Entropy production and Fluctuation theorems with odd-parity variables

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[PRL 110, 050602 (2013) + on-going]

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Talk at GGI, Florence (June 4, 2014)
Workshop on Advances in nonequilibrium statistical mechanics: large deviations and long-range correlations, extreme value statistics, anomalous transport and long-range interactions

Dynamic processes with odd-parity variables?

¶ underdamped Brownian dynamics with potential $V(\mathbf{x})$

$$\dot{\mathbf{x}} = \mathbf{v}$$

$$\dot{\mathbf{v}} = -\gamma \mathbf{v} + \boldsymbol{\xi} - \nabla \mathbf{V}(\mathbf{x})$$

$$\langle \boldsymbol{\xi}(t) \boldsymbol{\xi}^T(t') \rangle = 2D\mathsf{I}\delta(t-t')$$

$$\gamma = \beta D \text{ (Einstein relation)}$$

$\mathbf{v}$: odd-parity variable ($\mathbf{v} \to -\mathbf{v}$ under time-rev. op.)

$\mathbf{x}$: even-parity variable

¶ underdamped dynamics with general NEQ forces

$$\dot{\mathbf{v}} = -\gamma \mathbf{v} + \boldsymbol{\xi} - \nabla \mathbf{V}(\mathbf{x}; \lambda(t)) + \mathbf{f}_{nc}(\mathbf{x}) + \mathbf{g}(\mathbf{x}, \mathbf{v})$$

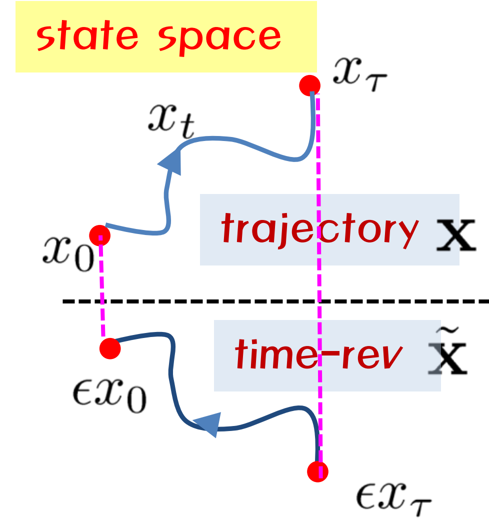
$\lambda(t)$: time-dep. protocol

$\mathbf{f}_{nc}(\mathbf{x})$: non-conserv. force like swirling force, nano-heat engine

$\mathbf{g}(\mathbf{x}, \mathbf{v})$: $\mathbf{v}$-dep. force in active matter dynamics, magnetic force

molecular refrigerator(cold damping), feedback control, ...

Stochastic process, Irreversibility & Total entropy production



¶ Dynamic trajectory in state space $(0 < t < \tau)$
with a set of state variables: $x = (s_1, s_2, \cdots)$

- under time-reversal operation: $s_i \to \epsilon_i s_i$ (ϵ_i : parity)
- odd-parity variable: $\epsilon_i = -1$ (momentum, ...)
  even-parity variable : $\epsilon_i = 1$ (position, ...)
- "time-reversed" (mirror) state : $\epsilon x = (\epsilon_1 s_1, \epsilon s_2, \cdots)$

¶ Irreversibility for a trajectory $\mathbf{x}$ (total entropy production)

$$\Delta S_{\rm tot}[\mathbf{x}] \;\; \mathrm{Irr} \;\; \equiv \ln \frac{\mathcal{P}[\mathbf{x}]}{\mathcal{P}[\tilde{\mathbf{x}}]}$$

$\mathcal{P}[\mathbf{x}]$:probability of traj. $\mathbf{x}$
$\tilde{\mathbf{x}}$: time-reversed traj.

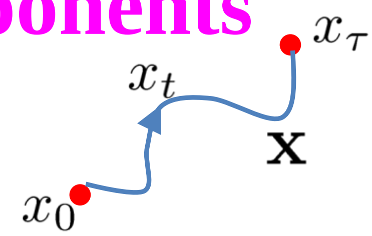
Time-reversed dynamics ??

[Sekimoto(1998)/Seifert(2005)]

- *integral* fluctuation theorem (FT) : automatic
  $\langle e^{-\Delta S_{\rm tot}} \rangle = \sum_{\mathbf{x}} \mathcal{P}[\mathbf{x}] e^{-\Delta S_{\rm tot}[\mathbf{x}]} = \sum_{\tilde{\mathbf{x}}} \mathcal{P}[\tilde{\mathbf{x}}] = 1$ (Jacobian $|\partial \tilde{\mathbf{x}} / \partial \mathbf{x}| = 1$).
  (valid for any finite-time "transient" process) $\langle \Delta S_{\rm tot} \rangle \geq 0$
- *detailed* fluctuation theorem (FT) : involution, i.c.-sensitive
  $P(\Delta S_{\rm tot}) / \tilde{P}(-\Delta S_{\rm tot}) = e^{\Delta S_{\rm tot}}$

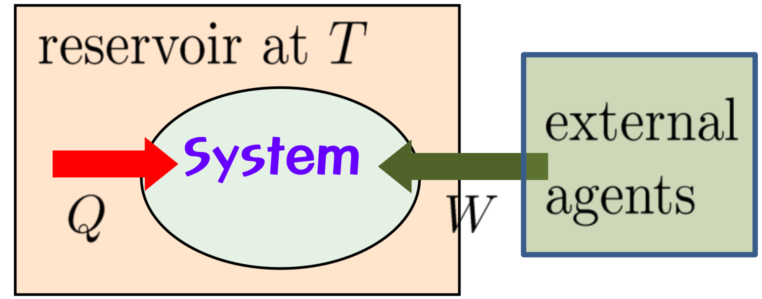
[Seifert(2005), Esposito/vdBroeck(2010)]

Total entropy production and its components



[A] $\Delta S_{\rm tot}[\mathbf{x}] = \Delta S_{\rm sys}[\mathbf{x}] + \Delta S_{\rm env}[\mathbf{x}]$

- $\Delta S_{\rm sys} = -\ln(p_{x_\tau}/p_{x_0})$ (Shannon entropy)
- $\Delta S_{\rm env} = \Delta S_{\rm res} = -\beta Q$ $\quad (\beta = 1/T)$



without measurement & feedback control

changing external agents (information entropy: Maxwell's demon)

- $\Delta S_{\rm sys}, \Delta S_{\rm res}$: not FT variables

[B] $\Delta S_{\rm tot}[\mathbf{x}] = \Delta S_{\rm hk}[\mathbf{x}] + \Delta S_{\rm ex}[\mathbf{x}]$

- $\Delta S_{\rm hk}$: EP to maintain the NESS [Hatano/Sasa(2001), Speck/Seifert(2005)]
- $\Delta S_{\rm ex}$: EP regarding transitions between steady states $(\lambda(t))$
- $\Delta S_{\rm ex}, \Delta S_{\rm hk}$: FT variables $\langle e^{-\Delta S_{\rm ex}}\rangle = 1, \langle e^{-\Delta S_{\rm hk}}\rangle = 1$ • 2nd laws
- $\Delta S_{\rm hk}$: adiabatic, $\Delta S_{\rm ex}$: non-adiabatic ($\Delta S_{\rm ex}$ vanishes in $\dot\lambda \to 0$ limit)

(mostly even-parity variable only: overdamped case) [Esposito/vdBroeck(2010)]

If odd-parity variables are introduced ???

[A] $\Delta S_{\rm env} = \Delta S_{\rm res} = -\beta Q$??

Not always !! $\Delta S_{\rm env} = \Delta S_{\rm res} + \Delta S_{\rm odd}$

[Kim/Qian(2004),CKwon/JYeo/HKLee/HP(2014)]

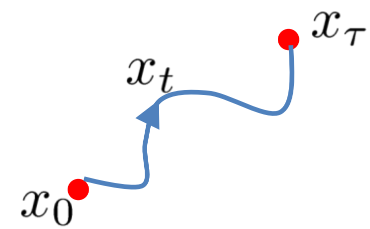
[B] $\Delta S_{\rm ex}, \Delta S_{\rm hk}$: FT variables ??

Yes & No !! $\Delta S_{\rm hk} = \Delta S_{\rm bDB} + \Delta S_{\rm as}$

not FT variable

[Spinney/Ford(2012),HKLee/CKwon/HP(2013)]

Langevin dynamics



¶ Brownian particle with (cons.+noncons.) force $f(x)$

$$\dot v = f(x) - \gamma v + \xi \quad (v = \dot x \ \& \ m = 1)$$

with $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$ and $\gamma = \beta D$ (Einstein relation)

- $\Delta S_{\rm tot}[\mathbf{x}] = \ln \dfrac{\mathcal{P}[\mathbf{x}]}{\mathcal{P}[\tilde{\mathbf{x}}]} = \ln \dfrac{p(x_0)\Pi(x_t)}{p(\tilde x_0)\Pi(\tilde x_t)}$

$\tilde x_t$: time-reverse path

$\Pi(x_t)$: conditional probability for path x_t

$p(\tilde x_0) = p(x_\tau)$

$$\Pi[x_t] \sim e^{-\int_0^\tau dt\left[\frac{1}{4D}(\dot v + \gamma v - f)^2 + \frac{1}{2}\partial_v(f - \gamma v)\right]}$$

(Onsager-Machlup)

(Stratonovich)

$$\Pi[\tilde x_t] \sim e^{-\int_0^\tau dt\left[\frac{1}{4D}(\dot v - \gamma v - f)^2 - \frac{1}{2}\gamma\right]}$$

$v \to -v$

$t \to \tau - t$

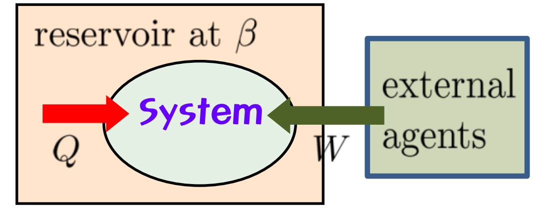
- $\ln \dfrac{\Pi(x_t)}{\Pi(\tilde x_t)} = -\dfrac{\gamma}{D}\displaystyle\int_0^\tau dt\ v(\dot v - f) = -\beta \int_0^\tau dt\ v(-\gamma v + \xi) = -\beta Q(x_t)$

$$= \Delta S_{\rm res}(x_t)$$

- $\ln \dfrac{p(x_0)}{p(\tilde x_0)} = \Delta S_{\rm sys}(x_t)$

- $\Delta S_{\rm tot}[\mathbf{x}] = \Delta S_{\rm sys}[\mathbf{x}] + \Delta S_{\rm res}[\mathbf{x}]$

[A] $\Delta S_{\rm env} = \Delta S_{\rm res}$??



¶ Brownian particle with velocity-dependent "external" force $g(x,v)$

$$\dot v = g(x,v) - \gamma v + \xi \quad (v = \dot x \ \& \ m = 1)$$
with $\langle \xi(t)\xi(t')\rangle = 2D\delta(t-t')$ and $\gamma = \beta D$ (Einstein relation)

$$g = g_e + g_o \text{ with } g_e(x,-v) = g_e(x,v) \ \& \ g_o(x,-v) = -g_o(x,v)$$

$$\mathcal{P}[\mathbf{x}] \sim p_{x_0} * e^{-\int_0^\tau dt[\frac{1}{4D}(\dot v + \gamma v - g_e - g_o)^2 + \frac{1}{2}\partial_v(g_e+g_o) - \frac{1}{2}\gamma]} * \delta(\dot x - v)$$

$$\mathcal{P}[\tilde{\mathbf{x}}] \sim p_{x_\tau} * e^{-\int_0^\tau dt[\frac{1}{4D}(\dot v - \gamma v - g_e + g_o)^2 + \frac{1}{2}\partial_v(-g_e+g_o) - \frac{1}{2}\gamma]} * \delta(\dot x - v)$$

$v \to -v$
$t \to \tau - t$

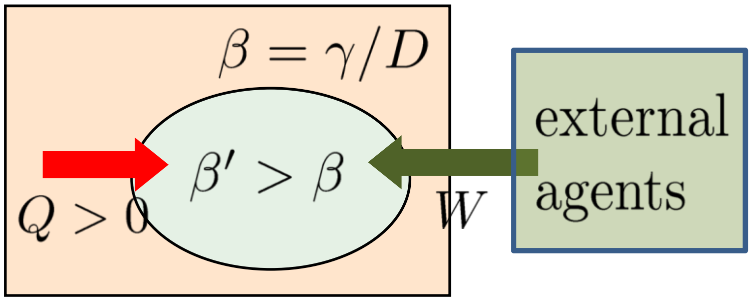
$$\Delta S_{\rm env} = -\int_0^\tau dt[\frac{1}{D}(\dot v - g_e)(\gamma v - g_o) + \partial_v g_e]$$
$$= -\int_0^\tau dt[\frac{1}{D}(-\gamma v + \xi)\gamma v - g_o(\dot v - g_e - \gamma v) + \partial_v g_e]$$
$$= -\beta Q + \int_0^\tau dt[\frac{1}{D} g_o(\dot v - g_e - \gamma v) - \partial_v g_e] = \Delta S_{\rm res} + \Delta S_{\rm odd}$$

¶ Simple example of cold damping with $g(x,v) = -\gamma' v$ $(\gamma' > 0)$

$$\dot{v} = -\gamma' v - \gamma v + \xi$$

- steady state distribution = EQ Boltzmann dis. with $\beta' = (\gamma' + \gamma)/D$

$$\sim e^{-\frac{1}{2}\beta' v^2}$$



$$\Delta S_{\rm tot} = \Delta S_{\rm env} \neq \Delta S_{\rm res} = -\beta Q < 0 \ (!)$$

$$\Delta S_{\rm odd} = \frac{1}{D}\int_0^\tau dt g_o(\dot{v} - g_e - \gamma v) = \beta Q - \beta' \Delta E$$

$$\langle \Delta S_{\rm tot}\rangle = \langle \Delta S_{\rm sys}\rangle + \langle \Delta S_{\rm res}\rangle + \langle \Delta S_{\rm odd}\rangle \geq 0$$

$$= \langle \Delta S_{\rm res}\rangle + \langle \Delta S_{\rm odd}\rangle = 0 \ ({\rm SS})$$

- cold damping as measure & feedback

$$\langle \Delta S_{\rm tot}\rangle = \langle \Delta S_{\rm res}\rangle + \langle \Delta S_{\rm odd}\rangle - \langle \Delta I\rangle > 0 \ ({\rm SS})$$

- No cancellation for general g_o or g_e (NEQ processes)

- $\dot{v} = g_e + g_o - \gamma v + \xi$

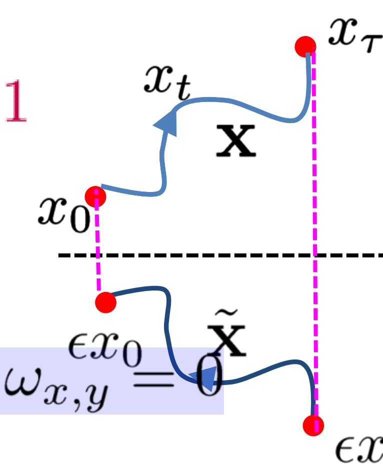
no v-dep. force

$\Delta S_{\rm odd} = 0$ only when no velocity-dependent force ($g_o = 0$ and $\partial_v g_e = 0$), but what is this extra odd term?? physical meaning? (not yet clear)

- entropy pumping [Kim/Qian(2004/2007),Ito/Sano(2011),Munakata/Rosinberg(2012)]

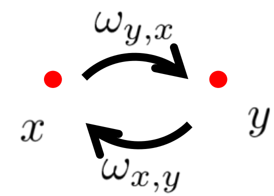
Markovian jump dynamics

¶ Markovian jump processes with odd-parity variables $\epsilon_i = -1$



$\dot{p}_x(t) = \sum_y \omega_{x,y}(\lambda_t) p_y(t)$ λ_t: time-dependent protocol.

$\omega_{x,y}$: transition rate for $y \to x$ for $x \neq y$



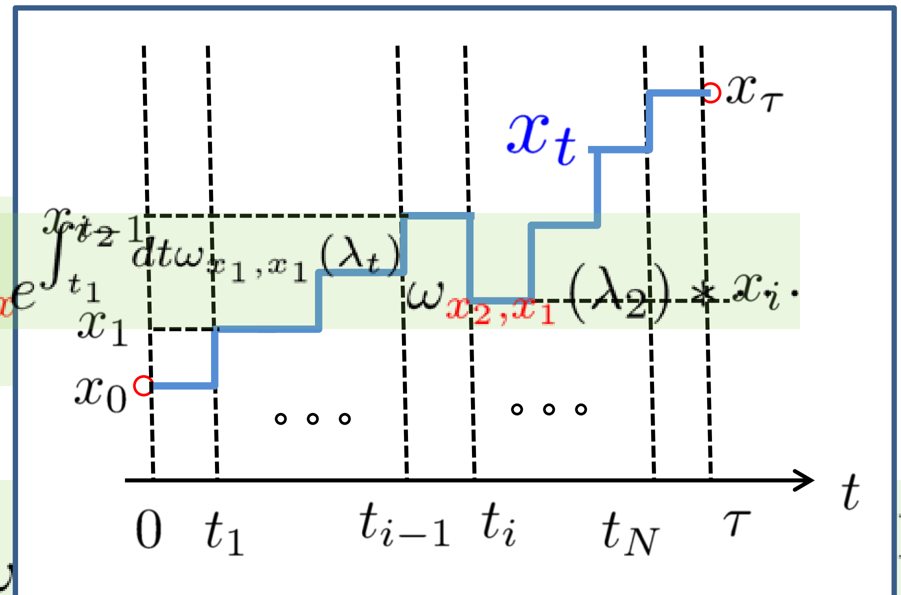
$\omega_{y,y} = -\sum_{x \neq y} \omega_{x,y} (< 0)$ $-$ escaping rate $\sum_x \omega_{x,y} = 0$

- $\Delta S_{\rm tot}[\mathbf{x}] = \ln \dfrac{\mathcal{P}_\omega[\mathbf{x}]}{\mathcal{P}_\omega[\tilde{\mathbf{x}}]}$ $[\tilde{\mathbf{x}}(t) = \epsilon \mathbf{x}(\tau - t)]$

N jump transitions at $(t_1, \cdots, t_N)$

$\dot{p}_{\tilde{x}}(t) = \sum_{\tilde{y}} \omega_{\tilde{x},\tilde{y}}(\tilde{\lambda}_t) p_{\tilde{y}}(t)$ $(\ \tilde{\lambda}_t = \lambda_{\tau - t})$

with $p_{\tilde{x}}(0) = p_x(t = \tau)$ $\dot{p}_x = \omega_{x,x} p_x$



$$\mathcal{P}_\omega[\mathbf{x}] \sim p_{x_0} * \left(\prod_{i=1}^{N} e^{\int_{t_{i-1}}^{t_i} dt\, \omega_{x_{i-1},x_{i-1}}(\lambda_t)} \omega_{x_i,x_{i-1}}(\lambda_i) \right) \cdots$$

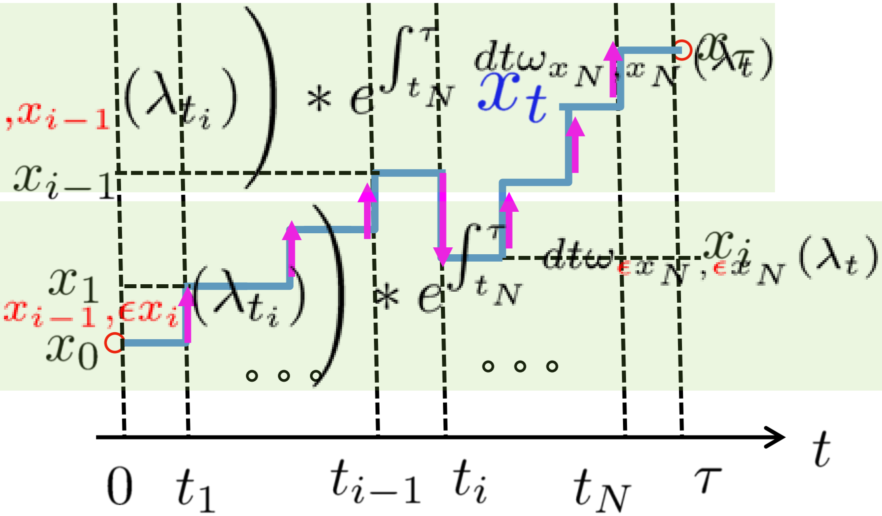
$$\mathcal{P}_\omega[\mathbf{x}] \sim p_{x_0} * e^{\int_0^{t_1} dt\, \omega_{x_0,x_0}(\lambda_t)} \omega_{x_1,x_0}(\lambda_1) * e^{\int_{t_1}^{t_2} dt\, \omega_{x_1,x_1}(\lambda_t)} \omega_{x_2,x_1}(\lambda_2) * \cdots$$

- time-reversed path with $\lambda_t \to \lambda_{\tau - t}$

$$\mathcal{P}_\omega[\tilde{\mathbf{x}}] \sim p_{x_\tau} * \left(\prod_{i=1}^{N} e^{\int_{t_{i-1}}^{t_i} dt\, \omega_{\epsilon x_{i-1},\epsilon x_{i-1}}(\lambda_t)} \omega \cdots \right)$$

$$\mathcal{P}_\omega[\mathbf{x}] \sim p_{x_0} * \left(\prod_{i=1}^{N} e^{\int_{t_{i-1}}^{t_i} dt\,\omega_{x_{i-1},x_{i-1}}(\lambda_t)}\, \omega_{x_i,x_{i-1}}(\lambda_{t_i})\right) * e^{\int_{t_N}^{\tau} dt\,\omega_{x_N,x_N}(\lambda_t)}$$

$$\mathcal{P}_\omega[\tilde{\mathbf{x}}] \sim p_{x_\tau} * \left(\prod_{i=1}^{N} e^{\int_{t_{i-1}}^{t_i} dt\,\omega_{\epsilon x_{i-1},\epsilon x_{i-1}}(\lambda_t)}\, \omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i})\right) * e^{\int_{t_N}^{\tau} dt\,\omega_{\epsilon x_N,\epsilon x_N}(\lambda_t)}$$



- $$\Delta S_{\rm tot}[\mathbf{x}] = \ln \frac{\mathcal{P}_\omega[\mathbf{x}]}{\mathcal{P}_\omega[\tilde{\mathbf{x}}]}$$

$$= \ln \frac{p_{x_0}}{p_{x_\tau}} + \sum_i \ln \frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\,[\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t)]$$

$= \Delta S_{\rm sys}$

$= \Delta S_{\rm res} = -\beta Q$

???

(Schnakenberg, 1976)

overdamped

$= \Delta S_{\rm odd}$

$$[\mathbf{A}]\quad \Delta S_{\rm tot} = \Delta S_{\rm sys} + \Delta S_{\rm res} + \Delta S_{\rm odd}$$

$$\Delta S_{\rm odd} = 0 \text{ when } \omega_{x,x} = \omega_{\epsilon x,\epsilon x}.$$

?

$\langle \Delta S_{\rm tot}\rangle \geq 0$ but $\langle \Delta S_{\rm sys}\rangle + \langle \Delta S_{\rm res}\rangle \not\geq 0$

[B] $\Delta S_{\rm ex}, \Delta S_{\rm hk}$: FT variables ?? (Lee/Kwon/HP, 2013)

¶ $\Delta S_{\rm tot}[\mathbf{x}] = \Delta S_{\rm ex}[\mathbf{x}] + \Delta S_{\rm hk}[\mathbf{x}]$

- $\Delta S_{\rm ex}$: EP regarding transitions between steady states $p_x^s(\lambda_t)$
  $0 = \sum_x \omega_{y,x}(\lambda_t) p_x^s(\lambda_t)$ (non-adiabatic EP)
  - $\Delta S_{\rm ex} = 0$ when $\dot\lambda = 0$ and start with NESS. (adiabatic limit)
- $\Delta S_{\rm hk}$: EP to maintain the NESS (adiabatic EP)
  - $\Delta S_{\rm hk} = 0$ for EQ (reversible) processes.

$$\Delta S_{\rm tot}[\mathbf{x}] = \ln\frac{p_{x_0}}{p_{x_\tau}} + \sum_i \ln\frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\,[\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t)]$$

$$= \ln\frac{p_{x_0}}{p_{x_\tau}} + \sum_i \ln\frac{p^s_{x_i}(\lambda_{t_i})}{p^s_{x_{i-1}}(\lambda_{t_i})} \equiv \Delta S_{\rm ex} = \Delta S_{\rm sys} - \beta Q_{\rm ex}$$

$$+ \sum_i \ln\frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i}) p^s_{x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i}) p^s_{x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\,[\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t)]$$

$$\equiv \Delta S_{\rm hk} = \beta Q_{\rm hk} ?$$

¶ Excess EP: $\Delta S_{\rm ex}$ $\ln\frac{\mathcal{P}_\omega[\mathbf{x}]}{\mathcal{P}_\omega[\tilde{\mathbf{x}}]} = \ln\frac{p_{x_0}}{p_{x_\tau}} + \sum_i \ln\frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\,[\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t)]$

FT ??

- $\star$-process: $\omega^\star_{x,y} = \omega_{y,x}\left(\frac{p^s_x}{p^s_y}\right)$ with $\hat{\mathbf{x}}(t) = \mathbf{x}(\tau - t)$ $\neq \epsilon\mathbf{x}(\tau - t)$

$$\ln\frac{\mathcal{P}_\omega[\mathbf{x}]}{\mathcal{P}_{\omega^\star}[\hat{\mathbf{x}}]} = \ln\frac{p_{x_0}}{p_{x_\tau}} + \sum_i \ln\frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i})}{\omega^\star_{x_{i-1},x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\,[\omega_{x_i,x_i}(\lambda_t) - \omega^\star_{x_i,x_i}(\lambda_t)]$$

$$= \ln\frac{p_{x_0}}{p_{x_\tau}} + \sum_i \ln\frac{p^s_{x_i}(\lambda_{t_i})}{p^s_{x_{i-1}}(\lambda_{t_i})} = \Delta S_{\rm ex} = \Delta S_{\rm sys} - \beta Q_{\rm ex}$$

- Stochasticity: $\sum_x \omega^\star_{x,y} = 0$
- IFT automatic: $\langle e^{-\Delta S_{\rm ex}}\rangle = 1$ $\Rightarrow$ $\langle \Delta S_{\rm ex}\rangle \geq 0$

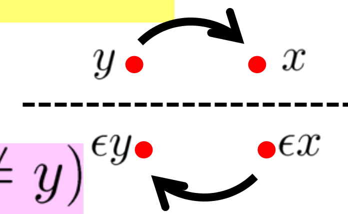
$$\langle e^{-\Delta S_{\rm ex}}\rangle = \int d\mathbf{x}\;\mathcal{P}_\omega(\mathbf{x}) e^{-\Delta S_{\rm ex}(\mathbf{x})} = \int d\mathbf{x}\;\mathcal{P}_\omega(\mathbf{x})\frac{\mathcal{P}_{\omega^\star}(\hat{\mathbf{x}})}{\mathcal{P}_\omega(\mathbf{x})} = 1$$

- $\langle \Delta S_{\rm sys}\rangle \geq \beta Q_{\rm ex}$ $\Leftrightarrow$ Clausis 2nd law : $\langle \Delta S_{\rm sys}\rangle \geq \beta Q$ (equilibrium SS)
- Equality holds in the adiabatic (quasi-static) limit : $\langle \Delta S_{\rm ex}\rangle = 0$

¶ House-keeping EP: $\Delta S_{\rm hk}$ $\ln\frac{\mathcal{P}_\omega[\mathbf{x}]}{\mathcal{P}_\omega[\tilde{\mathbf{x}}]} = \ln\frac{p_{x_0}}{p_{x_\tau}} + \sum_i \ln\frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\,[\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t)]$

$$\Delta S_{\rm hk} = \sum_i \ln\frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i})p^s_{x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i})p^s_{x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\,[\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t)]$$

- $\Delta S_{\rm hk} = 0$ for EQ (reversible) processes.



⋆ EQ conditions ?? (time-reversal sym)

- detailed balance : $\omega_{x,y}p^s_y = \omega_{\epsilon y,\epsilon x}p^s_{\epsilon x} (x \neq y)$ ⇒ $\omega_{x,x}p^s_x = \omega_{\epsilon x,\epsilon x}p^s_{\epsilon x}$
- mirror symmetry of SSD : $p^s_x = p^s_{\epsilon x}$

$\Big\}$ $\omega_{x,x} = \omega_{\epsilon x,\epsilon x}$ ⇒ $\Delta S_{\rm hk} = 0$

¶ EP due to breakage of DB: $\Delta S_{\rm bDB}$

Spinney/Ford: $\omega^\dagger_{x,y} = \omega_{\epsilon y,\epsilon x}\frac{p^s_{\epsilon x}}{p^s_{\epsilon y}}$

- †-process: $\omega^\dagger_{x,y} = \omega_{\epsilon y,\epsilon x}\frac{p^s_{\epsilon x}}{p^s_y}$ With DB satisfied, $\omega^\dagger_{x,y} = \omega_{x,y}$

$$\Delta S_{\rm bDB} \equiv \ln\frac{\mathcal{P}_\omega[\mathbf{x}]}{\mathcal{P}_{\omega^\dagger}[\mathbf{x}]} = \sum_i \ln\frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i})}{\omega^\dagger_{x_i,x_{i-1}}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\,[\omega_{x_i,x_i}(\lambda_t) - \omega^\dagger_{x_i,x_i}(\lambda_t)]$$

$$= \sum_i \ln\frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i})p^s_{x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i})p^s_{\epsilon x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt\left[\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t)\frac{p^s_{\epsilon x_i}(\lambda_{t_i})}{p^s_{x_i}(\lambda_{t_i})}\right]$$

- IFT automatic: $\langle e^{-\Delta S_{\rm bDB}}\rangle = 1$ $\langle \Delta S_{\rm bDB}\rangle \geq 0$ $(\sum_x \omega^\dagger_{x,y} = 0)$
- Equality holds for processes satisfying DB : $\langle \Delta S_{\rm bDB}\rangle = 0$ (no dynamic limit)

¶ Leftover EP: $\Delta S_{\rm as} \equiv \Delta S_{\rm hk} - \Delta S_{\rm bDB}$

$$\Delta S_{\rm hk} = \sum_i \ln \frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i}) p^s_{x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i}) p^s_{x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt \, [\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t)]$$

$$\Delta S_{\rm bDB} = \sum_i \ln \frac{\omega_{x_i,x_{i-1}}(\lambda_{t_i}) p^s_{x_{i-1}}(\lambda_{t_i})}{\omega_{\epsilon x_{i-1},\epsilon x_i}(\lambda_{t_i}) p^s_{\epsilon x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt \left[\omega_{x_i,x_i}(\lambda_t) - \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t) \frac{p^s_{\epsilon x_i}(\lambda_{t_i})}{p^s_{x_i}(\lambda_{t_i})}\right]$$

$$\Rightarrow \quad \Delta S_{\rm as} = \sum_i \ln \frac{p^s_{\epsilon x_i}(\lambda_{t_i})}{p^s_{x_i}(\lambda_{t_i})} + \sum_i \int_{t_{i-1}}^{t_i} dt \; \omega_{\epsilon x_i,\epsilon x_i}(\lambda_t) \left[\frac{p^s_{\epsilon x_i}(\lambda_t)}{p^s_{x_i}(\lambda_t)} - 1\right]$$

- $\Delta S_{\rm as} = 0$ with mirror symmetry of SSD ($p^s_x = p^s_{\epsilon x}$).

⋆ EP for broken mirror symmetry of SSD

- $\langle e^{-\Delta S_{\rm as}} \rangle \neq 1$ $\qquad$ $\langle \Delta S_{\rm as} \rangle \not\geq 0$

¶ House-keeping entropy: $\Delta S_{\rm hk} = \Delta S_{\rm bDB} + \Delta S_{\rm as}$

- $\langle e^{-\Delta S_{\rm hk}} \rangle \neq 1$ $\qquad$ $\langle \Delta S_{\rm hk} \rangle \not\geq 0$

If odd-parity variables are introduced ???

[A] $\Delta S_{\mathrm{env}} = \Delta S_{\mathrm{res}}$?? Not Always !!

$\Delta S_{\mathrm{env}} = \Delta S_{\mathrm{res}} + \Delta S_{\mathrm{odd}}$

- $g(v)$

In the steady state, $\langle \Delta S_{\mathrm{env}} \rangle \geq 0$ $\langle \Delta S_{\mathrm{res}} \rangle \not\geq 0$

- mirror symmetry of decaying rate $(\omega_{x,x} = \omega_{\epsilon x,\epsilon x}) \overset{?}{\Longrightarrow} \Delta S_{\mathrm{odd}} = 0$

[B] $\Delta S_{\mathrm{ex}}, \Delta S_{\mathrm{hk}}$: FT variables ?? Yes & No !!

ΔS_{ex}: FT ΔS_{hk}: not FT in general $(\ p_x^s \neq p_{\epsilon x}^s)$

$\Delta S_{\mathrm{hk}} = \Delta S_{\mathrm{bDB}} + \Delta S_{\mathrm{as}}$ $\langle \Delta S_{\mathrm{hk}} \rangle$ can be negative.

ΔS_{bDB}: FT ΔS_{as}: not FT

- mirror symmetry of SSD $(p_x^s = p_{\epsilon x}^s) \Longrightarrow \Delta S_{\mathrm{as}} = 0$