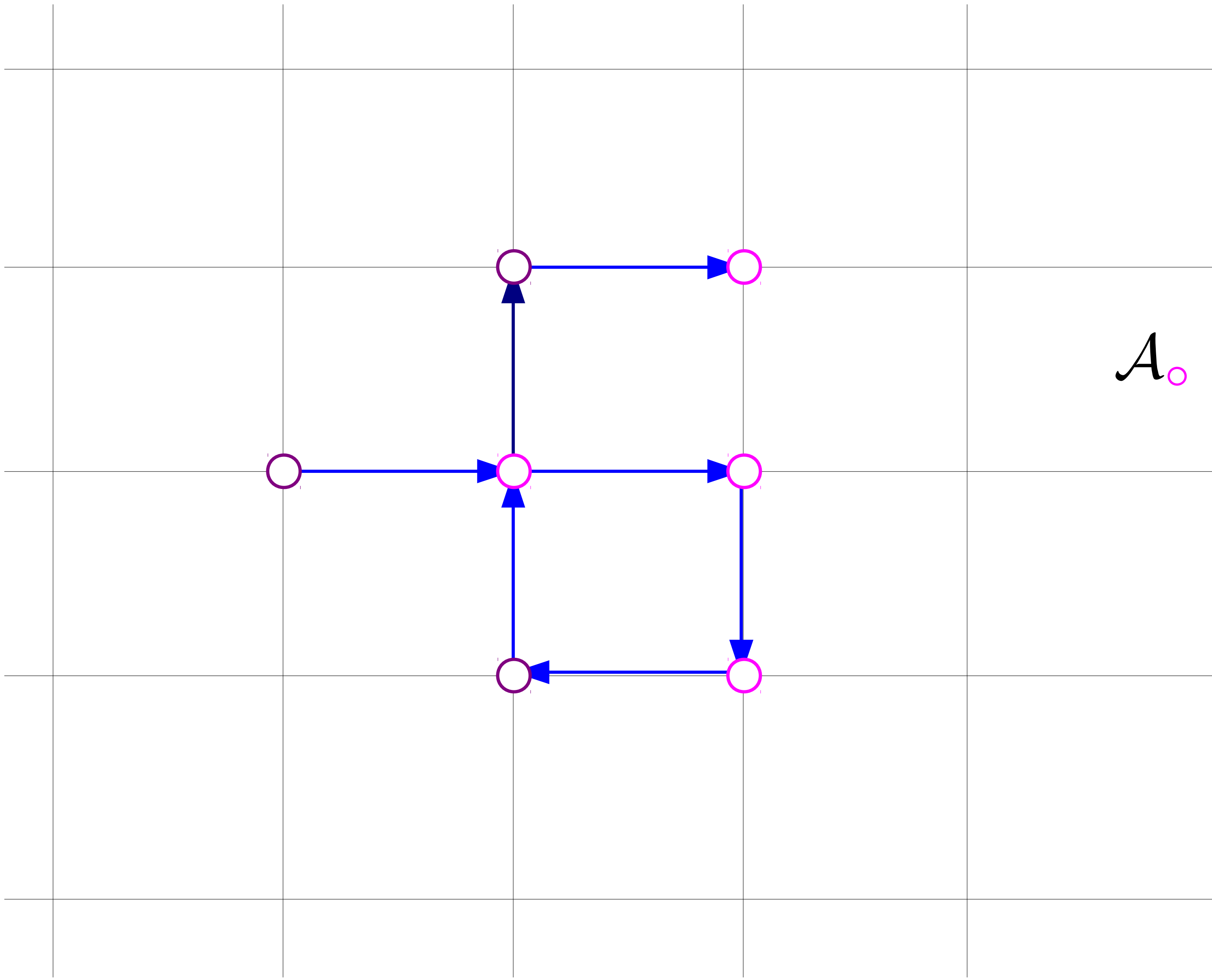


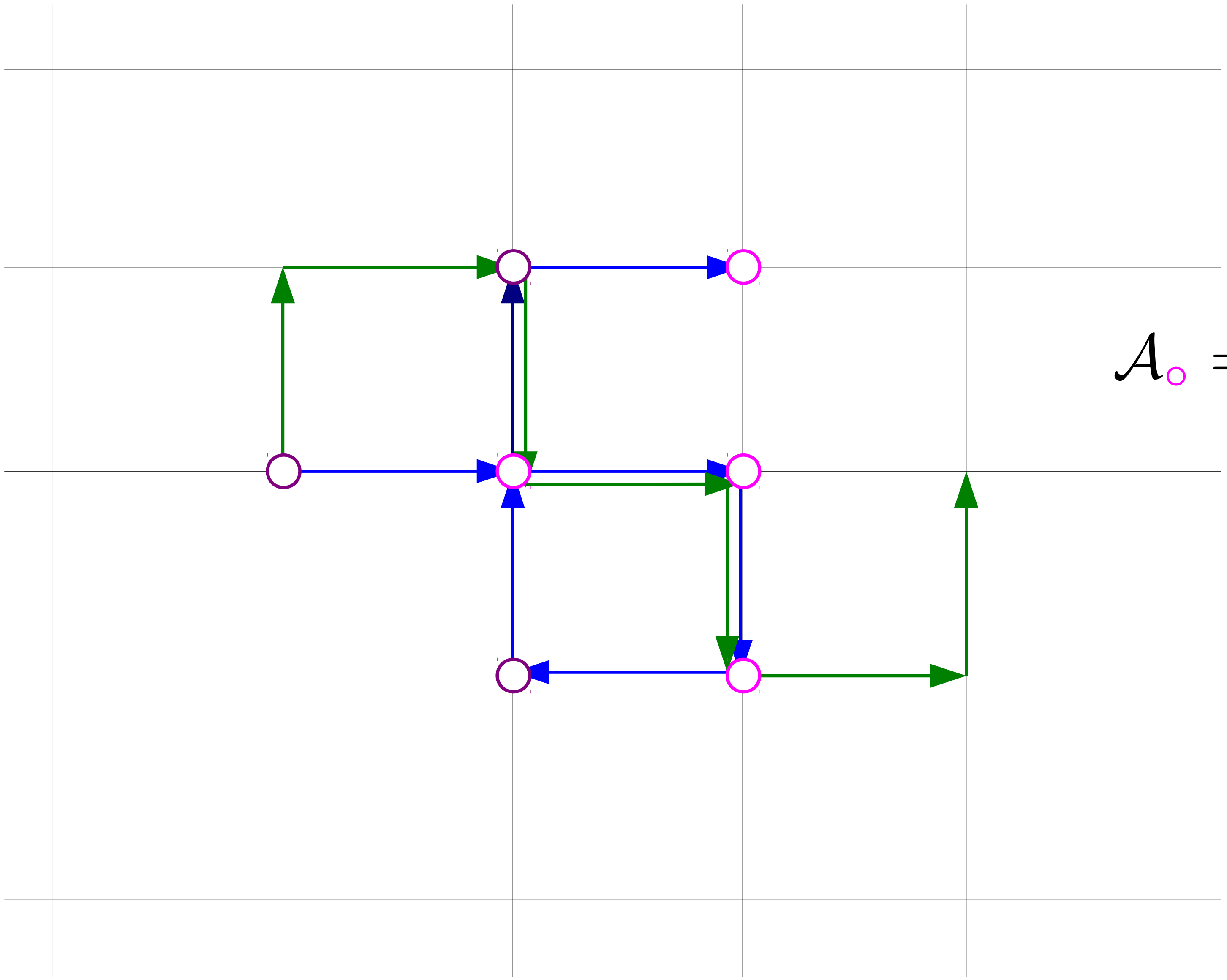
Distinct visited site

Distinct site :- Site visited by the walker



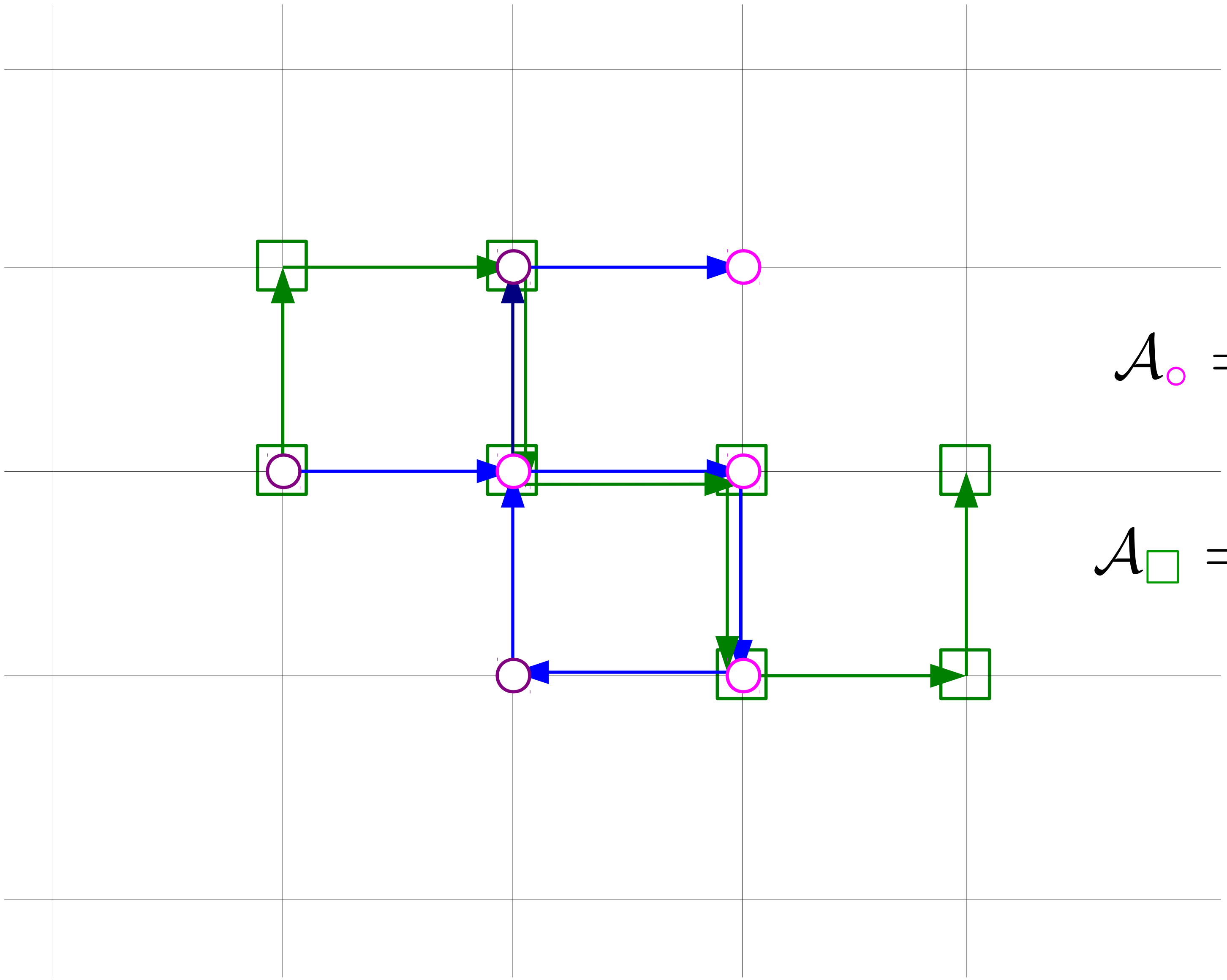
$\mathcal{A}_\circ = \{\circ\};$

Distinct visited site



$$\mathcal{A}_{\circ} = \{\circ\};$$

Distinct visited site



$$\mathcal{A}_{\circ} = \{\circ\};$$

$$\mathcal{A}_{\square} = \{\square\};$$

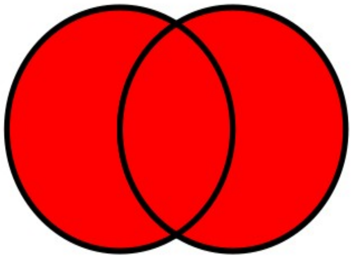
Distinct visited site

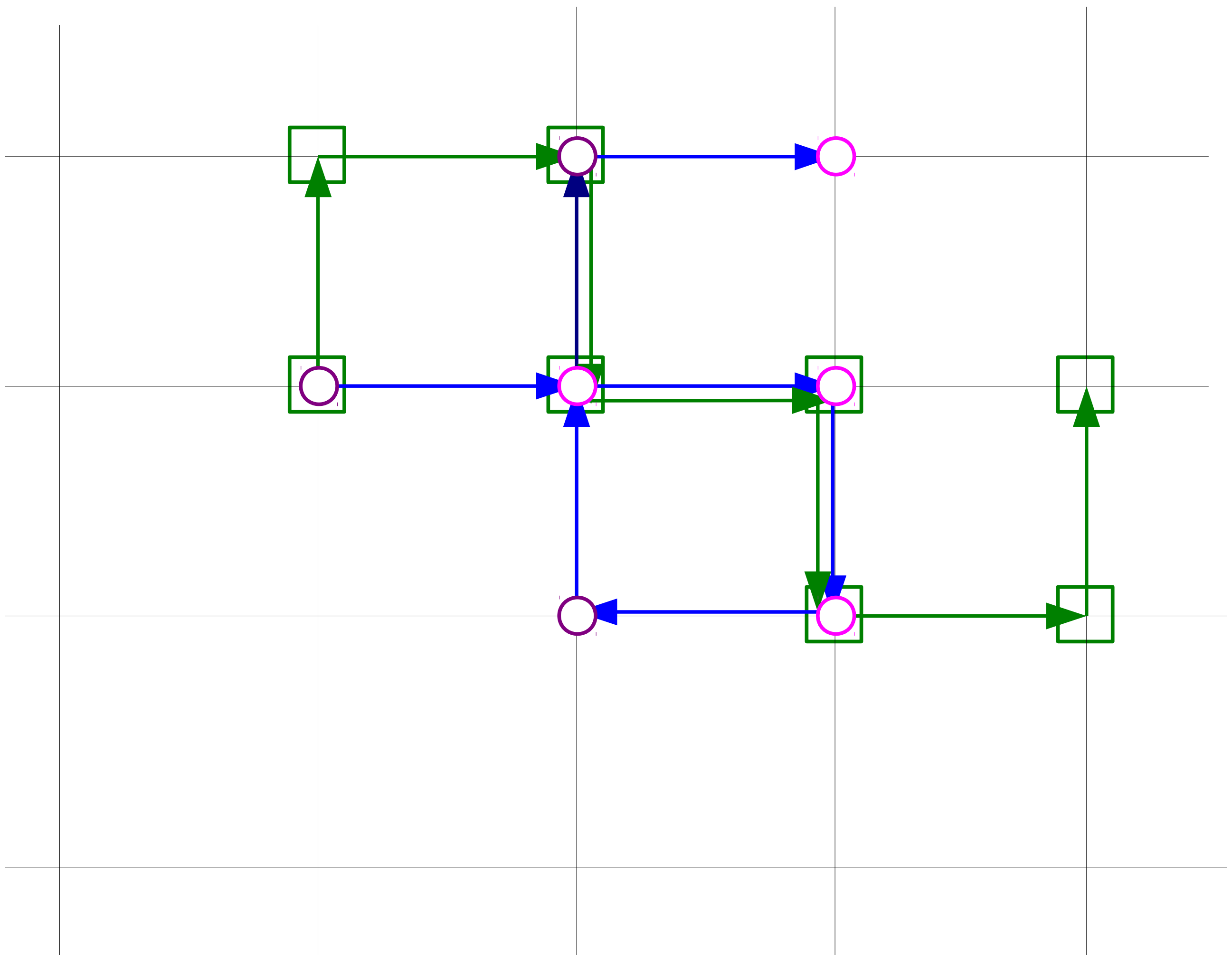
Distinct site :- Site visited by any walker

$\mathcal{A}_{\circ} = \{\circ\};$

$\mathcal{A}_{\square} = \{\square\};$

$\mathcal{U}_2 = \mathcal{A}_{\circ} \cup \mathcal{A}_{\square};$

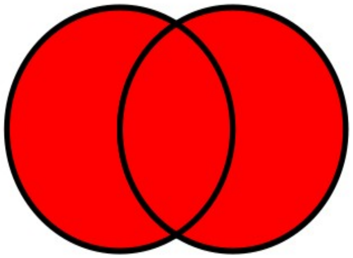


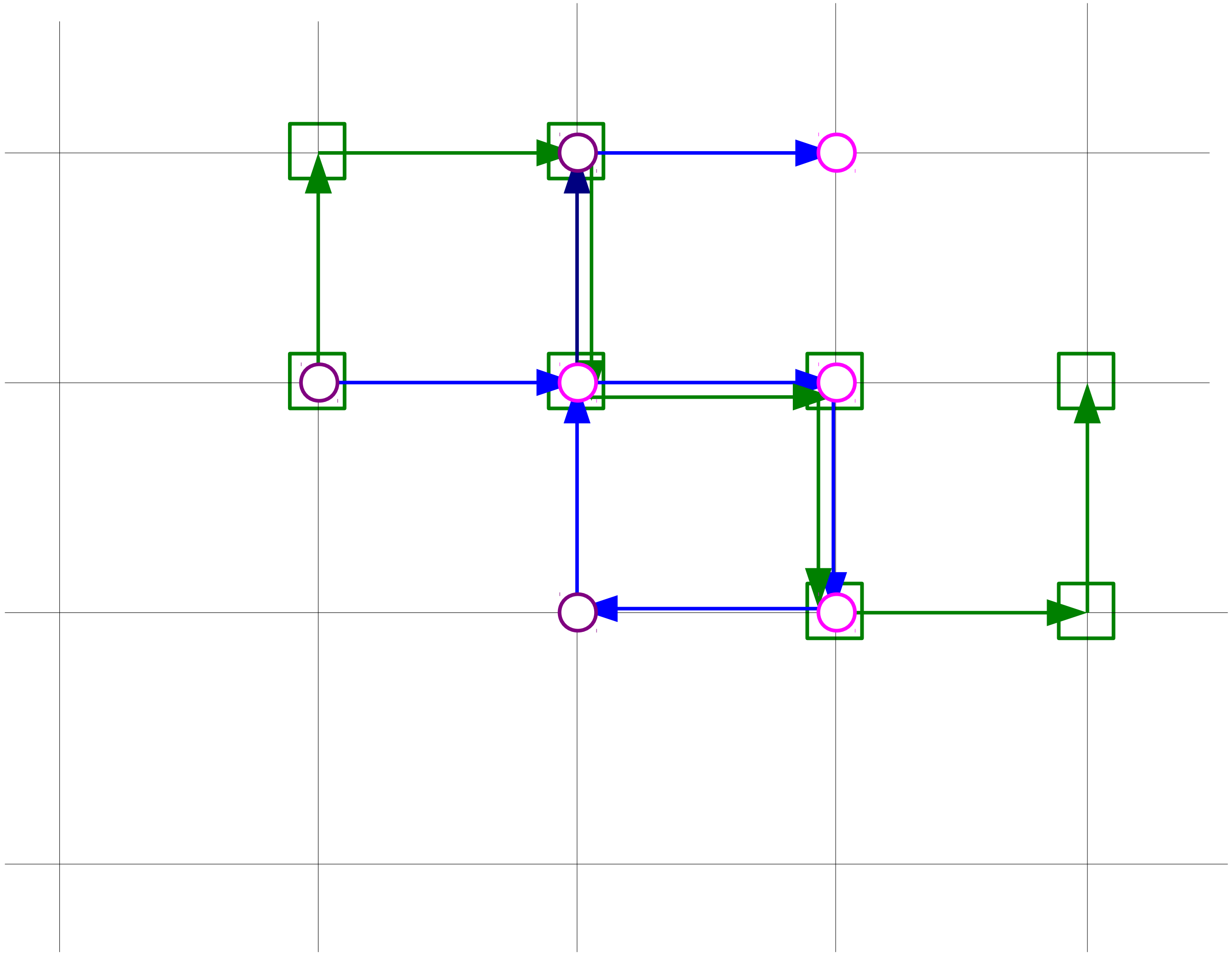


Distinct visited site

$$\mathcal{U}_2 = \mathcal{A}_{\circ} \cup \mathcal{A}_{\square};$$

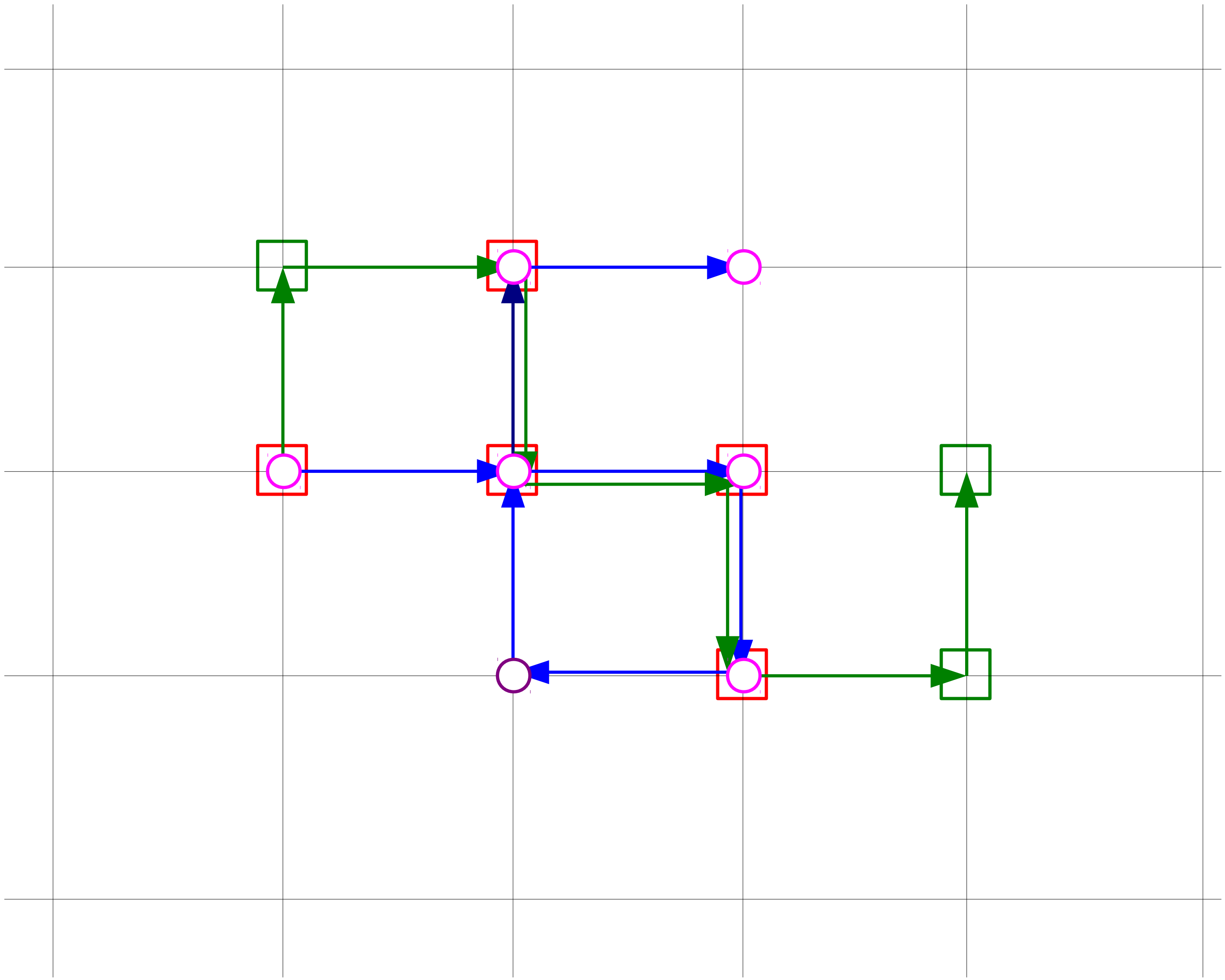
$$S_2 = \# \text{ of elements in the set } \mathcal{U}_2$$





Common visited site

Common site :- Site visited by all the walkers

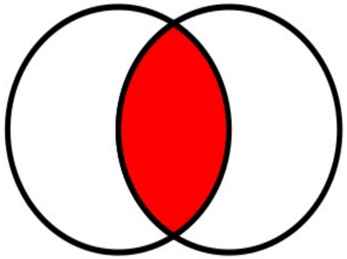


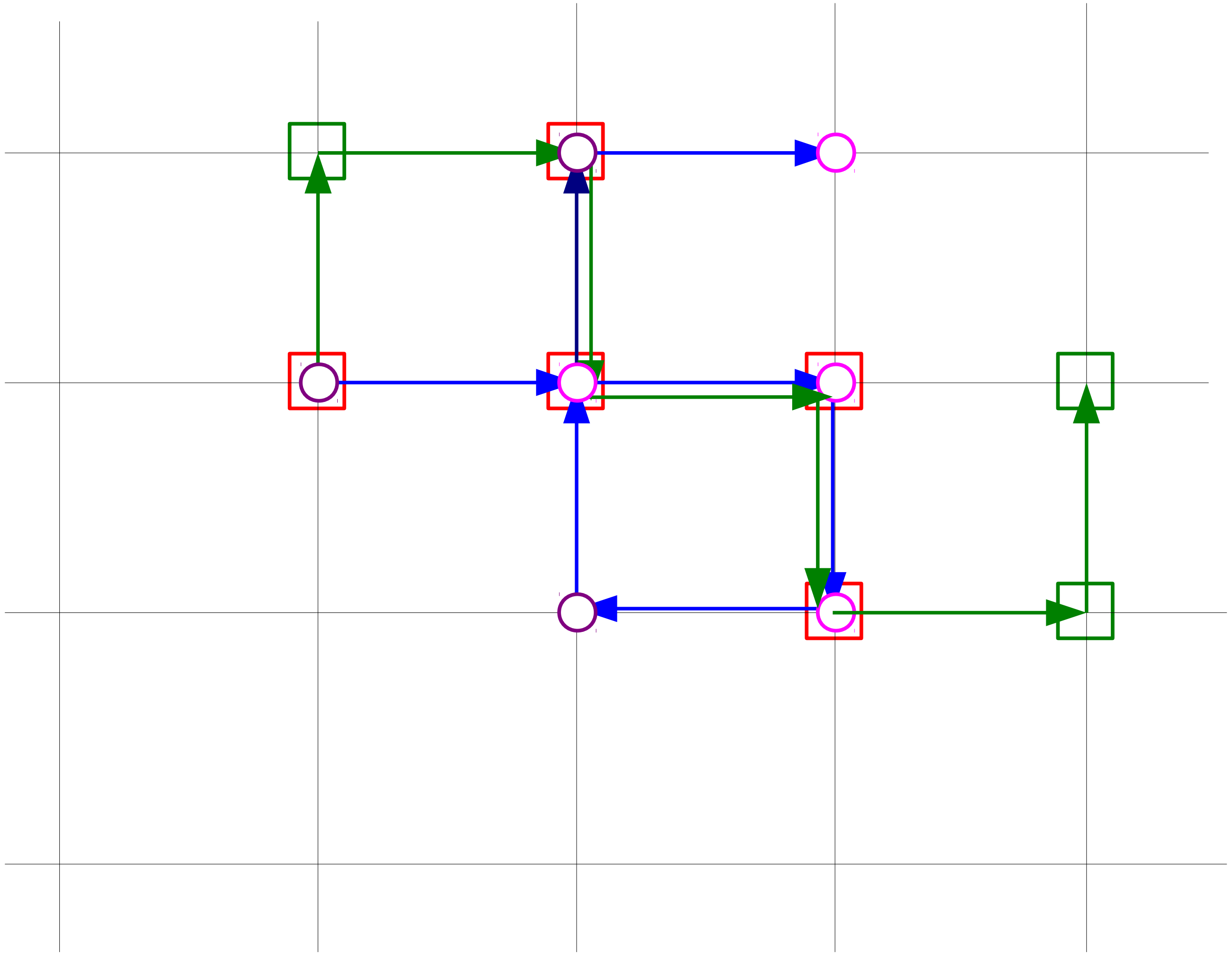
Common visited site

$$\mathcal{A}_{\circ} = \{\circ\}; \qquad \mathcal{I}_2 = \mathcal{A}_{\circ} \cap \mathcal{A}_{\square};$$

$$\mathcal{A}_{\square} = \{\square\};$$

$$W_2 = \#\ \text{of sites in the set}\ \mathcal{I}_2$$





Number of distinct & common sites

of distinct sites visited by N walkers in time step t = $\mathrm{S}_N(t)$

# of common sites visited by N walkers in time step t = $\mathrm{W}_N(t)$

$$\langle S_N(t)\rangle = \ ?$$

$$\langle W_N(t)\rangle = \ ?$$

Number of Distinct sites

- A. Dvoretzky and P. Erdos (1951) – for a single walker in d dimension.

$t \to \infty$

$$\langle S_1(t) \rangle \sim \begin{cases} \sqrt{t} & d = 1 \\ \frac{t}{\log(t)} & d = 2 \\ t & d > 2 \end{cases}$$

B. H. Hughes

- Later studied by Vineyard, Montroll, Weiss ….

Number of Distinct sites

- Larralde et al. : N independent random walkers in d dimension

Nature, 355, 423 (1992)

Territory covered by *N* diffusing particles

Hernan Larralde*, Paul Trunfio*, Shlomo Havlin*†, H. Eugene Stanley* & George H. Weiss†

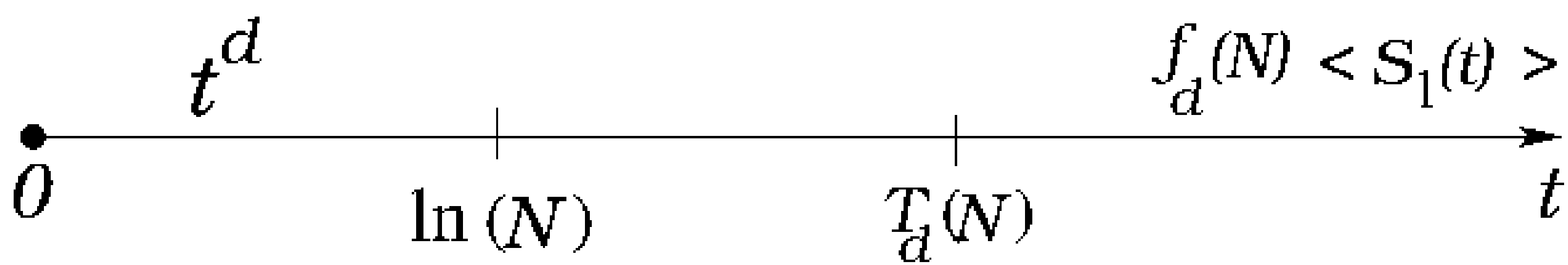
* Center for Polymer Studies and Department of Physics, Boston University, Boston, Massachusetts 02215, USA
† Physical Sciences Laboratory, Division of Computer Research and Technology, National Institutes of Health, Bethesda, Maryland 20892, USA

Number of Distinct sites

- Larralde et al. : N independent random walkers in d dimension

Nature, 355, 423 (1992)

- Three different growths of $\langle S_N(t) \rangle$ separated by two time scales



Number of Distinct sites

- Larralde et al. : N independent random walkers in d dimension

Nature, 355, 423 (1992)

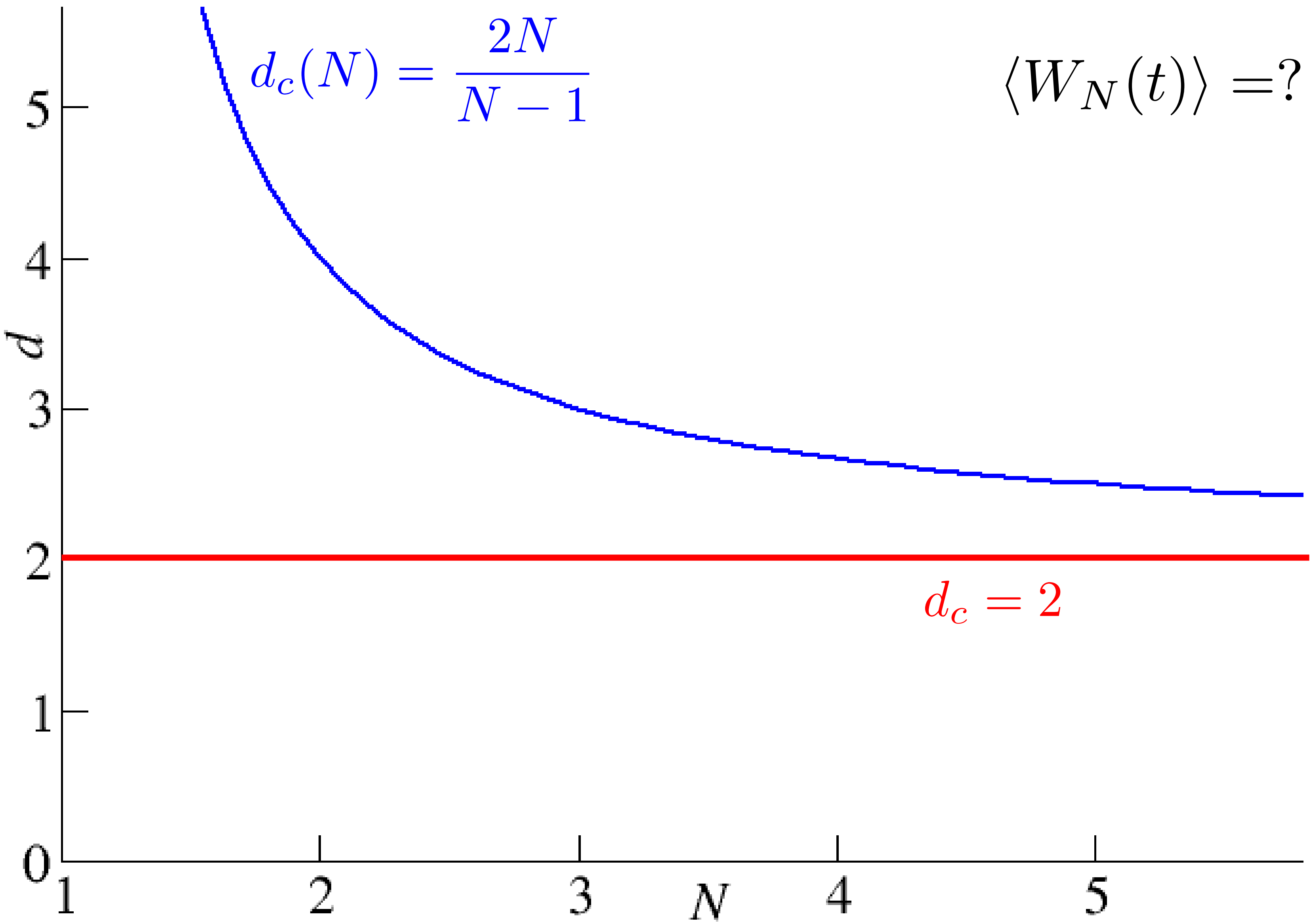
- Three different growths of $\langle S_N(t)\rangle$ separated by two time scales

$$\langle S_N(t)\rangle \sim \begin{cases} \sqrt{\log(N)}\sqrt{t} & d=1 \\ N\ \frac{t}{\log(t)} & d=2 \\ N\ t & d>2 \end{cases}$$

$t \to \infty$

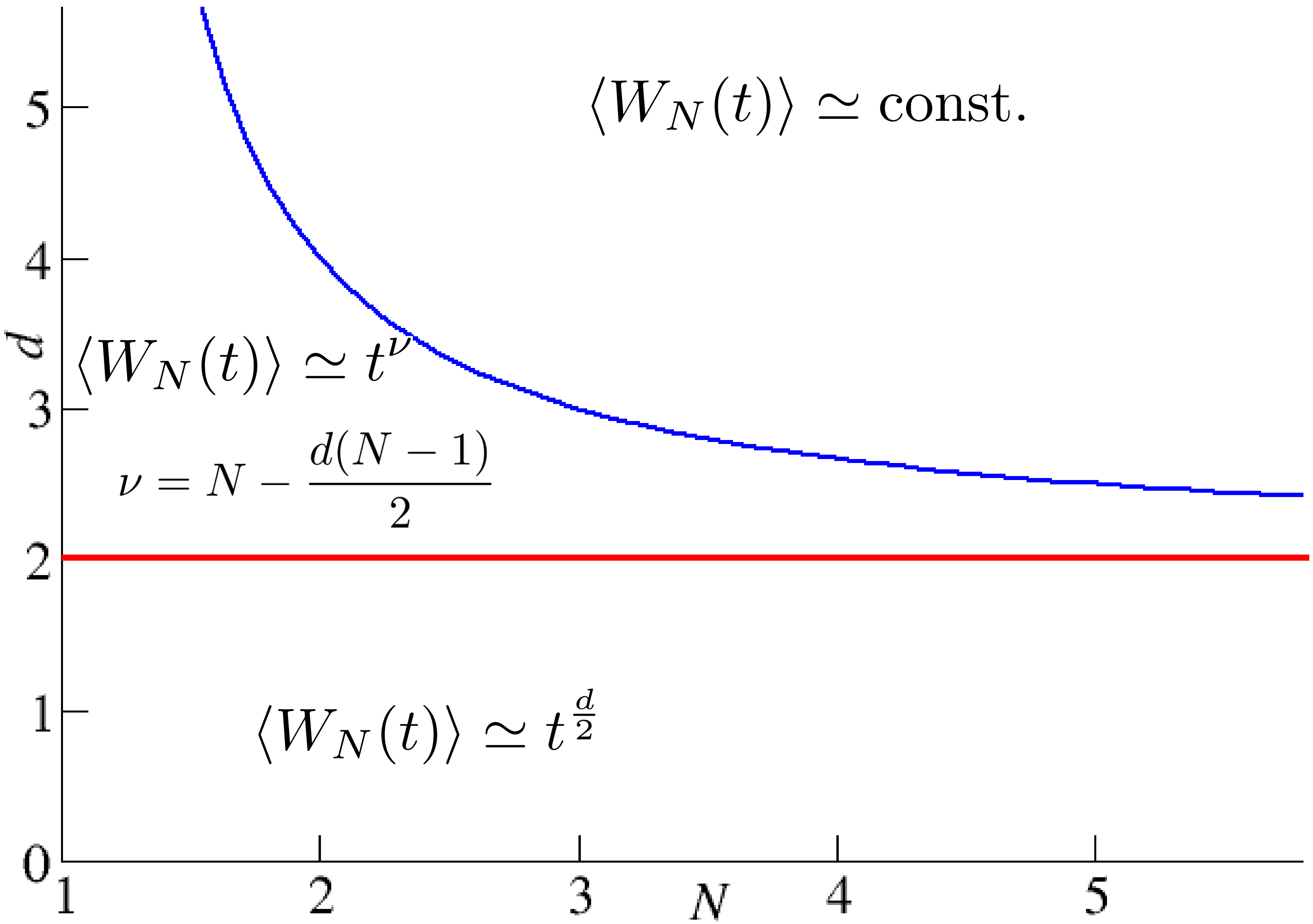
Number of Common sites

Majumdar and Tamm - *Phys. Rev. E 86, 021135,* (2012)



Number of Common sites

$d_c(N) = \frac{2N}{N-1}$ $\quad d_c = 2$



Majumdar and Tamm - *PRE* (2012)

Number of Common sites

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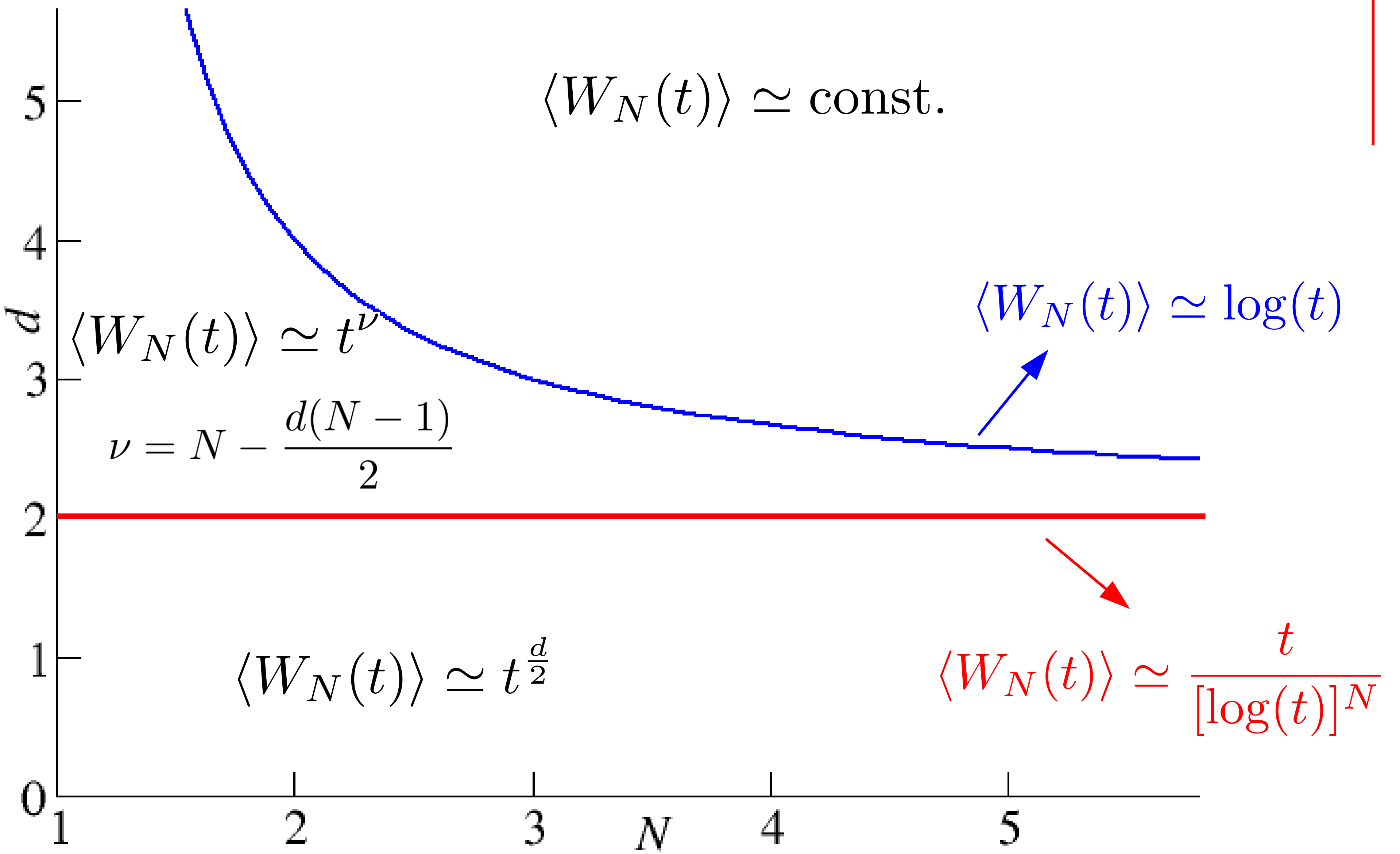

Majumdar and Tamm - *PRE* (2012)

Number of Common and distinct sites

$$t \to \infty$$

$$\langle W_N(t)\rangle \sim \begin{cases} t^{\frac{d}{2}} & d < 2 \\ \frac{t}{[\log(t)]^N} & d = 2 \\ t^{\nu} & 2 < d < d_c(N) \\ \log(t) & d = d_c(N) \\ \text{const.} & d > d_c(N) \end{cases}$$

$$\mathrm{d}_c(N) = \frac{2N}{N-1}$$

$$\nu = N - \frac{d(N-1)}{2}$$

$$\langle S_N(t)\rangle \sim \begin{cases} \sqrt{\log(N)}\sqrt{t} & d = 1 \\ N\,\frac{t}{\log(t)} & d = 2 \\ N\,t & d > 2 \end{cases}$$

Probability Distributions

- $\mathrm{P}_N(S,t)$ = Distribution function of the number of distinct sites visited by N walkers in time step t
- $\mathrm{Q}_N(W,t)$ = Distribution function of the number of common sites visited by N walkers in time step t

Probability Distributions

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Applications :

- Territory of animal population of size N
- Popular tourist place visited by all the tourists in a city
- Diffusion of proteins along DNA
- Annealing of defects in crystal
- Popular “hub” sites in a multiple user network

Probability Distributions

- $P_N(S,t)$ = Distribution function of the number of distinct sites visited by N walkers in time step t
- $Q_N(W,t)$ = Distribution function of the number of common sites visited by N walkers in time step t

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- One dimension $\langle S_N(t)\rangle = N\langle S_1(t)\rangle \ ; \ d > 1$
- Maximum overlap
- Connection with extreme value statistics : exactly solvable
- $S_N(t) =$ Total # of distinct sites = range or span
- $W_N(t) =$ # of common sites = common range or common span

Model

- N one dimensional t-step Brownian walkers

- Each of them starts at the origin and have diffusion constants D

$$
\begin{aligned}
\frac{dx_i}{d\tau} &= \eta_i(\tau),\ \forall i = 1 \ldots N \\
\langle \eta_i(\tau) \rangle &= 0,\ \langle \eta_i(\tau)\eta_j(\tau') \rangle = 2D\delta_{ij}\delta(\tau - \tau')
\end{aligned}
$$

Scaling

- All displacements are scaled by $\sqrt{4Dt}$

$$s = \frac{S_N}{\sqrt{4Dt}}, w = \frac{W_N}{\sqrt{4Dt}}$$

- Probability distributions take following scaling forms :

$$\begin{aligned} P_N(S,t) &= \frac{1}{\sqrt{4Dt}} p_N\left(\frac{S}{\sqrt{4Dt}}\right) \\ Q_N(W,t) &= \frac{1}{\sqrt{4Dt}} q_N\left(\frac{W}{\sqrt{4Dt}}\right) \end{aligned}$$

Scaling

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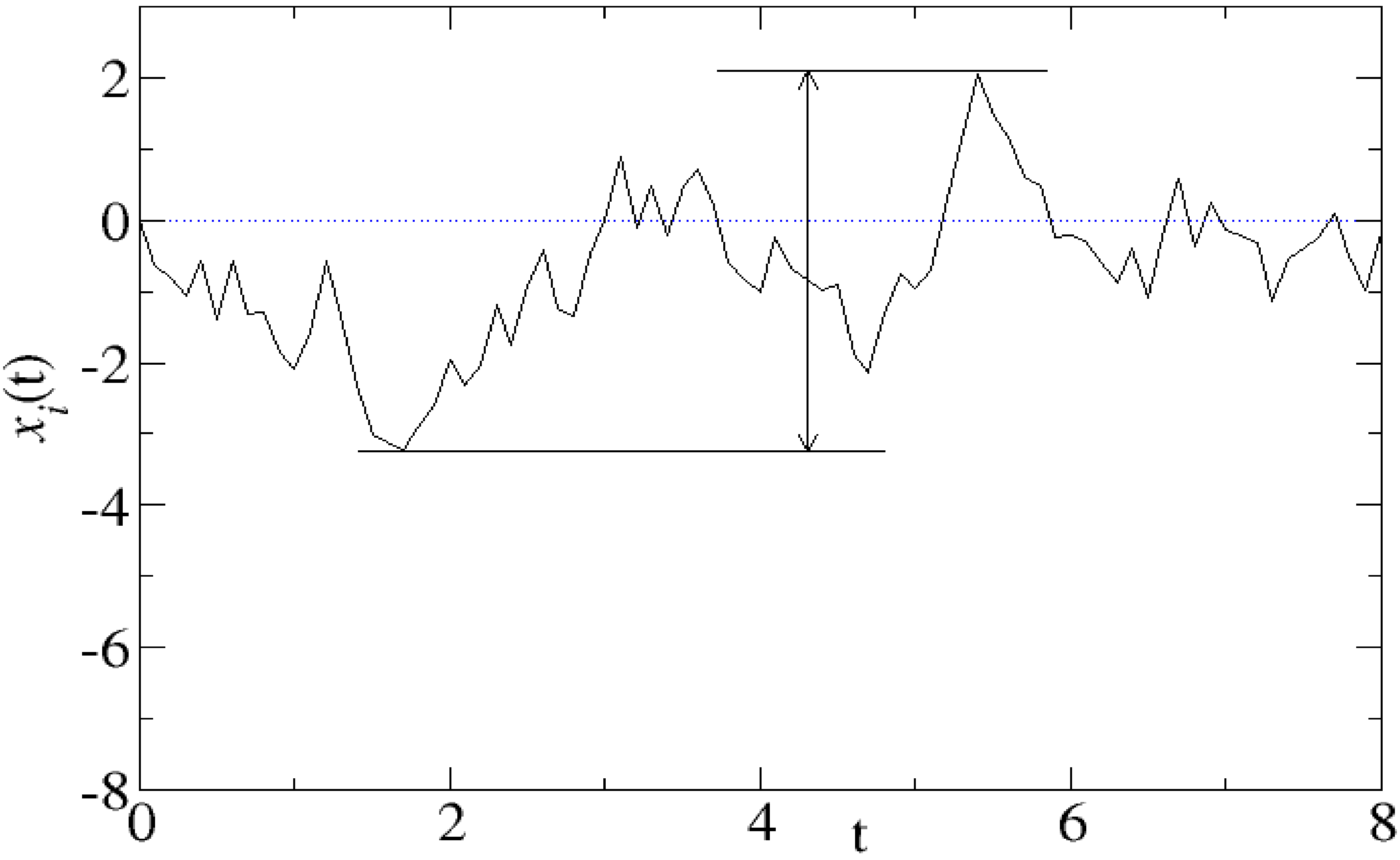
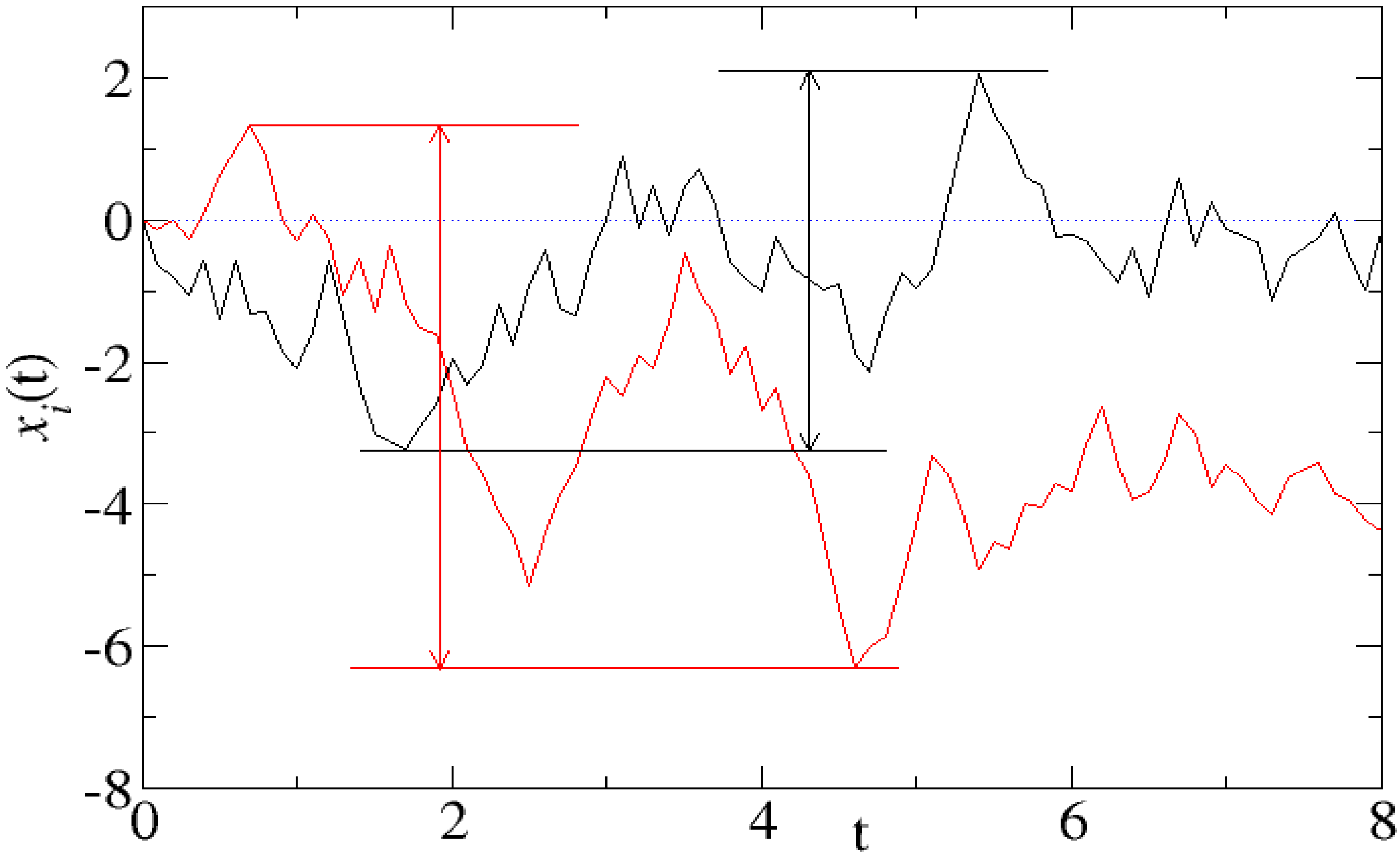
Range: Single particle

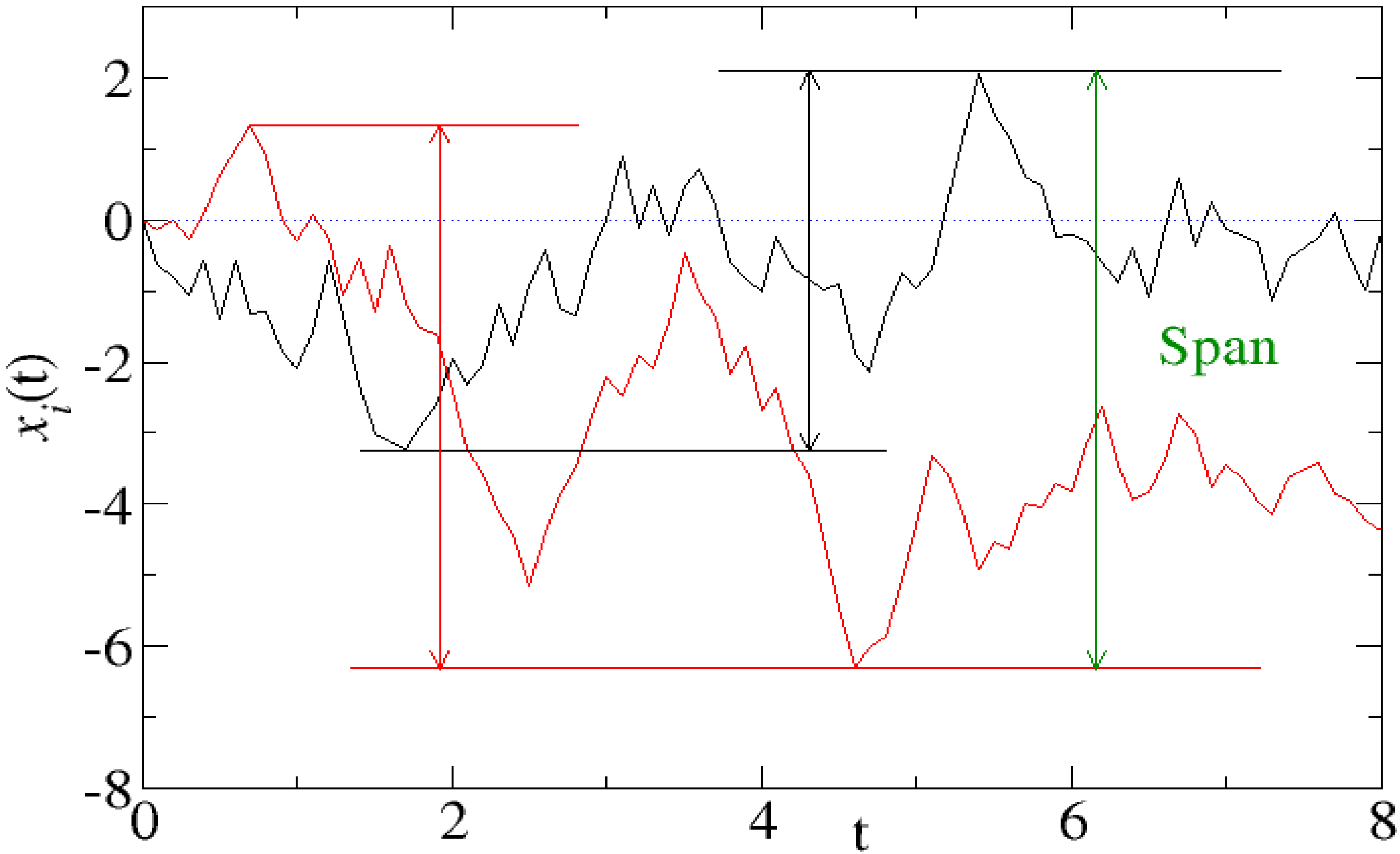


Range: Many particles



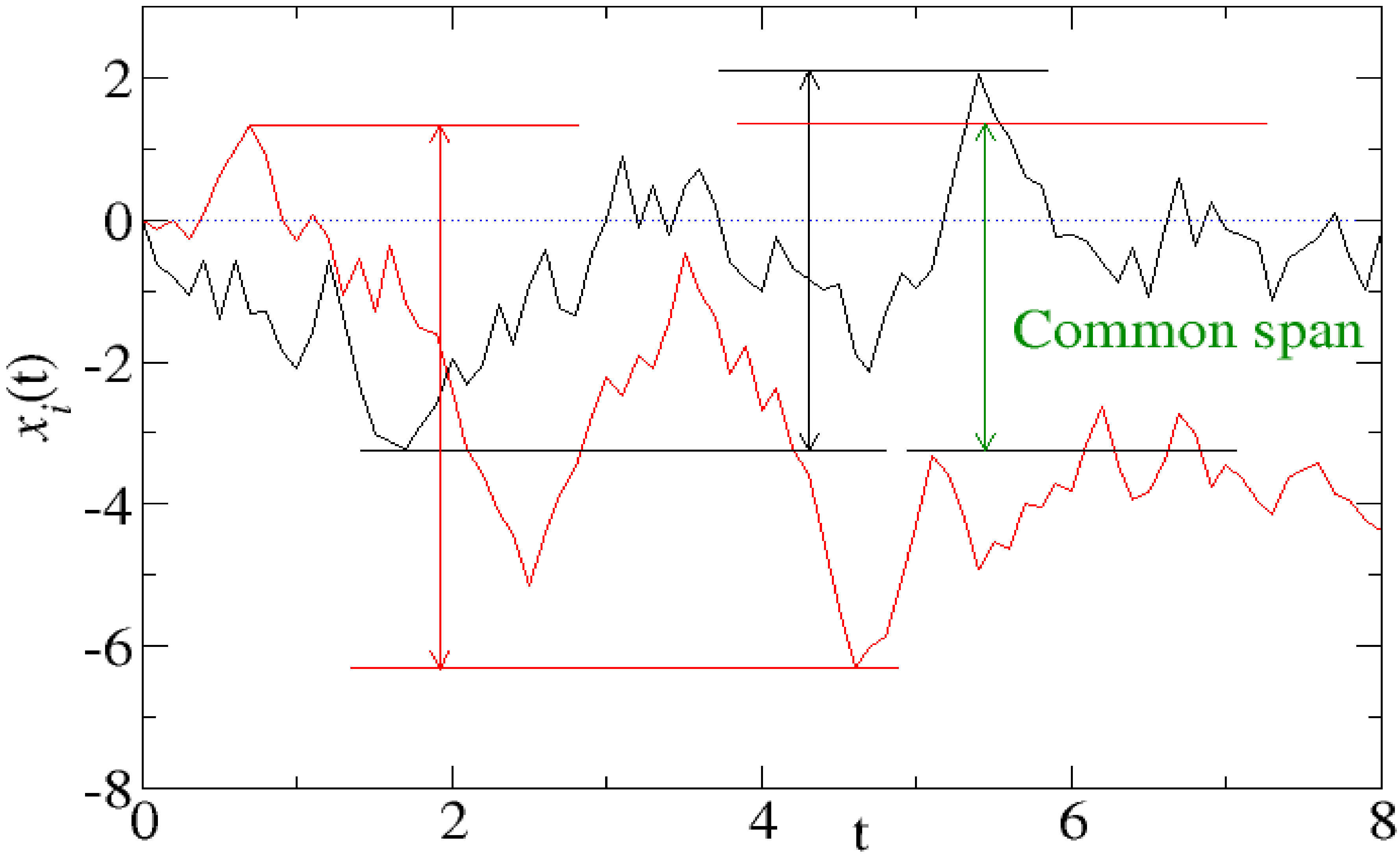
Span

Union $\Rightarrow$ Span

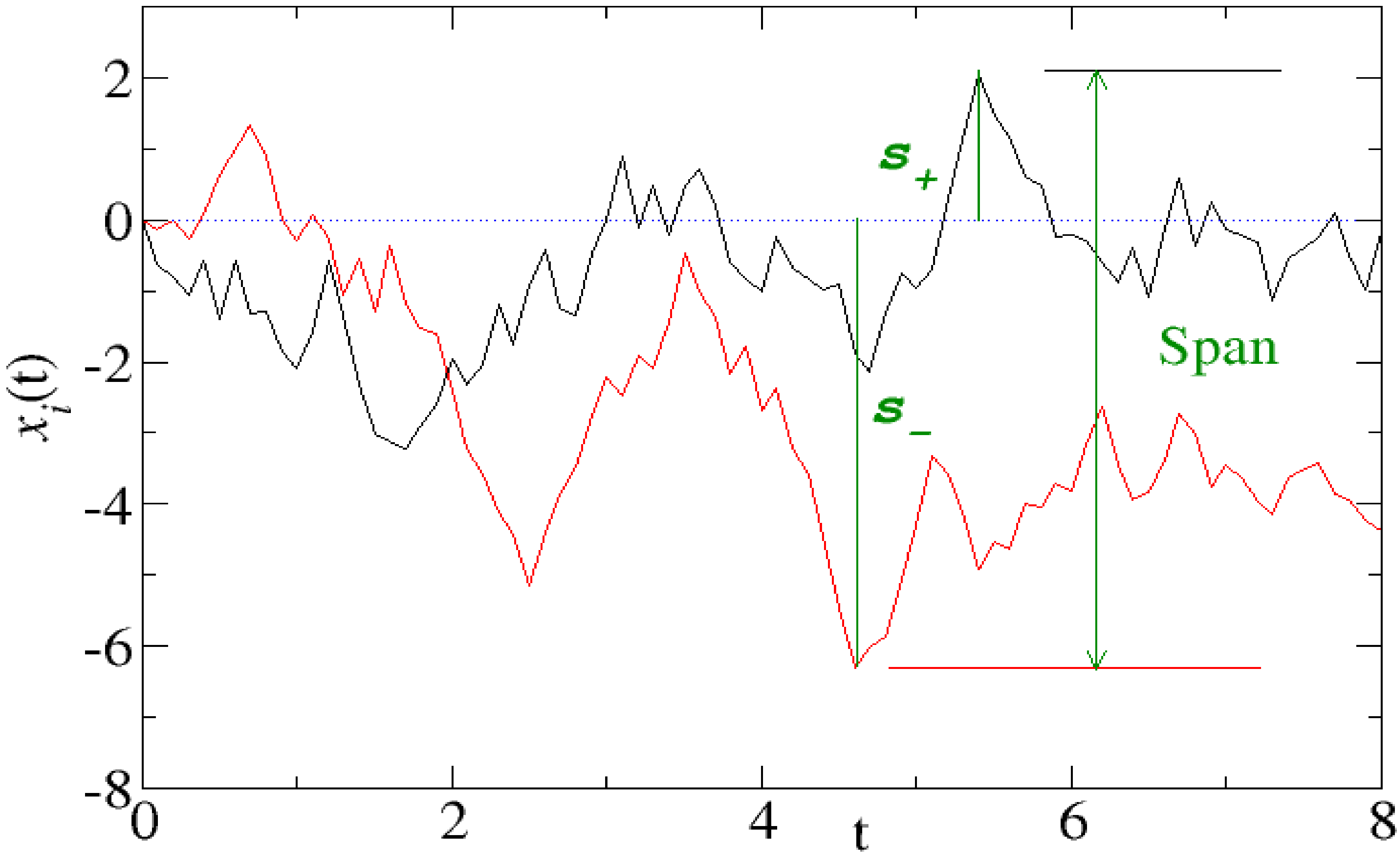


Common span

Intersection ⟹ Common span

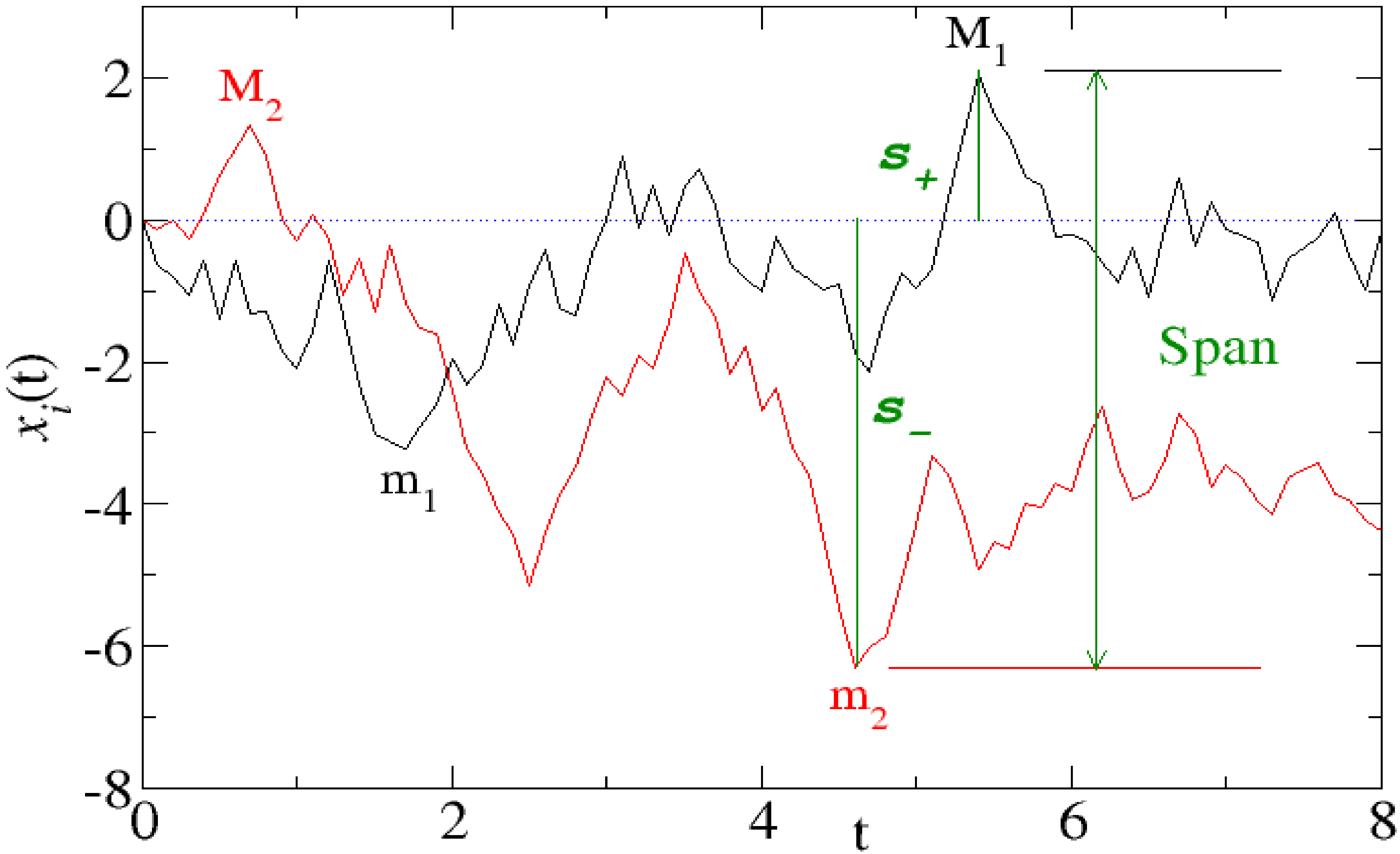


$\text{Span} = s = s_+ + s_-$



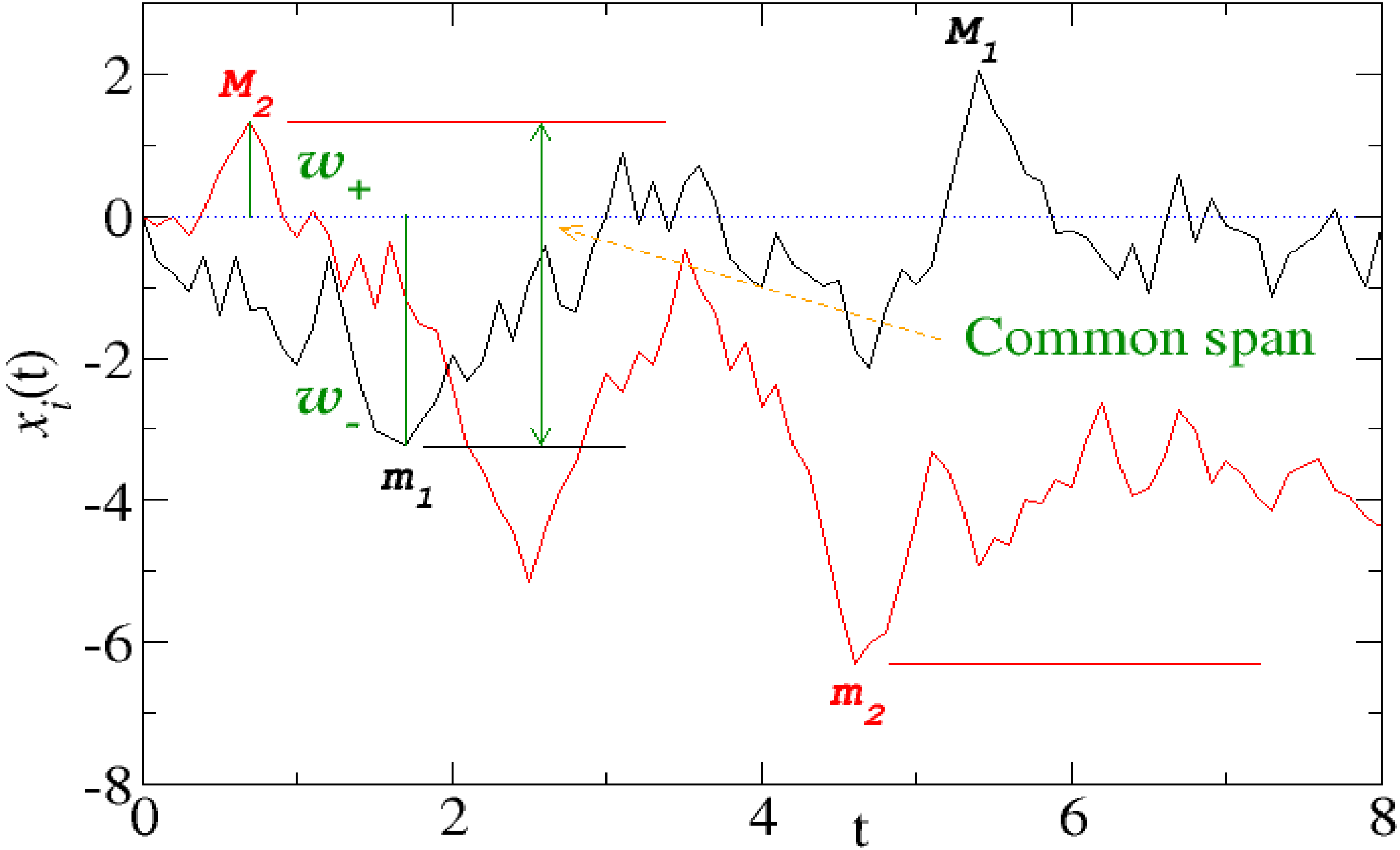
Span $= \; s = s_+ + s_-$

$s_+ = \max\left(M_1,\ M_2\right)$ $\qquad\qquad$ $s_- = \max\left(-m_1,\ -m_2\right)$



Common span = $w = w_+ + w_-$

$w_+ = \min(M_1, \; M_2)$ $\qquad$ $w_- = \min(-m_1, \; -m_2)$



Connection with extreme values

$$s_+ = \max\left(M_1, M_2\right)$$

$$s = s_+ + s_-$$

$$s_- = \max\left(-m_1, -m_2\right)$$

$$w_+ = \min\left(M_1, M_2\right)$$

$$w = w_+ + w_-$$

$$w_- = \min\left(-m_1, -m_2\right)$$

Connection with extreme values

$$s_+ = \max\left(M_1, M_2, \ldots, M_N\right)$$

$$s = s_+ + s_-$$

$$s_- = \max\left(-m_1, -m_2, \ldots, -m_N\right)$$

$$w_+ = \min\left(M_1, M_2, \ldots, M_N\right)$$

$$w = w_+ + w_-$$

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Connection with extreme values

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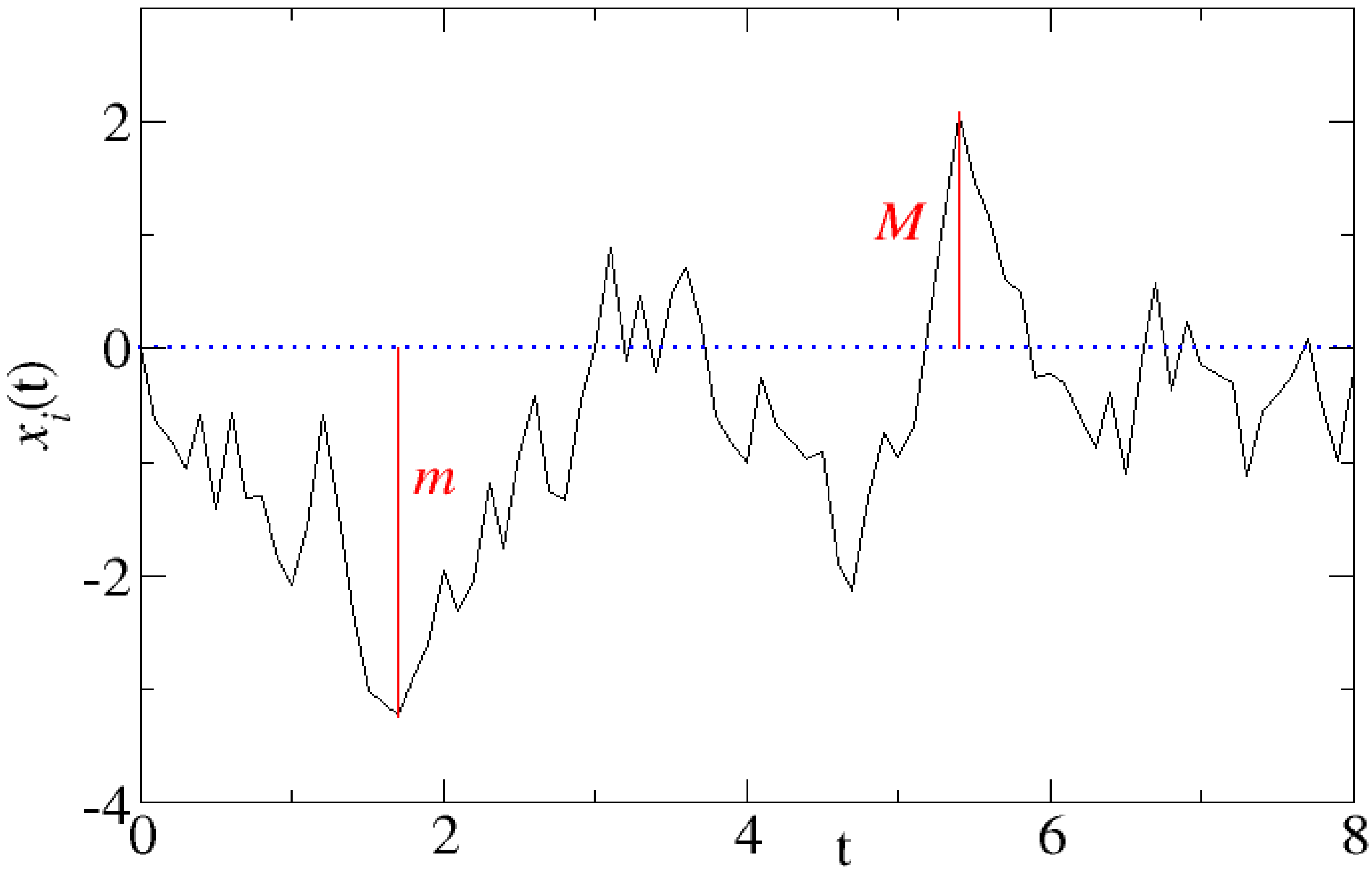
$$w_+ = \min\left(M_1, M_2, \ldots, M_N\right)$$

$$w_- = \min\left(-m_1, -m_2, \ldots, -m_N\right)$$

$$w = w_+ + w_-$$

- The variables s_+ & s_- are correlated random variables
- Similarly the variables w_+ & w_- are also correlated random variables
- We need joint probability distributions

Single particle

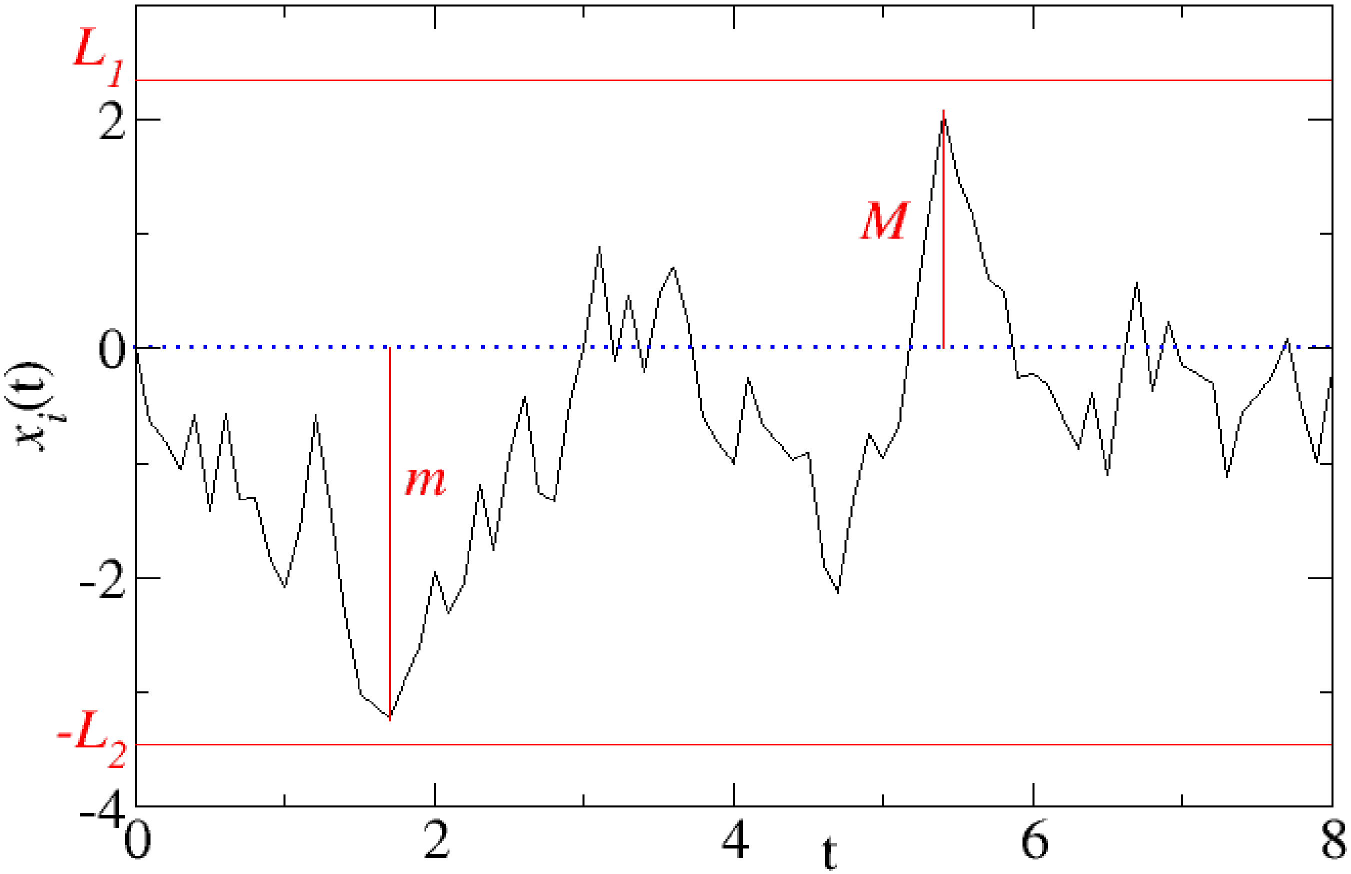


$$\text{Span} = S_1 = M + m$$

M, m are correlated random variables

Particle inside the box

$$\mathrm{Prob}\,(\mathrm{M} \le L_1, m \le L_2) \Rightarrow g\left(l_1 = \frac{L_1}{\sqrt{4Dt}},\ l_2 = \frac{L_2}{\sqrt{4Dt}}\right)$$



$$g(l_1, l_2) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n + \frac{1}{2}} \sin\left(\frac{(2\,n + 1)\pi l_2}{l_1 + l_2}\right) e^{-\left(\frac{(n+\frac{1}{2})\pi}{l_1+l_2}\right)^2} .$$

Distribution of the span : N=1

$$\mathrm{Prob}\,(\mathrm{M} \le L_1, m \le L_2) \Rightarrow g\left(l_1 = \frac{L_1}{\sqrt{4Dt}},\ l_2 = \frac{L_2}{\sqrt{4Dt}}\right)$$

$$g(l_1, l_2) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \sin\left(\frac{(2\,n+1)\pi l_2}{l_1+l_2}\right) e^{-\left(\frac{(n+\frac{1}{2})\pi}{l_1+l_2}\right)^2} .$$

$$p_1(s) = \int_0^{\infty} dl_1 \int_0^{\infty} dl_2\ \delta(s - l_1 - l_2) \frac{\partial^2 g(l_1, l_2)}{\partial l_1 \partial l_2}$$

$$p_1(s) \quad = \quad \frac{8}{\sqrt{\pi}} \sum_{m=1}^{\infty} (-1)^{m+1} m^2\ e^{-m^2 s^2}$$

Span for $N > 2$

$$s_+ = \max\left(M_1, M_2, \ldots, M_N\right)$$

$$s = s_+ + s_-$$

$$s_- = \max\left(-m_1, -m_2, \ldots, -m_N\right)$$

$$\mathbf{P}_d\left(l_1, l_2\right) = \mathrm{Prob}\left(s_+ \leq l_1, s_- \leq l_2\right) = \left[g(l_1, l_2)\right]^N$$

Span for $N > 2$

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$$\mathcal{P}_d\left(s_+ = l_1, s_- = l_2\right) = \frac{\partial^2 \; \left[g(l_1, l_2)\right]^N}{\partial l_1, \partial l_2}$$

$$\boxed{p_N(s) = \int_0^\infty dl_1 \int_0^\infty dl_2 \; \delta(s - l_1 - l_2) \left[\frac{\partial^2 \; \left[g(l_1, l_2)\right]^N}{\partial l_1, \partial l_2}\right]}$$

Common Span for $N > 2$

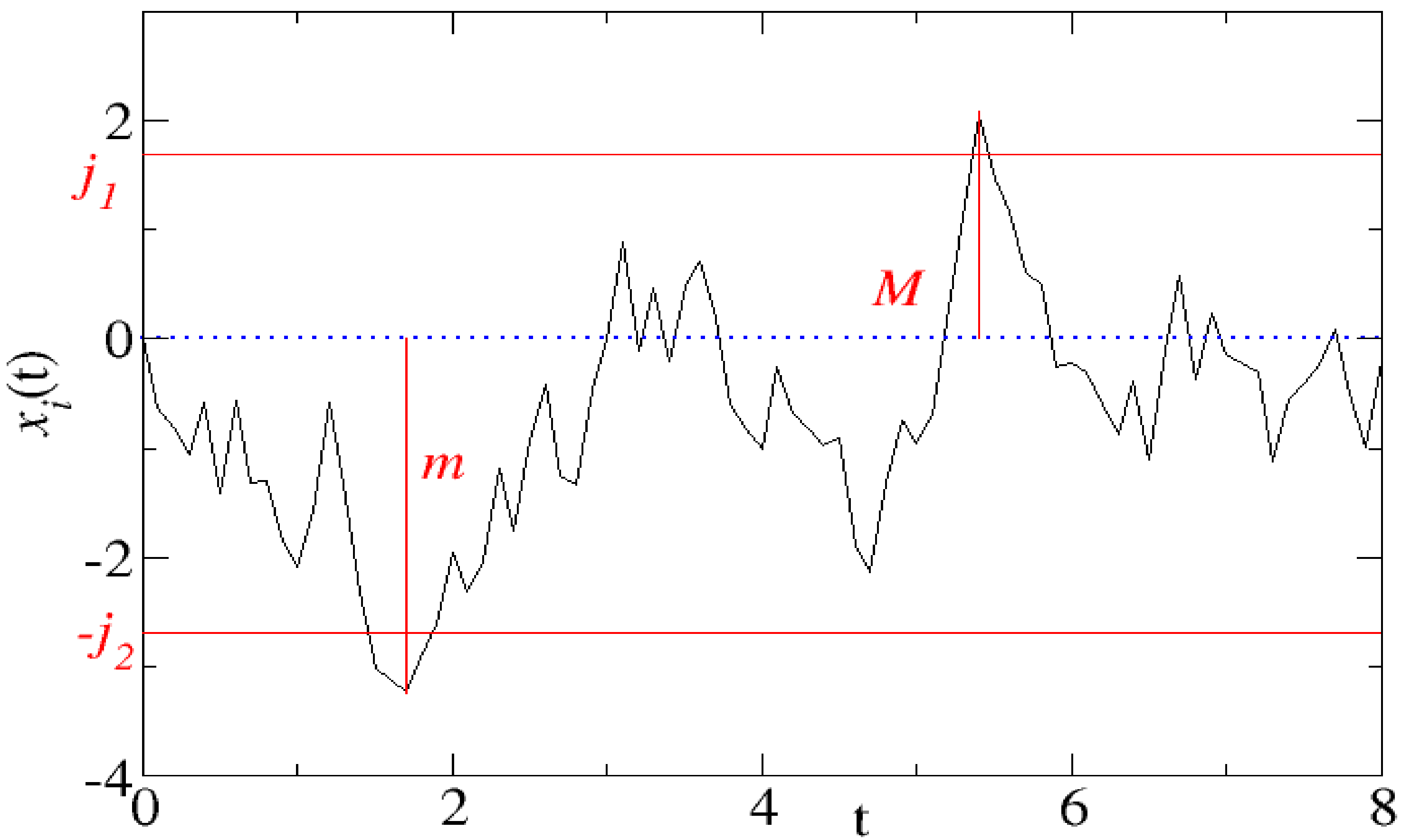
$$w_+ = \min\left(M_1, M_2, \ldots, M_N\right)$$
$$w_- = \min\left(-m_1, -m_2, \ldots, -m_N\right)$$

$$w = w_+ + w_-$$

$$\mathbf{P}_c\left(j_1, j_2\right) = \mathrm{Prob}\left(w_+ \geq j_1, w_- \geq j_2\right) = \left[h(j_1, j_2)\right]^N$$

Cumulative distribution of w_+ and w_-

$$\mathbf{P}_c\,(j_1, j_2) = \mathrm{Prob}\,(w_+ \geq j_1, w_- \geq j_2) = [h(j_1, j_2)]^N$$



$$\begin{aligned} h(j_1, j_2) &= 1 - \mathrm{erf}\,(j_1) - \mathrm{erf}\,(j_2) \\ &+ \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n+\frac{1}{2}} \sin\left(\frac{(2\,n+1)\pi j_2}{j_1+j_2}\right) e^{-\left(\frac{(n+\frac{1}{2})\pi}{j_1+j_2}\right)^2} \end{aligned}$$

Common Span for $N > 2$

$$w_+ = \min\left(M_1, M_2, \ldots, M_N\right)$$
$$w_- = \min\left(-m_1, -m_2, \ldots, -m_N\right)$$

$$w = w_+ + w_-$$

$$\mathbf{P}_c\left(j_1, j_2\right) = \mathrm{Prob}\left(w_+ \geq j_1, w_- \geq j_2\right) = \left[h(j_1, j_2)\right]^N$$

$$\begin{aligned} h(j_1, j_2) &= 1 - \mathrm{erf}\left(j_1\right) - \mathrm{erf}\left(j_2\right) \\ &+ \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{n+\frac{1}{2}}\sin\left(\frac{(2n+1)\pi j_2}{j_1+j_2}\right)e^{-\left(\frac{(n+\frac{1}{2})\pi}{j_1+j_2}\right)^2}\end{aligned}$$

$$\boxed{q_N(w) = \int_0^\infty dj_1 \int_0^\infty dj_2\ \delta(w - j_1 - j_2)\left[\frac{\partial^2\ \left[h(j_1, j_2)\right]^N}{\partial j_1, \partial j_2}\right]}$$

Distribution of span & common span

$$\boxed{p_N(s) = \int_0^\infty dl_1 \int_0^\infty dl_2 \; \delta(s - l_1 - l_2) \left[\frac{\partial^2 \; [g(l_1, l_2)]^N}{\partial l_1, \partial l_2} \right]}$$

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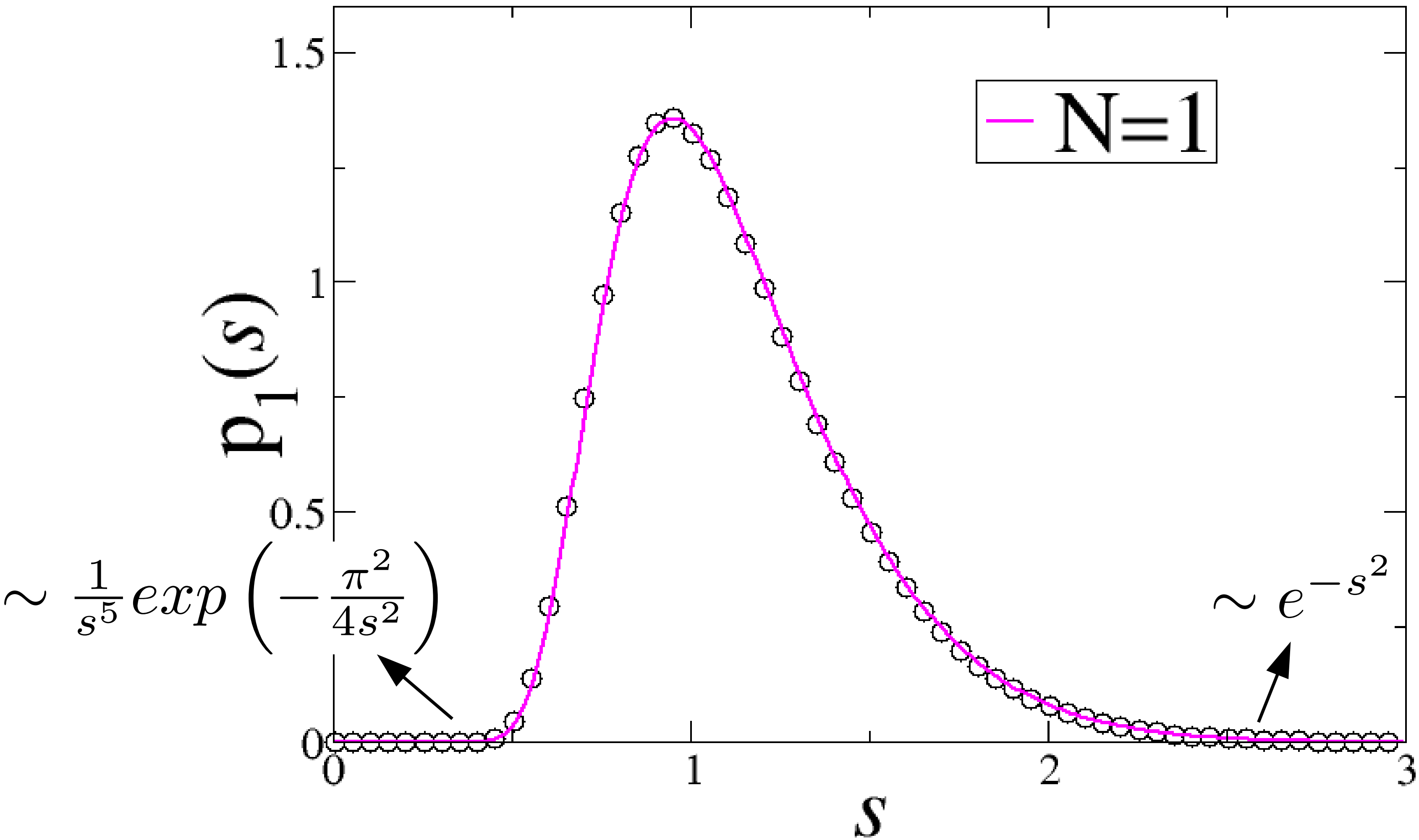
Exact Distributions for N=1

- Distribution of span or common span

N=1

$$p_1(s) = \frac{8}{\sqrt{\pi}} \sum_{m=1}^{\infty} (-1)^{m+1} m^2 \, e^{-m^2 s^2}$$



A. K, Majumdar & Schehr, PRL (2013)

Exact Distributions for N=1 & N=2

- Distribution of span

N=1

$$p_1(s) \quad = \quad \frac{8}{\sqrt{\pi}} \sum_{m=1}^{\infty} (-1)^{m+1} m^2 \; e^{-m^2 s^2}$$

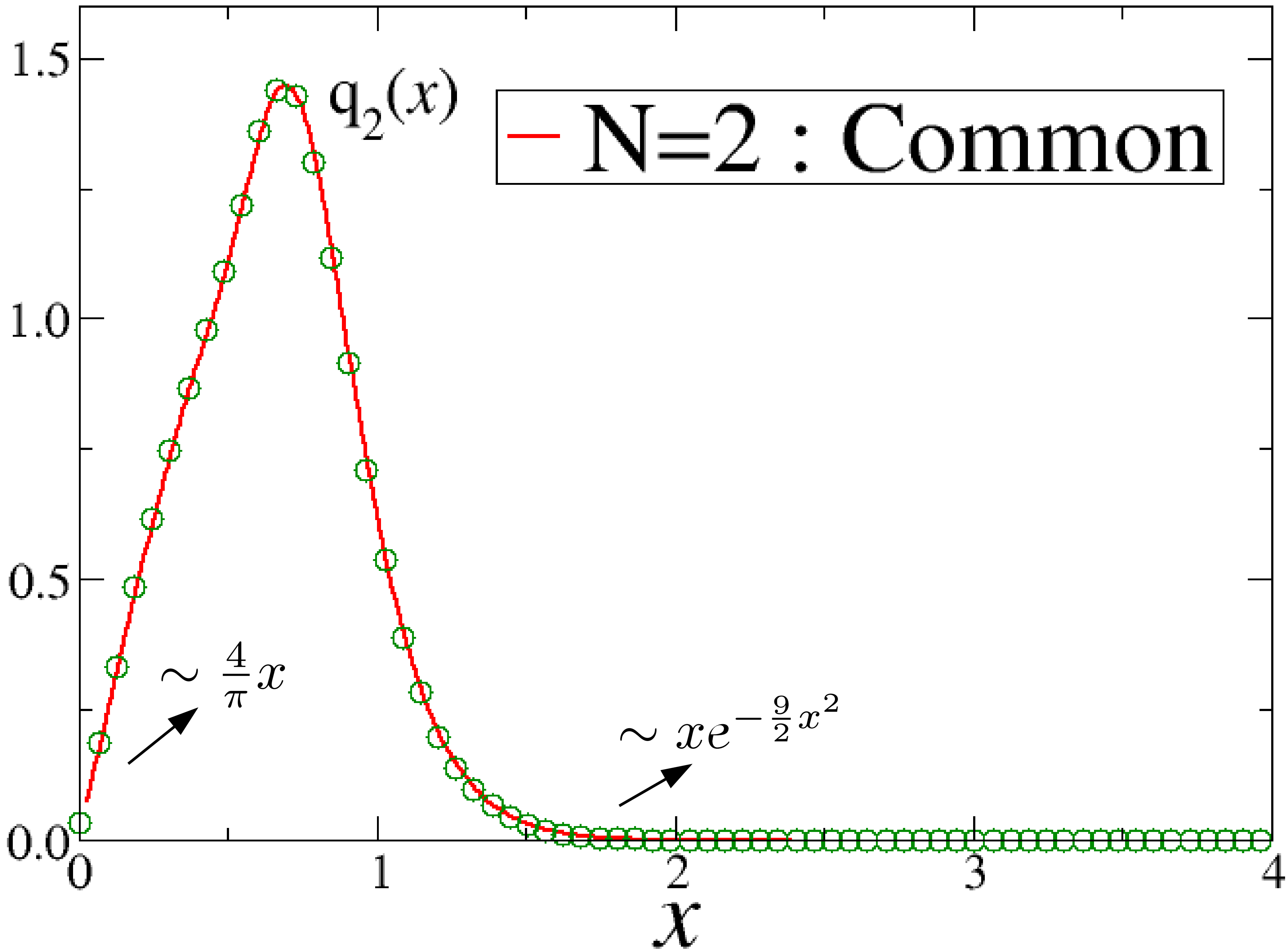
N=2

- Distribution of common span

$$q_2(w) = 2 \; \mathrm{erfc}(w) \; p_2\left(\sqrt{2} w\right) \; + \; p_2(w) + \frac{8}{\sqrt{2\pi}} \,\mathrm{erf}\left(\frac{w}{\sqrt{2}}\right) \; e^{-\frac{w^2}{2}}$$
$$+ \frac{8}{\sqrt{2\pi}} \sum_{m=1}^{\infty} (-1)^m \; m^2 \; e^{-\frac{m^2 w^2}{2}} \left(\mathrm{erf}\left[(m+2)\frac{w}{\sqrt{2}}\right] - \mathrm{erf}\left[(m-2)\frac{w}{\sqrt{2}}\right]\right).$$

A. K, Majumdar & Schehr, PRL (2013)

Distributions : $N = 2$



Exact Distributions for N=1 & N=2

- Distribution of span

N=1

$$p_1(s) \;=\; \frac{8}{\sqrt{\pi}} \sum_{m=1}^{\infty} (-1)^{m+1} m^2 \; e^{-m^2 s^2}$$

N=2

- Distribution of common span

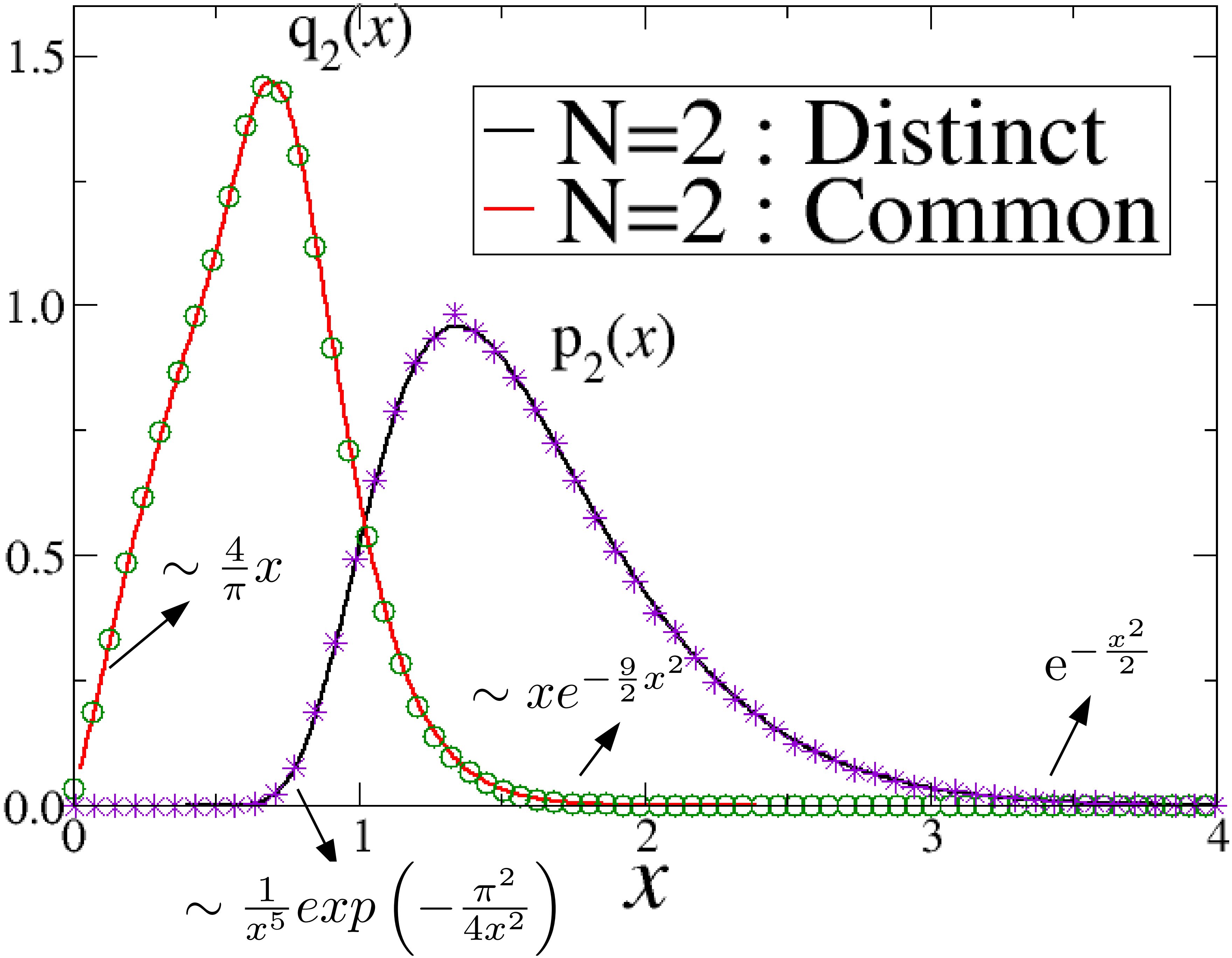
$$q_2(w) = 2\,\mathrm{erfc}(w)\; p_2\left(\sqrt{2}w\right) \;+\; p_2(w) + \frac{8}{\sqrt{2\pi}}\,\mathrm{erf}\left(\frac{w}{\sqrt{2}}\right)\; e^{-\frac{w^2}{2}}$$
$$+ \frac{8}{\sqrt{2\pi}} \sum_{m=1}^{\infty} (-1)^m \; m^2 \; e^{-\frac{m^2 w^2}{2}} \left(\mathrm{erf}\left[(m+2)\frac{w}{\sqrt{2}}\right] - \mathrm{erf}\left[(m-2)\frac{w}{\sqrt{2}}\right]\right).$$

- Distribution of span

$$p_2(s) = \frac{8}{\sqrt{2\pi}} \sum_{m=1}^{\infty} (-1)^{m+1} m^2 \; e^{-m^2 \frac{s^2}{2}}$$

A. K, Majumdar & Schehr, PRL (2013)

Distributions : $N = 2$



Distributions

$$\boxed{p_N(s) = \int_0^\infty dl_1 \int_0^\infty dl_2 \; \delta(s - l_1 - l_2) \left[\frac{\partial^2 \; [g(l_1, l_2)]^N}{\partial l_1, \partial l_2} \right]}$$

$$\boxed{q_N(w) = \int_0^\infty dj_1 \int_0^\infty dj_2 \; \delta(w - j_1 - j_2) \left[\frac{\partial^2 \; [h(j_1, j_2)]^N}{\partial j_1, \partial j_2} \right]}$$

Are there any limiting forms of these two distributions for large N ?

Moments

- 1st moment

$$\langle s \rangle_N = 2\int_0^\infty x \frac{d}{dx}[\mathrm{erf(x)}]^N dx \, ,$$

$$\langle w \rangle_N = 2\int_0^\infty [\mathrm{erfc(x)}]^N \, dx \, ,$$

- 2nd moment

$$\langle s^2 \rangle_N = 2\int_0^\infty x^2 \frac{d}{dx}[\mathrm{erf}(x)]^N dx + 2\int_0^\infty\int_0^\infty xy \, \frac{\partial [g(x,y)]^N}{\partial x \partial y} \, dx \, dy$$

$$\langle w^2 \rangle_N = 4\int_0^\infty x \, [\mathrm{erfc}(x)]^N dx + \int_0^\infty\int_0^\infty [h(x,y)]^N \, dx \, dy$$

Moments : $N \to \infty$

- Span :

$$\langle s \rangle \sim 2\sqrt{\log N} + \frac{\gamma}{\sqrt{\log N}}$$

$$\sqrt{\langle s^2 \rangle - \langle s \rangle^2} \sim \frac{1}{\sqrt{\log N}}$$

- Common span :

$$\langle w \rangle \sim \frac{\sqrt{\pi}}{N}$$

$$\sqrt{\langle w^2 \rangle - \langle w \rangle^2} \sim \frac{1}{N}$$

Moments : $N \to \infty$

- Span :

$$\langle s \rangle \sim 2\sqrt{\log N} + \frac{\gamma}{\sqrt{\log N}}$$

$$\sqrt{\langle s^2 \rangle - \langle s \rangle^2} \sim \frac{1}{\sqrt{\log N}}$$

$$\longrightarrow \quad s = 2\sqrt{\log N} + \frac{x}{\sqrt{\log N}}$$

Random variable x has N independent distribution $\mathcal{D}(x)$

- Common span :

$$\langle w \rangle \sim \frac{\sqrt{\pi}}{N}$$

$$\sqrt{\langle w^2 \rangle - \langle w \rangle^2} \sim \frac{1}{N}$$

$$\longrightarrow \quad w = \frac{y}{N}$$

Random variable y has N independent distribution $\mathcal{C}(y)$

Moments : $N \to \infty$

- Span :

$$\langle s \rangle \sim 2\sqrt{\log N} + \frac{\gamma}{\sqrt{\log N}}$$

$$\sqrt{\langle s^2 \rangle - \langle s \rangle^2} \sim \frac{1}{\sqrt{\log N}}$$

$$s = 2\sqrt{\log N} + \frac{x}{\sqrt{\log N}}$$

Random variable x has N independent distribution $\mathcal{D}(x)$

$$p_N(s) \sim \sqrt{\log(N)}\, \mathcal{D}\left[\sqrt{\log N}\left(s - 2\sqrt{\log N}\right)\right]$$

- Common span :

$$\langle w \rangle \sim \frac{\sqrt{\pi}}{N}$$

$$\sqrt{\langle w^2 \rangle - \langle w \rangle^2} \sim \frac{1}{N}$$

$$w = \frac{y}{N}$$

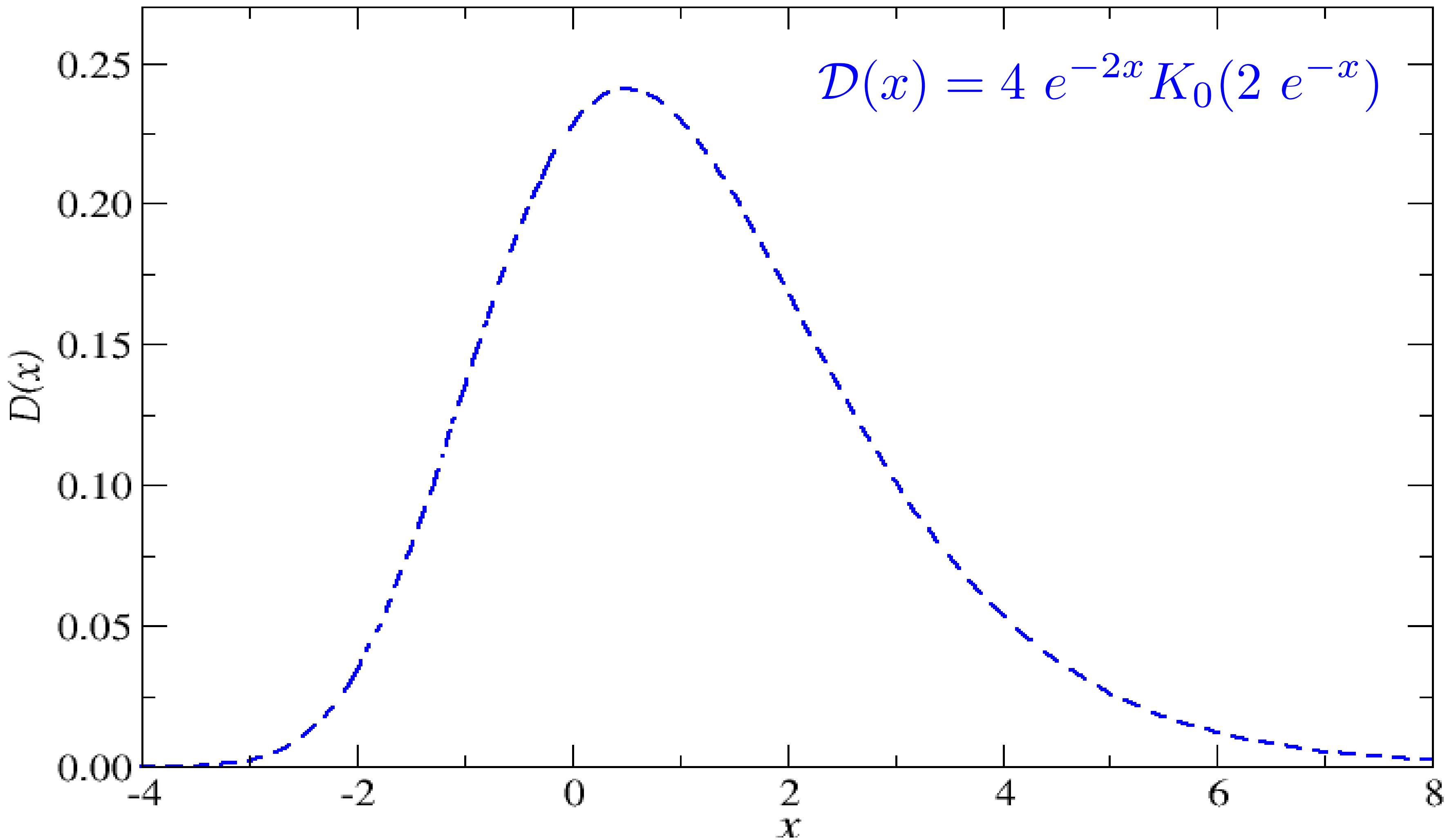
Random variable y has N independent distribution $\mathcal{C}(y)$

$$q_N(w) = N\, \mathcal{C}(N\, w)$$

Distributions : Large N

- Distribution of the number of distinct sites or the span

$$p_N(s) \sim \sqrt{\log(N)}\ \mathcal{D}\left[\sqrt{\log N}\left(s - 2\sqrt{\log N}\right)\right] \ ;$$

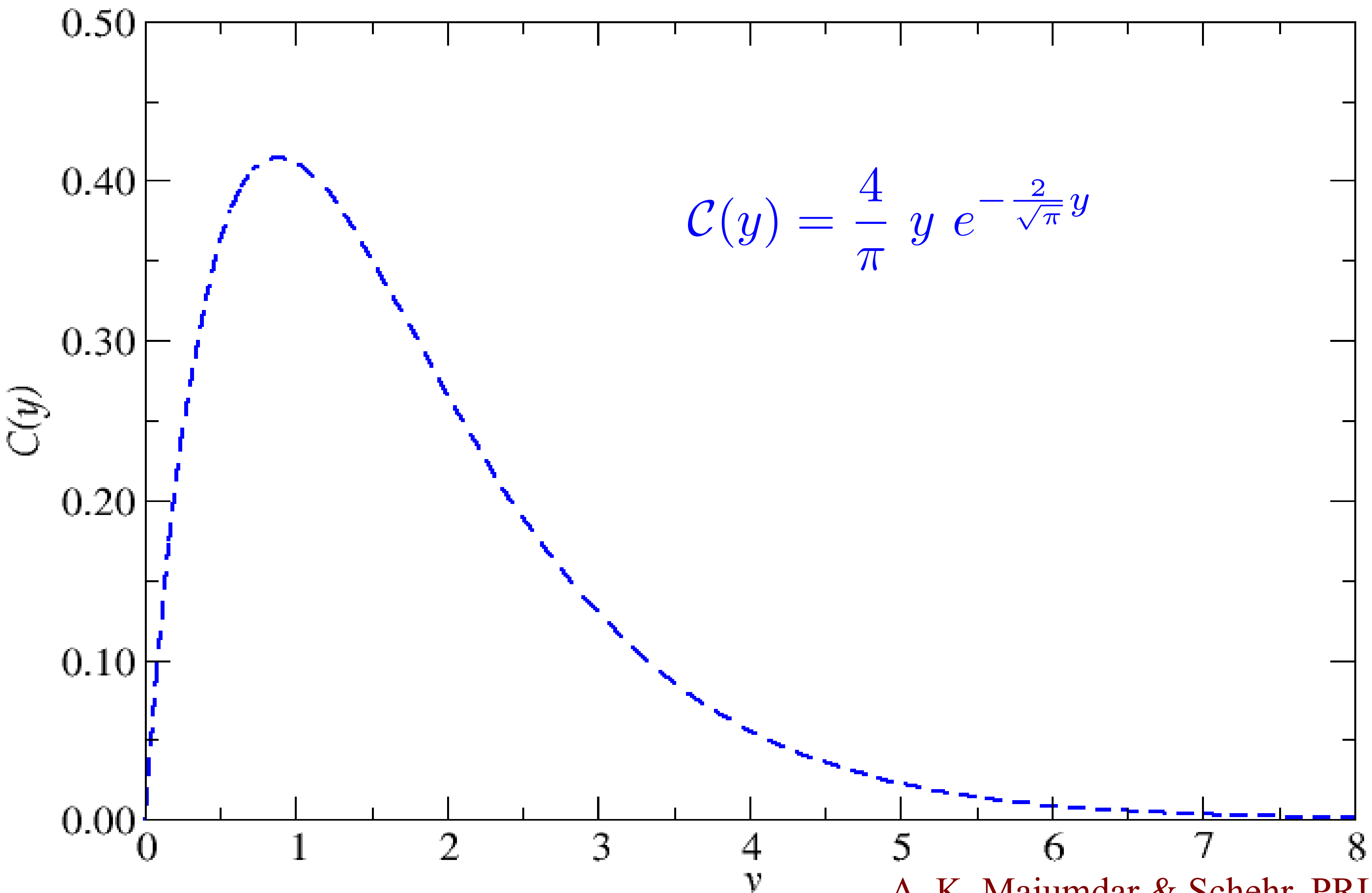


A. K, Majumdar & Schehr, PRL (2013)

Distributions : Large N

- Distribution of the number of common sites or the common span

$$q_N(w) = \; N\, \mathcal{C}(N\; w)\;;$$



A. K, Majumdar & Schehr, PRL (2013)

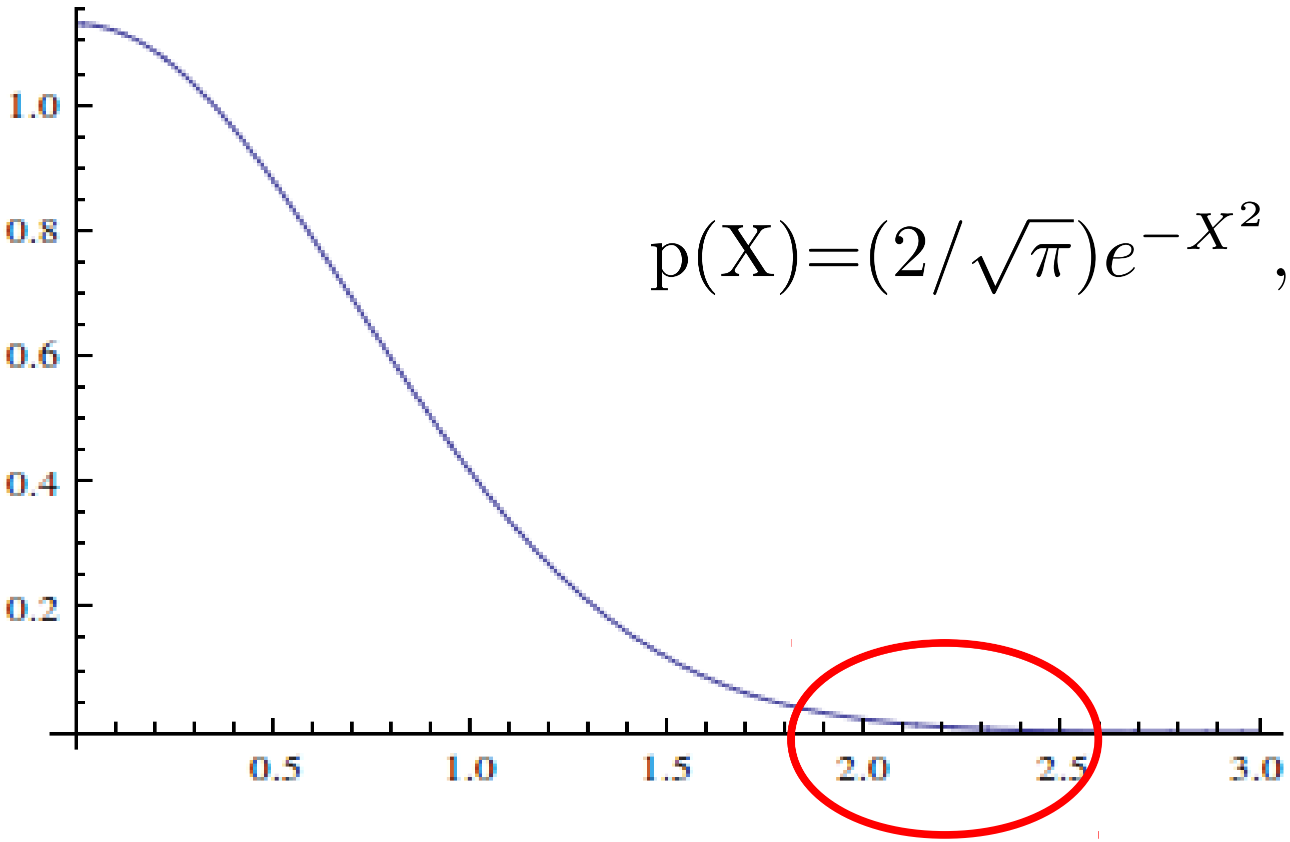
s_+ & s_- are Gumbel variables

- Span : $s = s_+ + s_-$

$$s_+ = \max_{1\leq i\leq N} M_i$$

$$s_- = \max_{1\leq i\leq N} -m_i$$

- The variables M_i 's are independent, positive random variables



$$\mathrm{p(X)}=(2/\sqrt{\pi})e^{-X^2}, X > 0$$

s_+ & s_- are Gumbel variables

- Span : $s = s_+ + s_-$ $\qquad s_+ = \max_{1\le i\le N} M_i$ $\qquad s_- = \max_{1\le i\le N} -m_i$

- For large N, both s_+ and s_- distributed according to Gumbel distribution :

$$\mathcal{P}(s_\pm) \approx 2\sqrt{\log N} e^{-2\sqrt{\log N}\left(s_\pm - \sqrt{\log N}\right)} e^{-e^{-2\sqrt{\log N}\left(s_\pm - \sqrt{\log N}\right)}}$$

$$\Longrightarrow \qquad \langle s_\pm \rangle \sim \sqrt{\log N} \qquad \sqrt{\langle \Delta s_\pm^2 \rangle} \sim \frac{1}{\sqrt{\log N}}$$

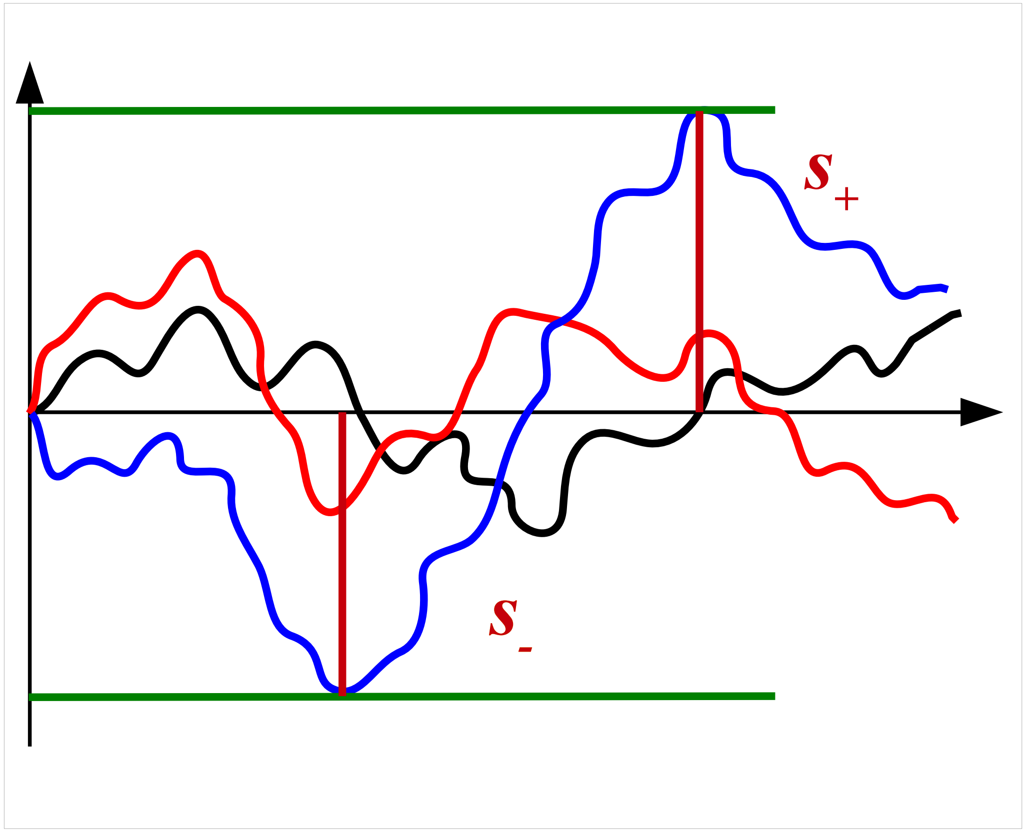
- For large N, both s_+ and s_- are of $\mathcal{O}\left(\sqrt{\log N}\right)$

Two ways of creating S

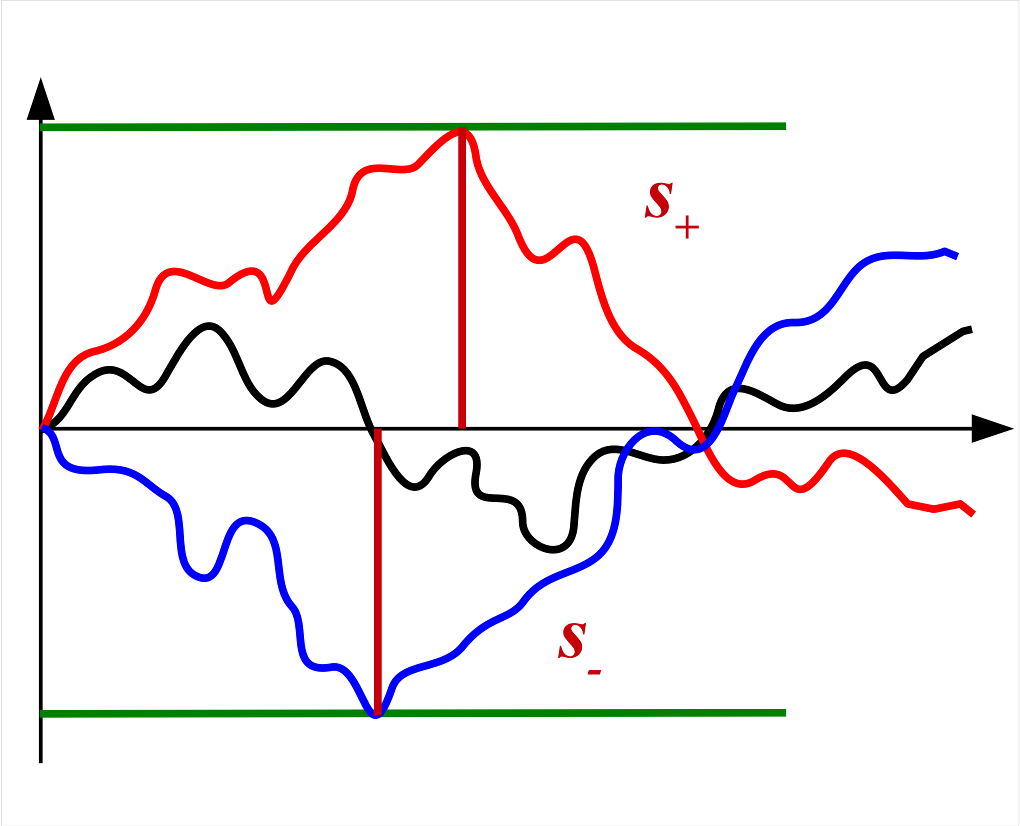
- Span : $s = s_+ + s_-$

- Two ways of creating s :

Single particle creating s_+ & s_-



Two particles creating s_+ & s_-

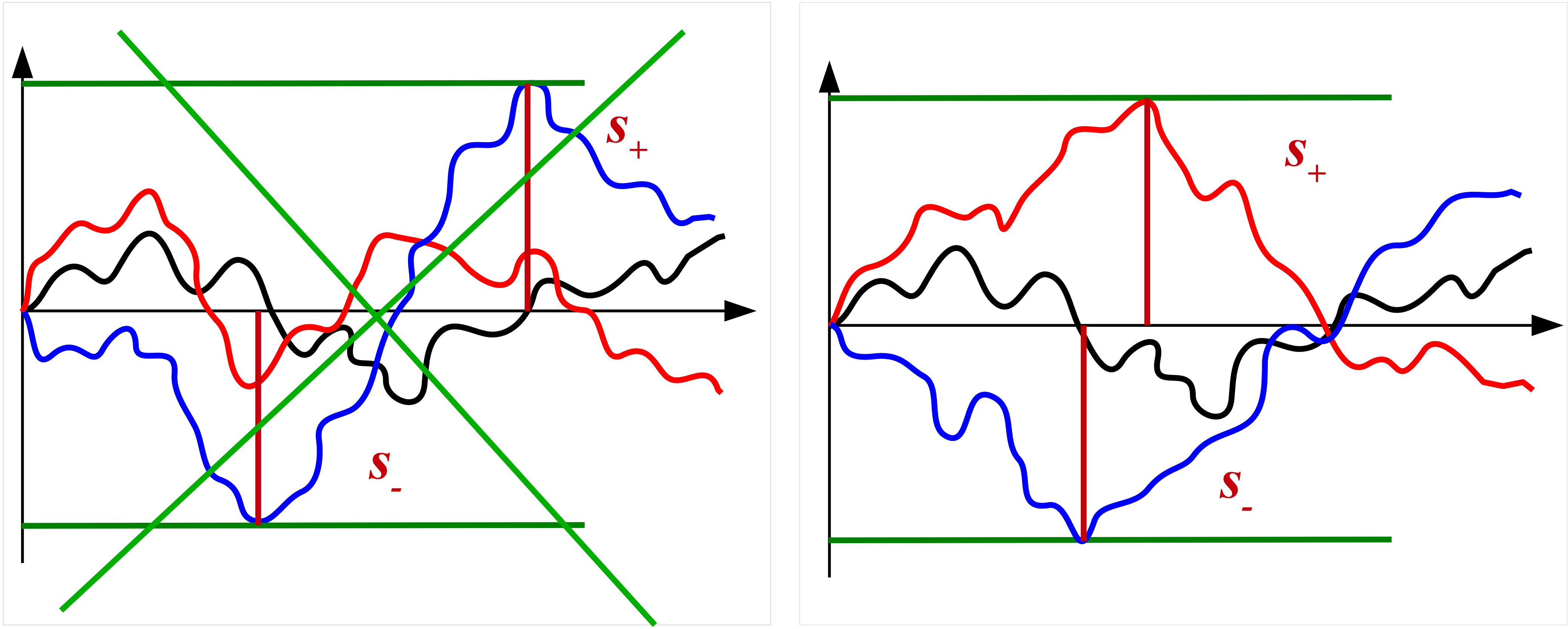


Two ways of creating S

- Span : $s = s_+ + s_-$

- Two ways of creating s :

Single particle creating s_+ & s_- Two particles creating s_+ & s_-



Distribution of the span

- So, when $N \to \infty$, s_+ and s_- become independent :

$$\mathcal{P}_d(s_+, s_-) \approx \mathcal{P}(s_+)\mathcal{P}(s_-) \qquad \text{where,}$$

$$\mathcal{P}(s_\pm) \approx 2\sqrt{\log N} e^{-2\sqrt{\log N}\left(s_\pm - \sqrt{\log N}\right)} e^{-e^{-2\sqrt{\log N}\left(s_\pm - \sqrt{\log N}\right)}}$$

$$\Downarrow$$

$$p_N(s) = \int_0^\infty ds_+ \int_0^\infty ds_- \; \delta(s - s_+ - s_-)\mathcal{P}_d(s_+, s_-)$$

$$\Downarrow$$

$$p_N(s) \sim \sqrt{\log(N)} \; \mathcal{D}\left[\sqrt{\log N}\left(s - 2\sqrt{\log N}\right)\right]$$

$$\mathcal{D}(x) = 4\; e^{-2x} K_0(2\; e^{-x})$$

Distribution of the span

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$$\Downarrow$$

$$p_N(s) = \int_0^\infty ds_+ \int_0^\infty ds_- \; \delta(s - s_+ - s_-)\mathcal{P}_d(s_+, s_-)$$

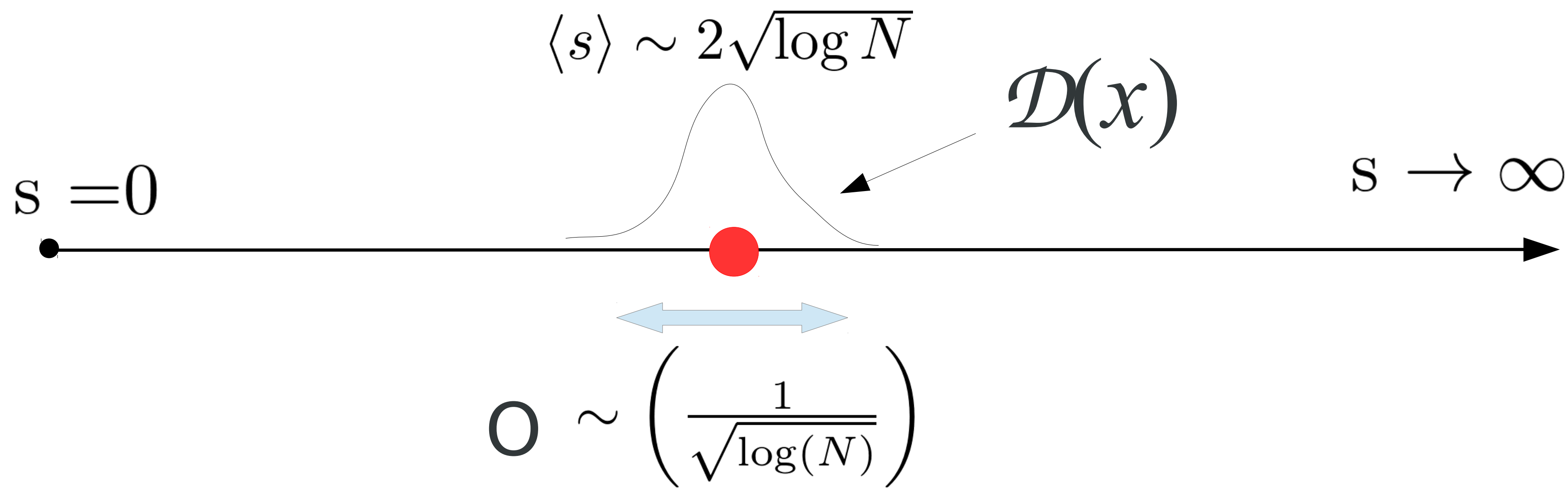
$$\Downarrow$$

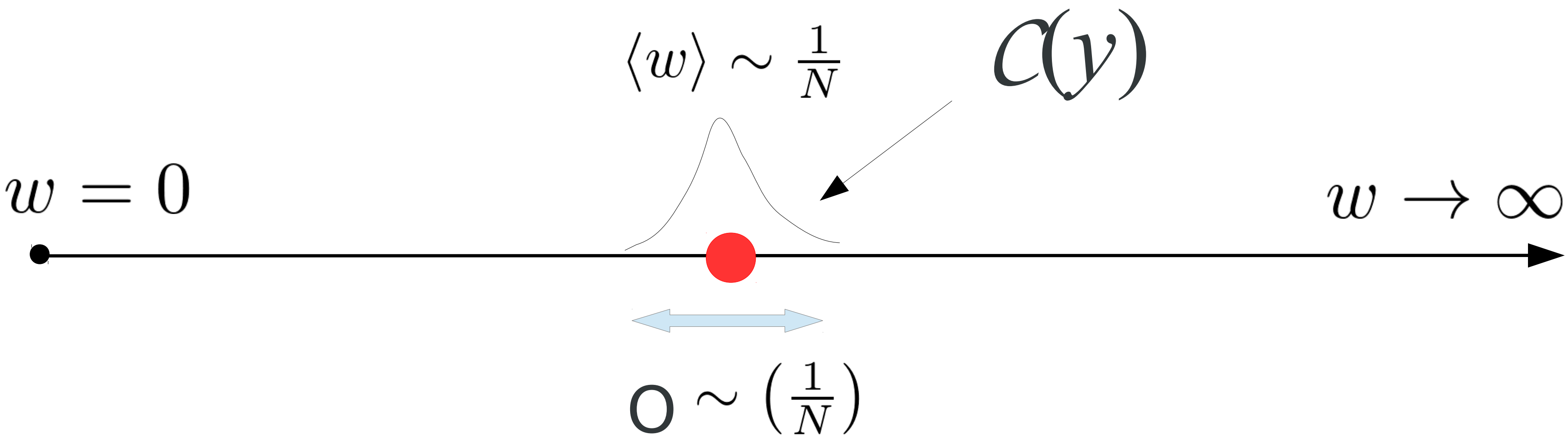
$$p_N(s) \sim \sqrt{\log(N)}\; \mathcal{D}\left[\sqrt{\log N}\left(s - 2\sqrt{\log N}\right)\right]$$

Span : $$\mathcal{D}(x) = 4\, e^{-2x} K_0(2\, e^{-x})$$

Common Span : $$q_N(w) = N\, \mathcal{C}(N\, w)\; ; \; \mathcal{C}(y) = \frac{4}{\pi}\, y\, e^{-\frac{2}{\sqrt{\pi}} y}$$

Asymptotes : finite N





Asymptotes : finite N

Span

$$p_N(s) \sim \begin{cases} a_N s^{-5} \exp\left[-N\pi^2/(4s^2)\right] , & s \to 0 , \\ b_N \exp\left(-s^2/2\right) , & s \to \infty , \end{cases}$$

$$\mathrm{a}_N = 4\pi^{3/2} N(N-1) \left(\frac{4}{\pi}\right)^{N-2} \frac{\Gamma(\frac{N-1}{2})}{\Gamma(\frac{N}{2})}$$

$$\mathrm{b}_N = 2\sqrt{2} N(N-1)/\sqrt{\pi}$$

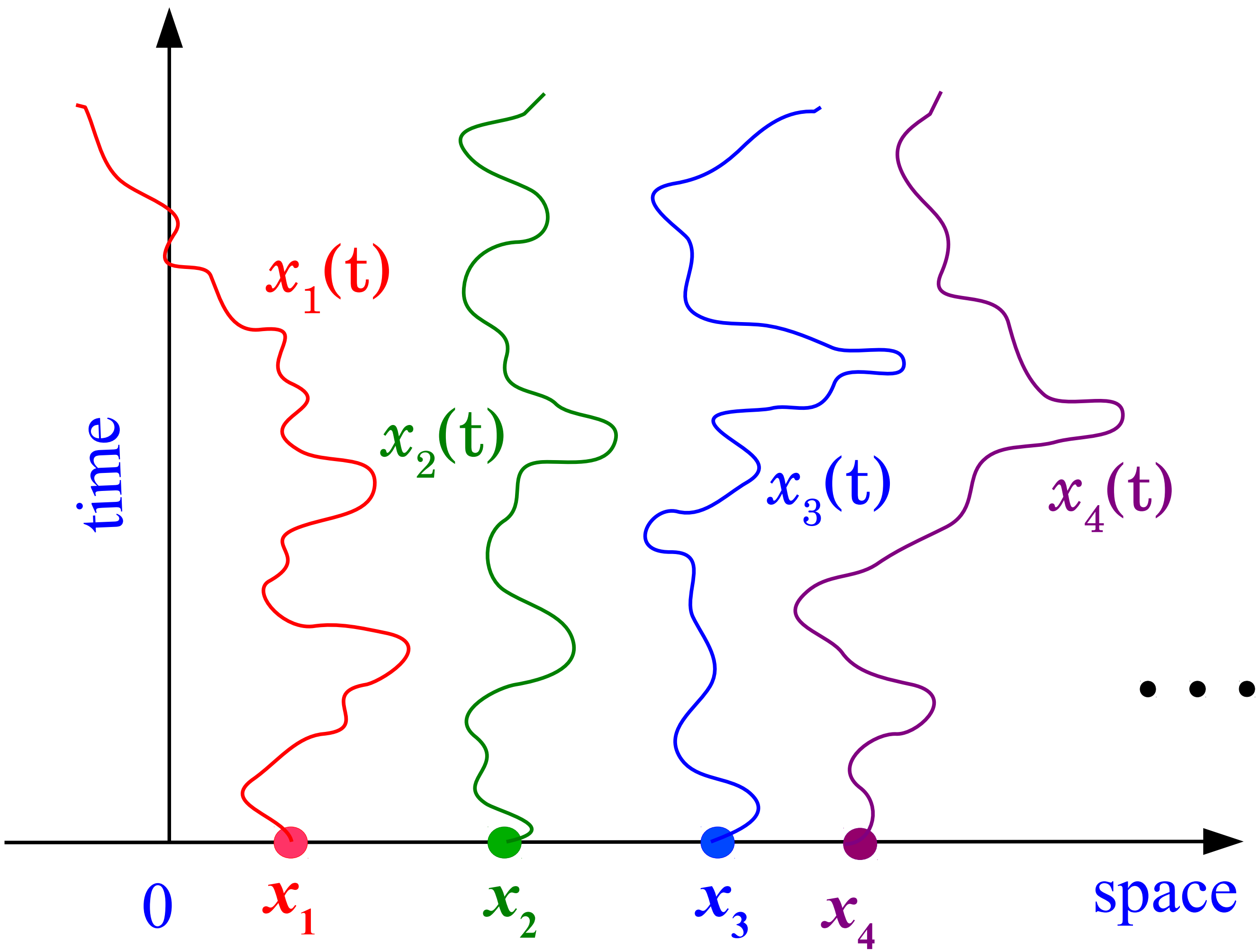
Common Span

$$q_N(w) \sim \begin{cases} c_N\, w , & w \to 0 \\ d_N w^{1-N} \exp\left(-N\, w^2\right) , & w \to \infty , \end{cases}$$

$$\mathrm{c}_N = 4N(N-1)/\pi \qquad \mathrm{d}_N = 8N/\pi^{N/2}$$

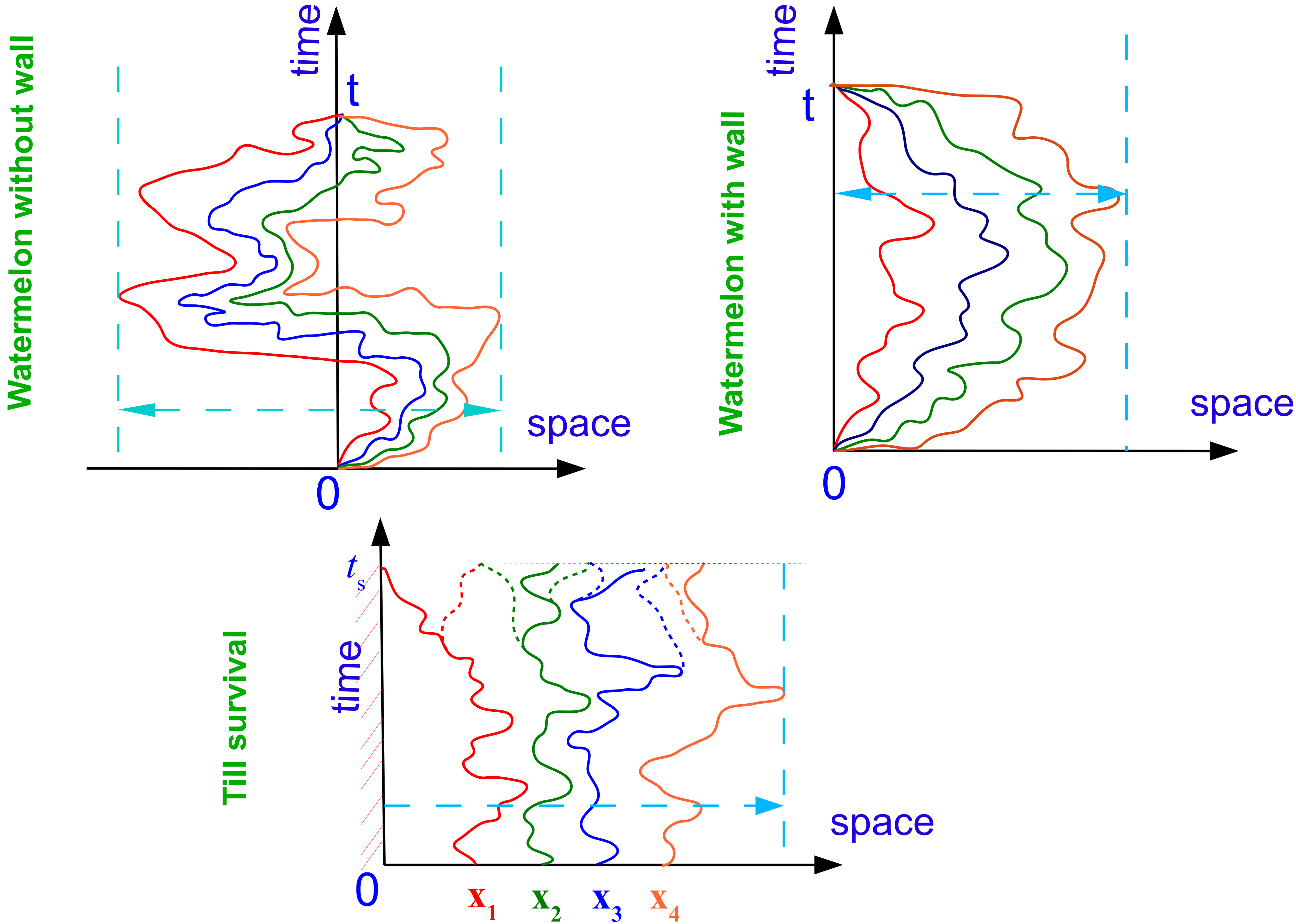
A. K, Majumdar & Schehr, PRL, 110, 220602, (2013) What happens when the walkers are interacting ?

Non-intersection ⇨ Interaction

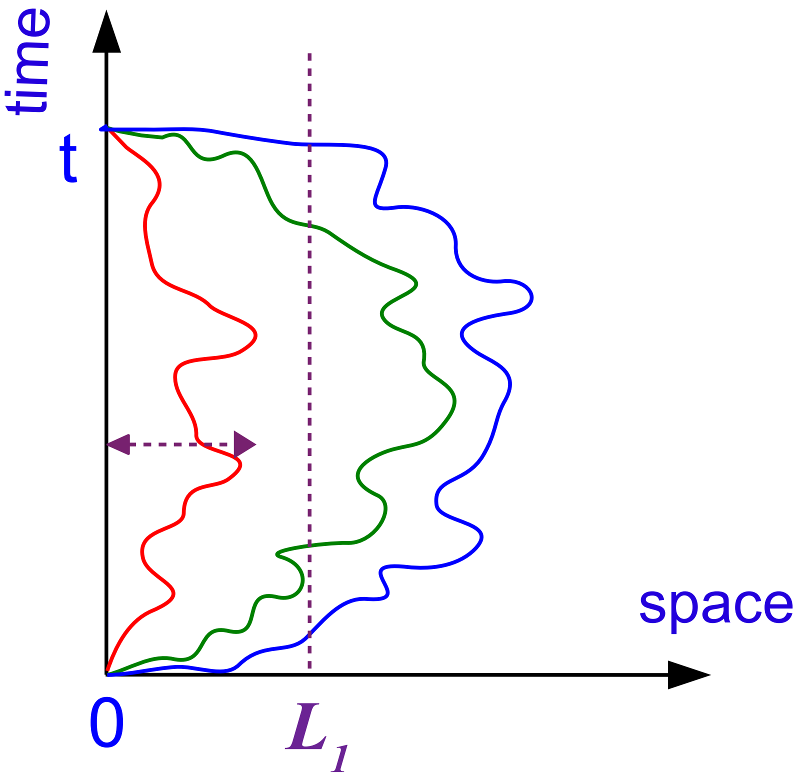


Vicious walkers

Span in different situations



Common Span



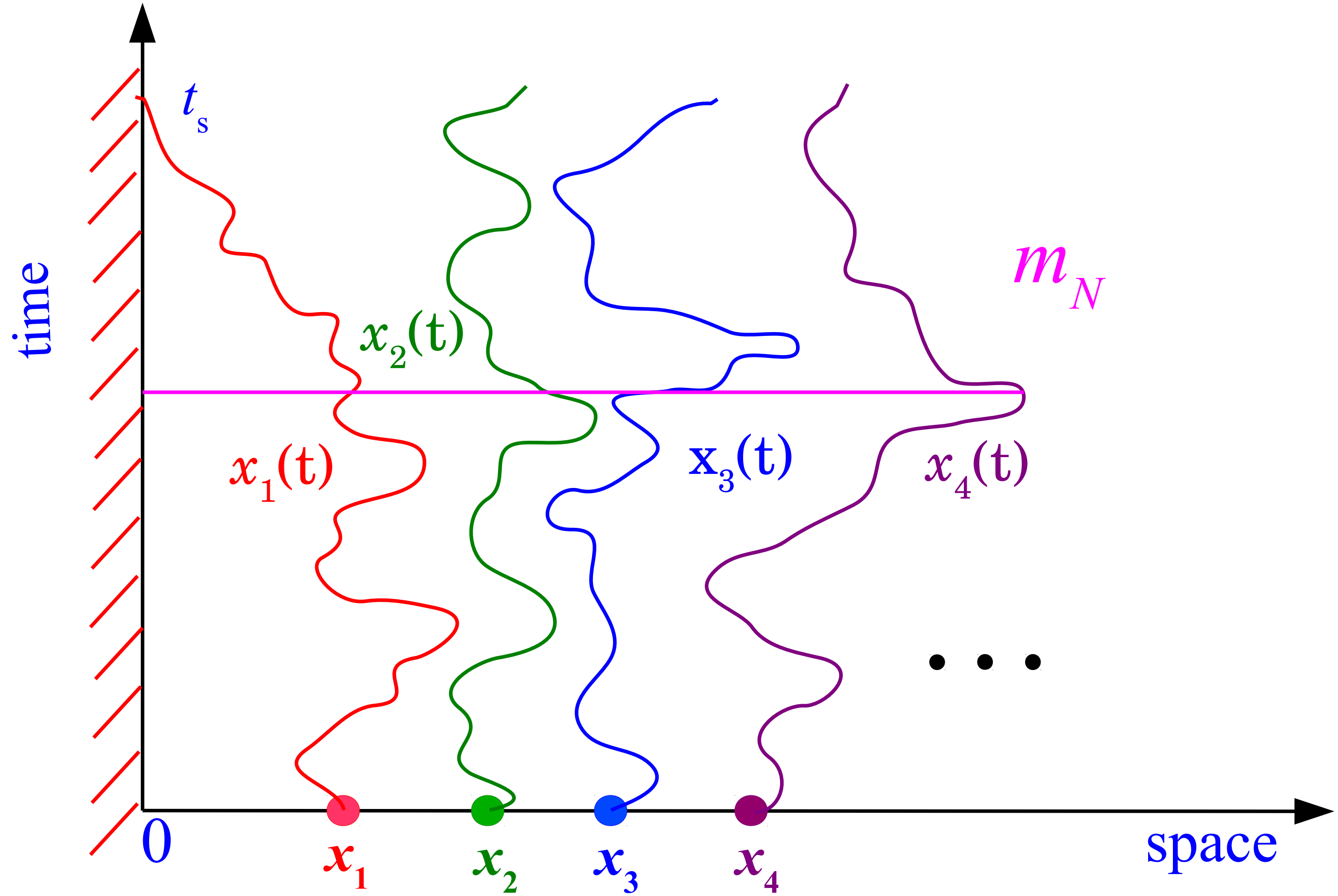
$$\mathbf{Q}(L_1) \quad = \quad \mathrm{Prob}\left(M_1 \le L_1\right)$$

$$\frac{\partial^2 G}{\partial x_1^2} + \frac{\partial^2 G}{\partial x_2^2} + \frac{\partial^2 G}{\partial x_3^2} = \frac{\partial G}{\partial t}$$

$$0 < x_1(t) < x_2(t) < x_3(t)$$

$$x_1(t) \le L_1$$

Span till survival



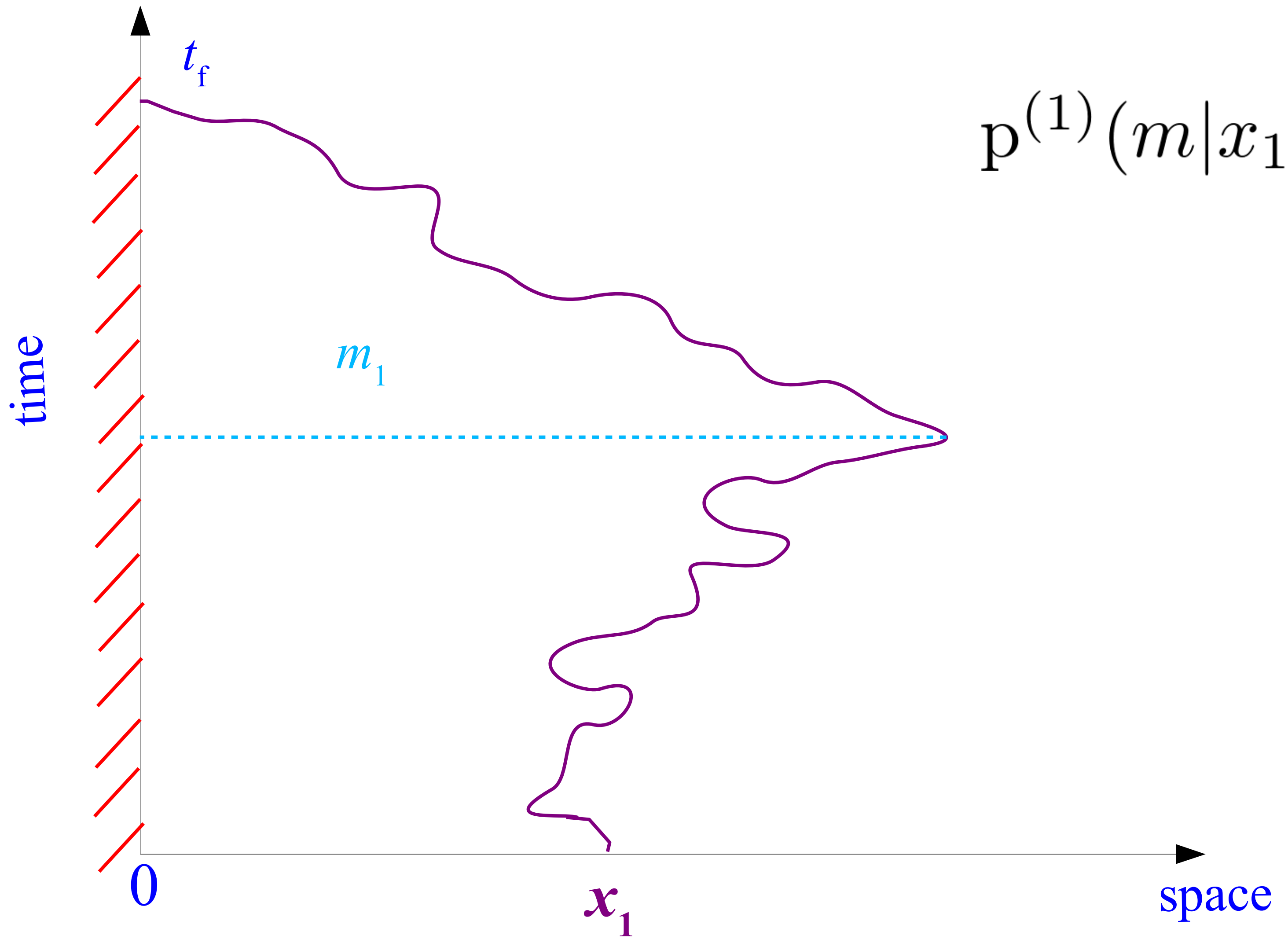
$$(x_{i+1} - x_i) = \text{small}$$

$$Q^{(N)}(L|\mathbf{x}) = \text{Prob.}\Big[\,\text{Global maximum } m_N \leq L \,|\mathbf{x}\,\Big] = \,?$$

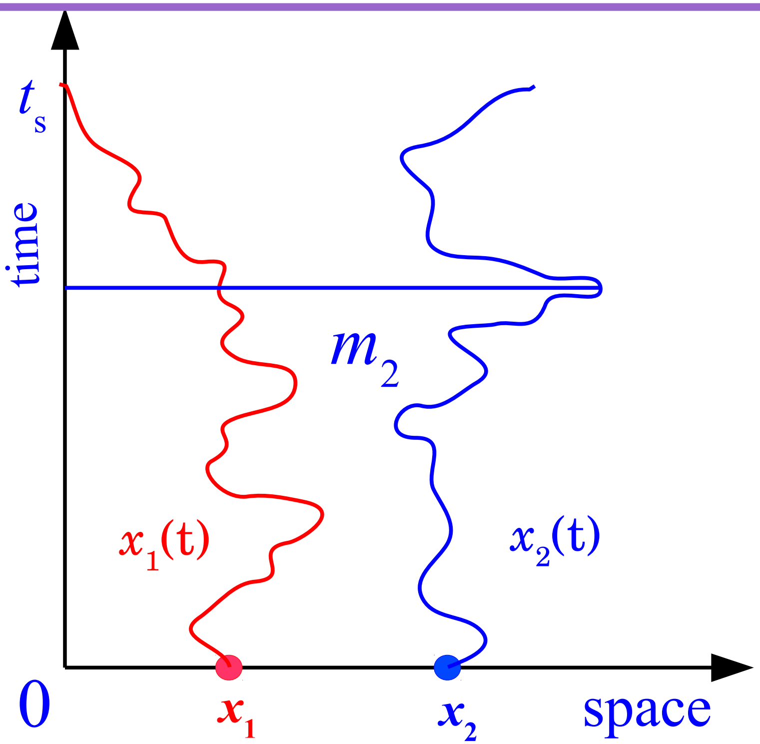
Single Brownian walker : N=1

$$\mathrm{Q}^{(1)}(L|x_1) = \mathrm{Prob}[m_1 \leq L|x_1] = 1 - \frac{x_1}{L}$$

$$\mathrm{p}^{(1)}(m|x_1) = \frac{x_1}{m^2}$$

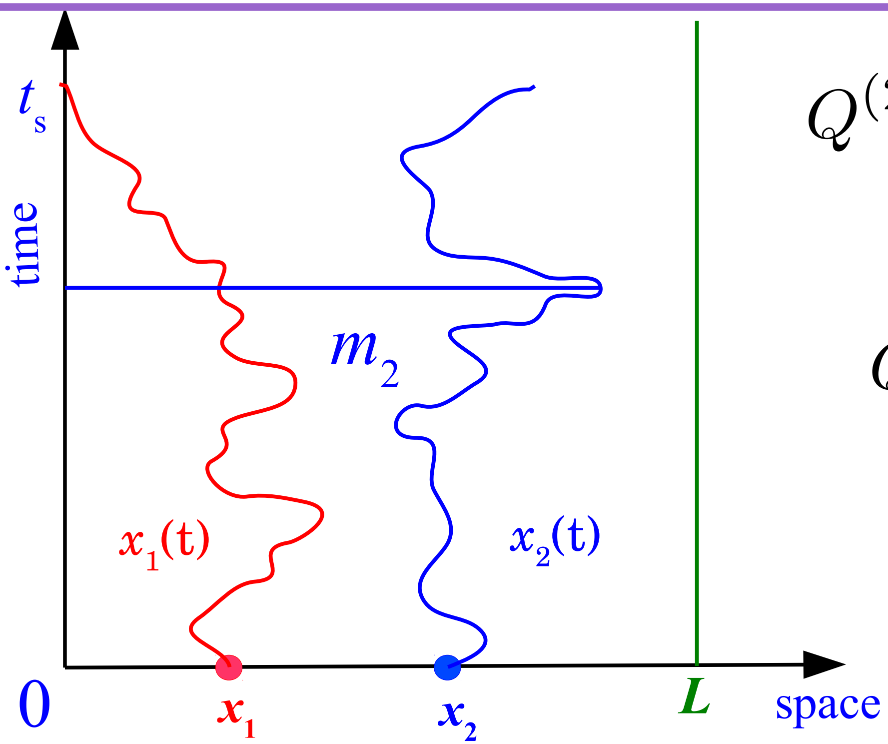


$N = 2$ particles



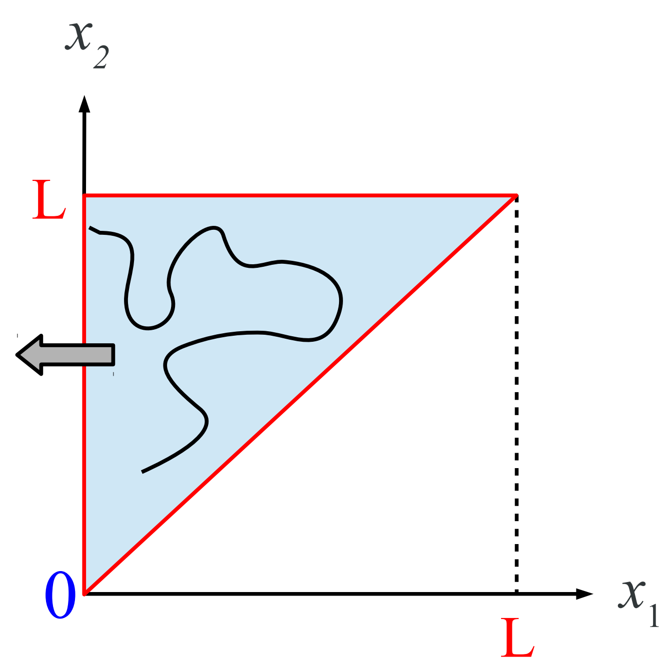
$$Q^{(2)}(L|x_1, x_2) = \text{Prob.}(m_2 \leq L|x_1, x_2)$$

$N = 2$ particles in a box



$$Q^{(2)}(L|x_1, x_2) = \text{Prob.}(m_2 \leq L|x_1, x_2)$$

$$Q^{(2)}(L|x_1, x_2) = \frac{F(L; x_1, x_2)}{F(L \to \infty; x_1, x_2)}$$

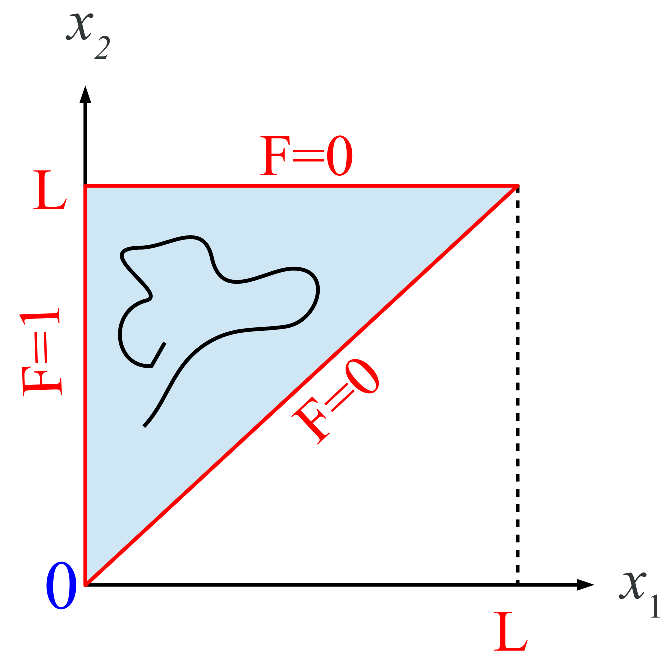


Exit probability

- $F(L; x_1, x_2) =$

Prob. $\left[\begin{array}{l} \text{The two walkers stay non-intersecting inside the box } [0, L] \\ \text{till the first walker crosses the origin for the first time} \end{array} \right]$

$N = 2$ particles in a box

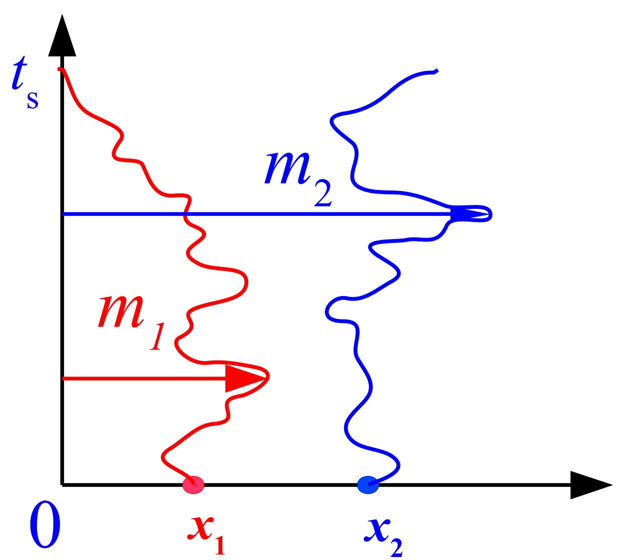


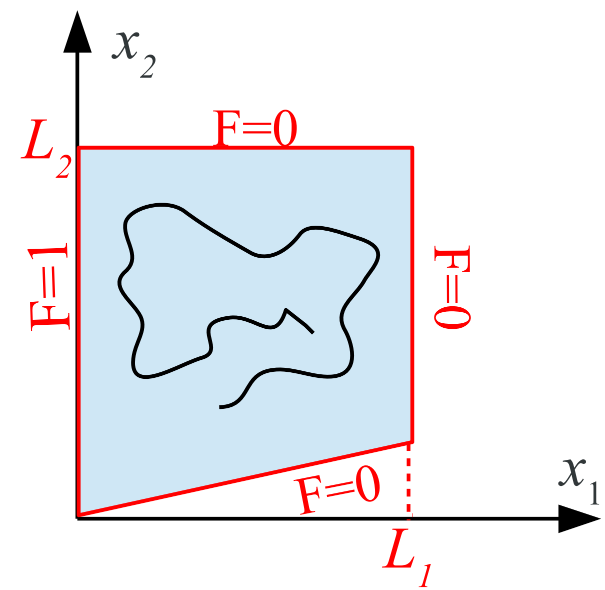
$$Q^{(2)}(L|x_1,x_2) = \mathrm{Prob.}(m_2 \leq L|x_1,x_2)$$

$$Q^{(2)}(L|x_1,x_2) = \frac{F(L;x_1,x_2)}{F(L\to\infty;x_1,x_2)}$$

$$\frac{\partial^2 F}{\partial x_1^2} + \frac{\partial^2 F}{\partial x_2^2} = 0$$

$$\mathcal{Q}^{(2)}(L_1,L_2|x_1,x_2) = \mathrm{Prob.}(m_2 \leq L_2,\ m_1 \leq L_1|x_1,x_2)$$

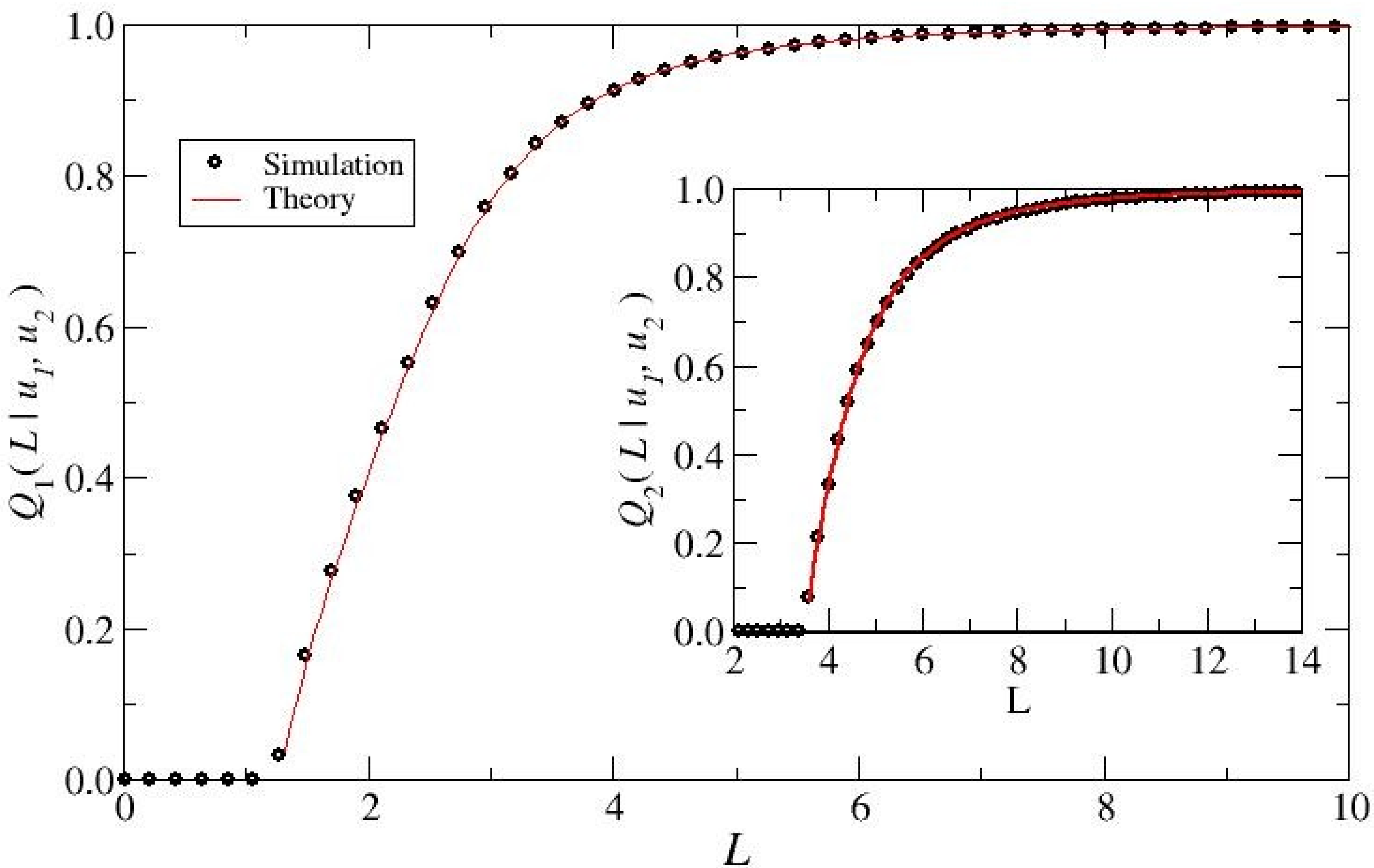




Marginal cumulative probabilities

$$Q_1(L|x_1, x_2) = \text{Prob.}(m_1 \leq L|x_1, x_2)$$

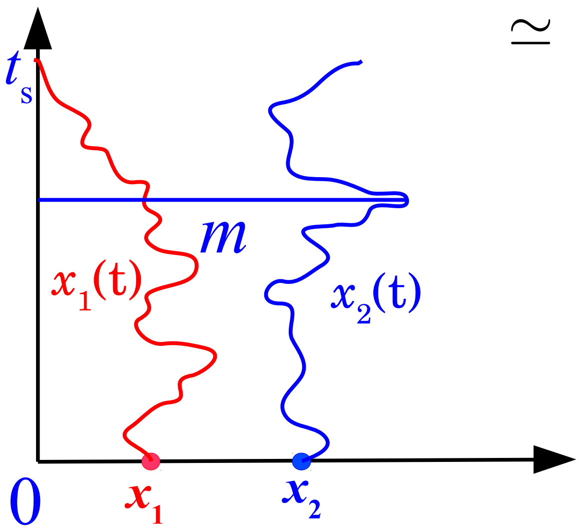
$$Q_2(L|x_1, x_2) = \text{Prob.}(m_2 \leq L|x_1, x_2)$$

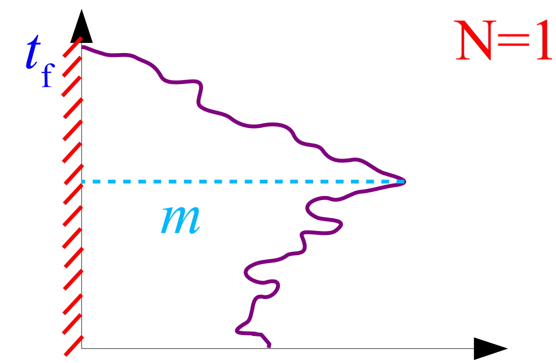


$$x_1 = 1.27,\ \ x_2 = 3.51,\ \ D_1 = 1.3,\ \ D_2 = 1.5;$$

$N = 2$ case

- $$p^{(2)}(m|x_1, x_2) = \left(\frac{\partial Q^{(2)}(L|x_1, x_2)}{\partial L} \right) \Bigg|_{L=m}$$
$$\simeq \mathcal{B}_2(x_1, x_2)\, \frac{1}{m^5}\ ; \quad m \gg x_2$$





$$\mathrm{p}^{(1)}(m|x_1) = \frac{x_1}{m^2}$$

- $$\mathcal{B}_2(x_1, x_2) = H_2\, \frac{\pi x_1 x_2 (x_2^2 - x_1^2)}{\left(4 \arctan\left(\frac{x_2}{x_1}\right) - \pi\right)}$$

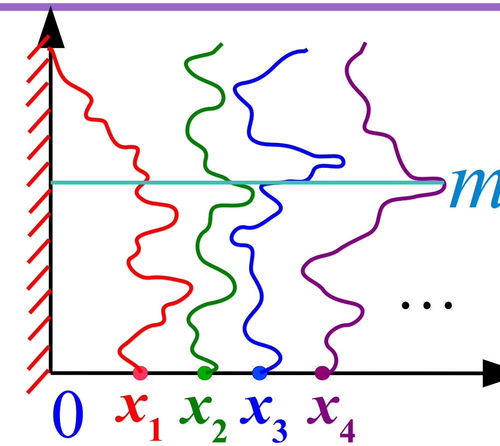
- $$H_2 = \frac{3}{5} \frac{\Gamma[\frac{1}{4}]^8}{4^4 \pi^5};$$

Kundu, Majumdar, Schehr (2014)

$N \geq 2$

- N Non-intersecting walkers :

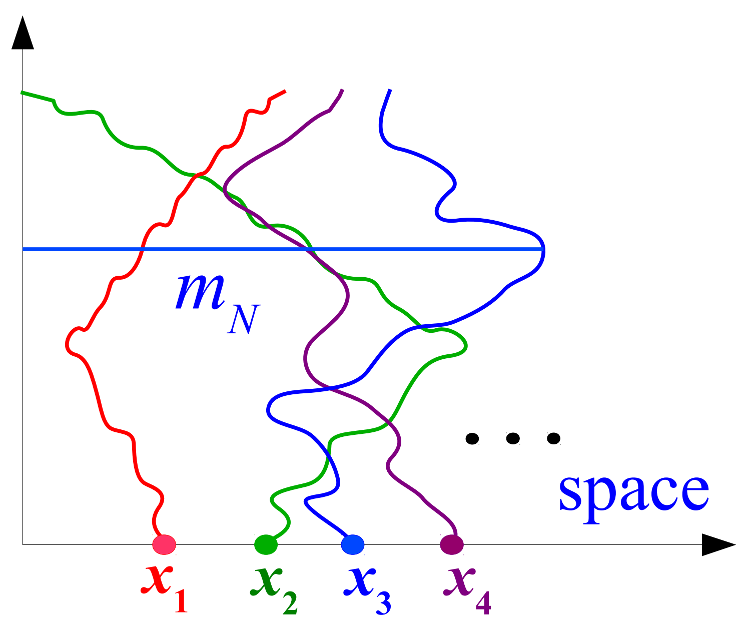
  For $\quad m \gg x_N$



$$p^{(N)}(m|\mathbf{x}) = \left(\frac{\partial Q^{(N)}(\mathbf{x}|L)}{\partial L}\right)\Bigg|_{L=m} \simeq \mathcal{B}_N(\mathbf{x})\ \frac{1}{m^{N^2+1}}$$

Kundu, Majumdar, Schehr (2014)

- N Non-interacting or independent walkers :



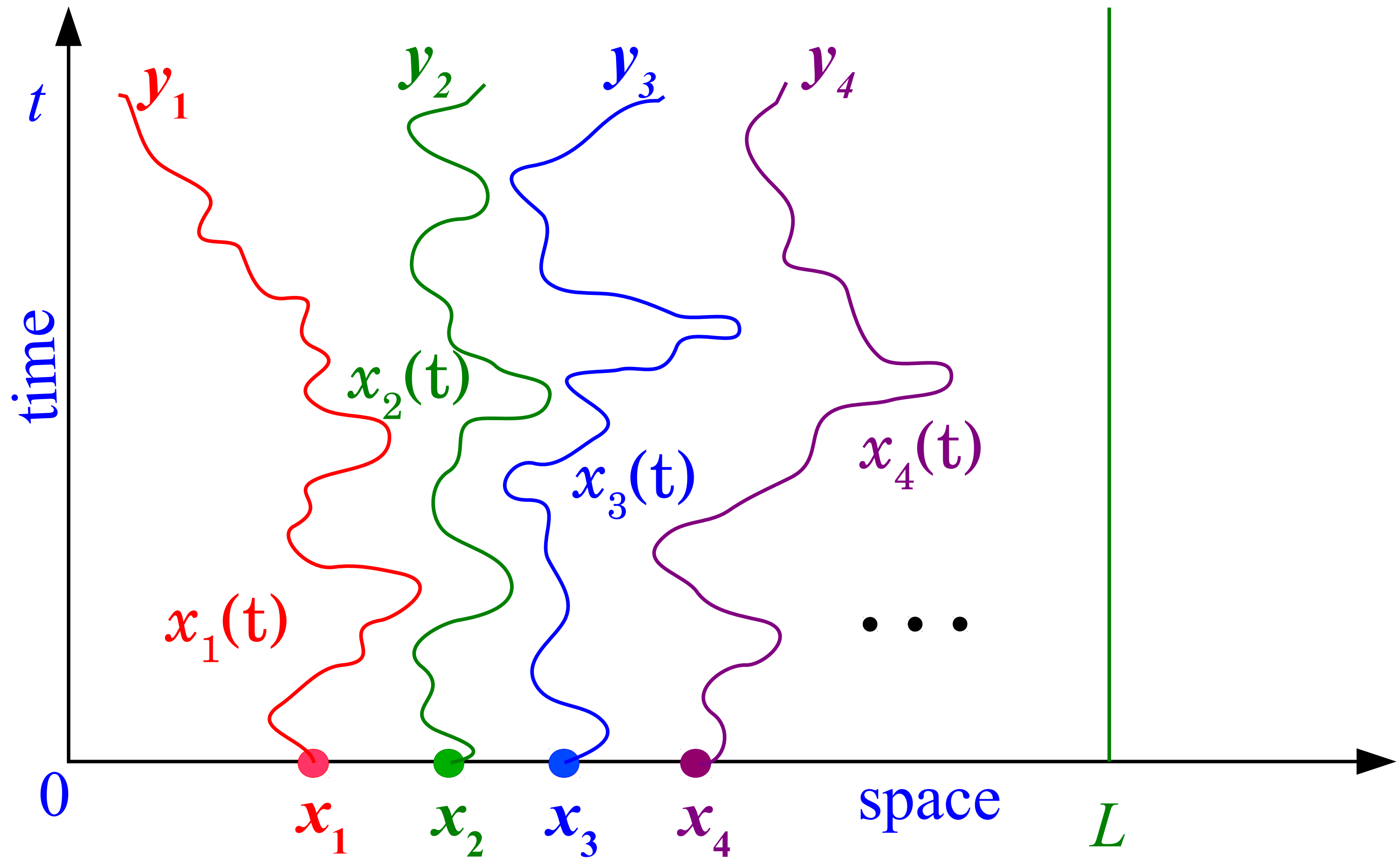
$$p^{(N)}(m|\mathbf{x}) \simeq \mathcal{A}_N(\mathbf{x})\ \frac{1}{m^{N+1}}; \quad m \gg x_N$$

Krapivsky, Majumdar, Rosso, J. Phys. A (2010)

$N \geq 2$ walkers : propagator

- Start with the N particle propagator in the box $[0, L]$:

$G_N(\mathbf{y}, t; \mathbf{x}, 0; L)$ = Probability density that particles starting from $(x_1, x_2, ..., x_N)$ reach $(y_1, y_2, ..., y_N)$ inside $[0,L]$ in time t.



$N \geq 2$ walkers : exit probability

- Start with the N particle propagator in the box $[0, L]$:

$G_N(\mathbf{y}, t; \mathbf{x}, 0; L)$ = Probability density that particles starting from $(x_1, x_2, \ldots, x_N)$ reach $(y_1, y_2, \ldots, y_N)$ in time t.

$\approx$ Slater Determinant

- Exit probability :

$$F^{(N)}(L; \mathbf{x}) = \text{Prob.} \left[\begin{array}{l} \text{The } N \text{ walkers stay non-intersecting inside the box } [0, L] \\ \text{till the first walker crosses the origin for the first time} \end{array} \right]$$

$$F^{(N)}(L; \mathbf{x}) = D \int_0^{\infty} dt \left(\int_0^{y_3} dy_2 \int_0^{y_4} dy_3 \ldots \int_0^{\infty} dy_N \left(\frac{\partial G_N(\mathbf{y}, t; \mathbf{x}, 0; L)}{\partial y_1} \right)_{y_1 = 0} \right)$$

$N \geq 2$ walkers : Distribution

- Start with the N particle propagator in the box [0, L]:

$G_N(\mathbf{y}, t; \mathbf{x}, 0; L) =$ Probability density that particles starting from $(x_1, x_2, \ldots, x_N)$ reach $(y_1, y_2, \ldots, y_N)$ in time t.

$\approx$ Slater Determinant

- Exit probability :

$F^{(N)}(L; \mathbf{x}) =$ Prob. [The N walkers stay non-intersecting inside the box [0, L] till the first walker crosses the origin for the first time]

$$Q^{(N)}(L|\mathbf{x}) = \mathrm{Prob}[m_N \leq L|\mathbf{x}] = \frac{F_N(L; \mathbf{x})}{F_N(L \to \infty; \mathbf{x})}$$

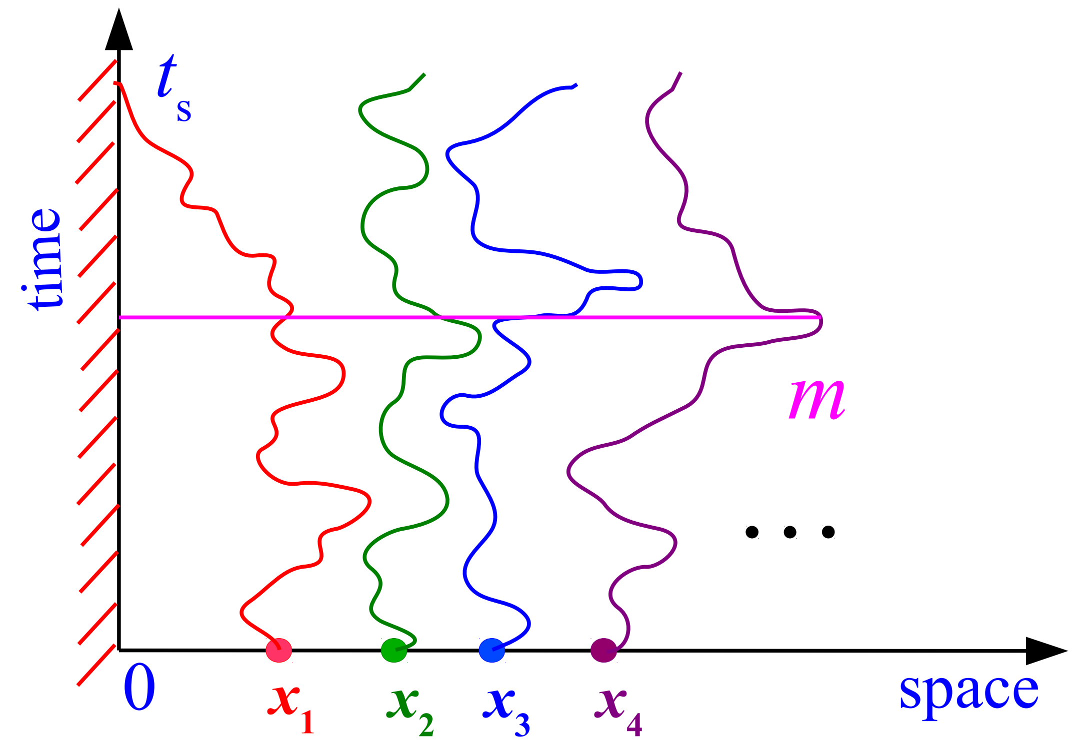
$$p^{(N)}(m|\mathbf{x}) = \left(\frac{\partial Q^{(N)}(\mathbf{x}|L)}{\partial L}\right)\Bigg|_{L=m} \simeq \mathcal{B}_N(\mathbf{x})\ \frac{1}{m^{N^2+1}}$$

Heuristic argument

- First passage time probability distribution :

$$f^{(N)}(t_s)|_{\text{large } t_s} \simeq \frac{1}{t_s^{\frac{N^2}{2}+1}}$$

Fisher 1984
Krattenthaler et al 2000
Bray, Winkler , 2004



- $\langle m \rangle \sim \sqrt{t\ N}$;

- $\sqrt{\langle \Delta m^2 \rangle}$ Decreases as N increases

$$\langle m \rangle \sim 2\sqrt{t \log N}$$

$$\sqrt{\langle \Delta m^2 \rangle} \sim \frac{\sqrt{t}}{\sqrt{\log N}}$$

Independent walkers

Heuristic argument

- First passage time probability distribution :

$$f^{(N)}(t)|_{\text{large } t} \simeq \frac{1}{t^{\frac{N^2}{2}+1}}$$

$$\simeq \frac{1}{t^{\frac{N}{2}+1}}$$

Independent walkers

$$\mu = N+1$$

- $$m \sim \sqrt{t}$$

For large N

- $$p^{(N)}(m|\mathbf{x})|_{\text{large } m} \simeq \frac{1}{m^{\mu}};$$

$$\mu = N^2+1$$

Distribution

- $$p^{(N)}(m|\mathbf{x}) \simeq \mathcal{B}_N(\mathbf{x}) \, \frac{1}{m^{N^2+1}}; \; m \gg x_N$$

Independent walkers

$$p^{(N)}(m|\mathbf{x}) \simeq \mathcal{A}_N(\mathbf{x}) \, \frac{1}{m^{N+1}}$$

Kundu, Majumdar, Schehr (2014)

Prefactor

- $$p^{(N)}(m|\mathbf{x}) \simeq \mathcal{B}_N(\mathbf{x})\ \frac{1}{m^{N^2+1}};\ m \gg x_N$$

Where

$$\mathcal{B}_N(\mathbf{x}) = E_N\ \frac{\prod_{i=1}^N x_i \prod_{1\leq i<j\leq N}(x_j^2 - x_i^2)}{S_N(\mathbf{x})}$$

and $\quad S_N(\mathbf{x}) = F_N(L \to \infty; \mathbf{x})$

$$\Downarrow$$

Prob. $\Bigg[$ The N walkers stay non-intersecting till the first walker crosses the origin for the first time $\Bigg]$

Independent walkers

$$p^{(N)}(m|\mathbf{x}) \simeq \mathcal{A}_N(\mathbf{x})\ \frac{1}{m^{N+1}}$$

$$\mathcal{A}_N(\mathbf{x}) = K_N \prod_{i=1}^N x_i$$

$$K_N \quad \approx N \quad \left[\frac{4}{\pi}\ln N\right]^{\frac{N}{2}}$$

$$N \gg 1$$

Krapivsky, Majumdar, Rosso,(2010)

Kundu, Majumdar, Schehr (2014)

Prefactor

- $$p^{(N)}(m|\mathbf{x}) \simeq \mathcal{B}_N(\mathbf{x})\ \frac{1}{m^{N^2+1}};\ m \gg x_N$$

Where

$$\mathcal{B}_N(\mathbf{x}) = E_N\ \frac{\prod_{i=1}^{N} x_i \prod_{1\le i<j\le N}(x_j^2 - x_i^2)}{S_N(\mathbf{x})}$$

and $S_N(\mathbf{x}) = F_N(L \to \infty; \mathbf{x})$

$$S_2(x_1, x_2) = \frac{4}{\pi}\arctan\left(\frac{x_2}{x_1}\right) - 1$$

$$S_3(\mathbf{x}) = \{\Psi(x_1,x_2,x_3) - \Psi(x_1,x_3,x_2)\} + \{\Psi(x_2,x_3,x_1) - \Psi(x_2,x_1,x_3)\} + \{\Psi(x_3,x_1,x_2) - \Psi(x_3,x_2,x_1)\}$$

$\Psi(x,y,z)$ has an explicit expression

Independent walkers

$$p^{(N)}(m|\mathbf{x}) \simeq \mathcal{A}_N(\mathbf{x})\ \frac{1}{m^{N+1}}$$

$$\mathcal{A}_N(\mathbf{x}) = K_N \prod_{i=1}^{N} x_i$$

$$K_N \approx N \left[\frac{4}{\pi}\ln N\right]^{\frac{N}{2}}$$

$$N \gg 1$$

Krapivsky, Majumdar, Rosso, J. Phys. A (2010)

Kundu, Majumdar, Schehr (2014)

E_N for large N

- The constant E_N can be computed for any given N.

$$E_N = \left(\frac{(2/\pi)^{N/2}}{2\prod(2i-1)!} \right) \int_0^\infty d\tau \; \frac{1}{\tau^{\frac{N^2+N+4}{2}}} \int_0^1 dx \; \exp\left[-\frac{(x-2)^2}{2\tau} \right] \; (2-x)^3 \; \mathcal{K}_{N-2}(x,\tau)$$

$$\mathcal{K}_N(x,\tau)|_{N\to\infty} \simeq N^{\frac{N^2}{2}} \; \Theta\left(\frac{x^2}{4N} - \tau \right)$$

$$\Downarrow$$

$$E_N|_{N\to\infty} \approx \exp\left(\frac{N^2}{2} [\; \log N + o(\log N) \;] \right)$$

Kundu, Majumdar, Schehr (2014)

Remarks & summary : independent walkers

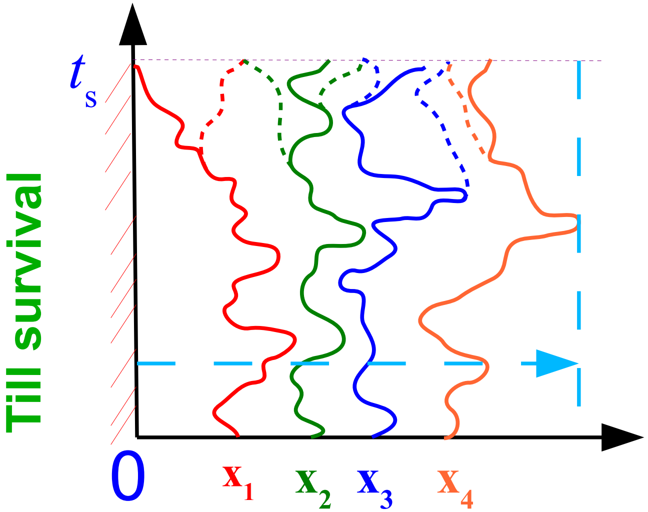
- Exact distribution of the number of distinct and common sites visited by N independent random walkers.

- Connection with extreme displacements => Exact limiting distributions for large N : $\mathcal{D}(x)$, $\mathcal{C}(y)$

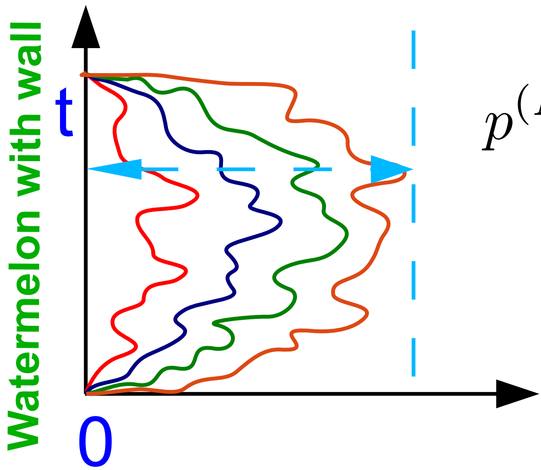
- Walkers moving in a globally bounded potential:

  $\mathcal{D}(x)$, $\mathcal{C}(y)$

Remarks & summary : Vicious walkers



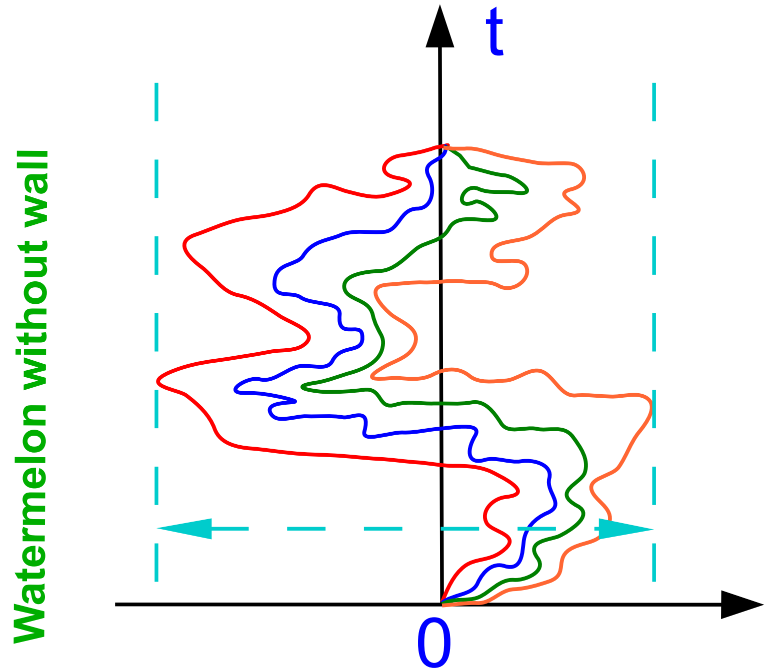
$$p^{(N)}(m|\mathbf{x}) \simeq \mathcal{B}_N(\mathbf{x})\ \frac{1}{m^{N^2+1}}; \quad m \gg x_N$$



$$\begin{aligned} p^{(N)}(m)|_{m\to\infty} &\simeq \sqrt{\frac{N}{2}}\Phi'\left(\frac{m}{\sqrt{2N}}\right)\mathrm{Exp}\left[-N\Phi\left(\frac{m}{\sqrt{2N}}\right)\right], \\ \Phi(x) &= \frac{16\sqrt{2}}{3}\,(x-1)^{3/2}. \end{aligned}$$

Schehr, Majumdar, Comtet, Forrester (2013)

Remarks & summary : Vicious walkers



- What about the distribution of the maximum displacement of the 1st walker ?

Thank You