Large deviation in single-file diffusion

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A joint work with P. L. Krapivsky and Kirone Mallick

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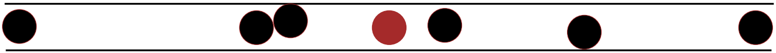
- 2390 published articles containing the word “single-file diffusion”; 69 published in this year

[Google scholar]

Tagged particle



- Motion of individual particles is sub-diffusive

$$\langle x_T^2 \rangle_{\text{single-file}} \simeq \mathcal{D}\sqrt{T} \qquad\qquad \langle x_T^2 \rangle_{\text{normal}} \simeq 2DT.$$

- Observed in experiment:
  - Transport of large molecules in Zeolite
    [Kukla et al, 1996]
  - Colloids in 1d channels
    [Wei et al, 2000][Lutz et al, 2004][Lin et al,2005]
  - Water molecules in carbon nanotube.
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theoretical studies

At large times the probability of tracer position is asymptotically Gaussian with variance $\mathcal{D}\sqrt{T}$

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- Colloidal system: [Kollmann, 2003]

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- Colloidal system: [Kollmann, 2003]
- Dependence on initial condition:[Leibovich et al, 2013]

$$\langle X^2 \rangle_{annealed} \neq \langle X^2 \rangle_{quenched}$$

Our work

What about higher cumulants? Any general framework?

Our work

- a general approach: macroscopic fluctuation theory.
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  - at large time T, cumulants

$$\langle X_T^n \rangle_c \simeq k_n \sqrt{T}$$

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depends on system

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  - $$Prob_T(X) \asymp e^{-\sqrt{T}\, \phi\left(\frac{X}{\sqrt{T}}\right)}$$

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large deviation function

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  - compute $\phi(\xi)$ for point particles
- An exact microscopic calculation for point particles.

A macroscopic approach

A macroscopic approach

- Continuous space: 
- On lattice (SEP): 

No bias

Infinite system size

At a hydrodynamic scale

$$\partial_t \, \rho(x,t) = \partial_x \left[\, D(\rho)\partial_x \rho(x,t) + \sqrt{\sigma(\rho)} \; \eta(x,t) \, \right]$$

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density

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white noise
$\langle \eta(x,t)\eta(x',t') \rangle = \delta(x-x')\delta(t-t')$

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mobility

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- For point particles: $D(\rho) = 1$ and $\sigma(\rho) = 2\rho$

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- For point particles: $D(\rho) = 1$ and $\sigma(\rho) = 2\rho$
- For SEP: $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$

tracer position

How is tracer position linked to density field $\rho(x,t)$?

No particle can cross each other $\Longrightarrow$ Number of particles on right of tracer is unchanged.

$$\boxed{\int_{X_T}^{\infty} \rho(z,T)dz = \int_0^{\infty} \rho(z,0)dz}$$

X_T is a functional of the density profile $\rho(x,t)$.

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How to compute large deviation function?

step 1 cumulant generating function

$$\mu(\lambda, T) = \log\left[\langle e^{\lambda X_T}\rangle\right]$$

$$\mu(\lambda, T) = \langle X_T\rangle_c\, \lambda + \frac{1}{2}\langle X_T^2\rangle_c\, \lambda^2 + \cdots$$

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first cumulant

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How to compute large deviation function?

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second cumulant

$$\mu(\lambda, T) = \log\left[\langle e^{\lambda X_T}\rangle\right]$$

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tracer position

How is tracer position linked to density field $\rho(x, t)$?

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step 2 Legendre transform $\mu(\lambda, T)$ to get large deviation function

Variational formulation

Macroscopic fluctuation theory:[Bertini et al,2001]

$$\langle e^{\lambda X_T} \rangle = \int \mathcal{D}[\rho, \hat{\rho}] \exp \left\{ \lambda X_T[\rho] - F[\rho(x,0)] - \int_0^T dt \int_{-\infty}^{\infty} dx \, [\hat{\rho} \partial_t \rho - H] \right\}$$

Let $X_T[q] = Y$ (the tracer position for least action path)

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Path integral
within time $[0, T]$

Let $X_T[q] = Y$ (the tracer position for least action path)

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$$\mathcal{P}[\rho(x,0)] = e^{-F[\rho(x,0)]}$$

for quench $F = 0$
for annealed $F =$free energy

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Variational formulation

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prob of each path

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$$H = \frac{\sigma(\rho)}{2} (\partial_x \hat{\rho})^2 - D(\rho)(\partial_x \rho)(\partial_x \hat{\rho})$$

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Action

$$\langle e^{\lambda X_T}\rangle = \int \mathcal{D}\,[\rho,\hat{\rho}]\,e^{-S[\rho,\hat{\rho},\lambda,T]}$$

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Action

$$\langle e^{\lambda X_T} \rangle = \int \mathcal{D}\,[\rho, \hat{\rho}]\, e^{-S[\rho, \hat{\rho}, \lambda, T]}$$

At large time, contribution from paths of least action $(\rho, \hat{\rho}) \equiv (q, p)$.

$$\mu(\lambda, T) = \log\left[\langle e^{\lambda X_T} \rangle\right] = -S[q, p, \lambda, T].$$

Let $X_T[q] = Y$ (the tracer position for least action path)

Least action

Paths of least action

$$\begin{aligned}\partial_t q &= \partial_x \left[D(q)\partial_x q - \sigma(q)\partial_x p\right]\\ \partial_t p &= -D(q)\partial_{xx} p - \frac{\sigma'(q)}{2}\left(\partial_x p\right)^2\end{aligned}$$

Boundary condition

Least action

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Boundary condition

annealed (initial density fluctuating)

$$\begin{aligned}p(x,T) &= \frac{\lambda}{q(Y,T)}\theta(x-Y)\\ p(x,0) &= \frac{\lambda}{q(Y,T)}\theta(x) + \frac{\delta F}{\delta q(x,0)}\end{aligned}$$

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from $\frac{\delta X_T}{\delta q(x,T)}$

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quenched (initial density fixed)

$$\begin{aligned}p(x,T) &= \frac{\lambda}{q(Y,T)}\theta(x-Y) \\ q(x,0) &= \rho\end{aligned}$$

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uniform initial density

Annealed

- least action equations

$$
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\end{aligned}
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- boundary condition

from $\frac{\delta F[q]}{\delta q(x,0)}$

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- cumulant generating function

$$\mu(\lambda,T) = \lambda Y - \int_{-\infty}^{\infty} dx \int_{\rho}^{q(x,0)} dr \frac{2D(r)}{\sigma(r)} \left(q(x,0) - r\right) - \int_0^T dt \int_{-\infty}^{\infty} dx \frac{\sigma(q)}{2} \left(\partial_x p\right)^2$$

- least action equations

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from $F[q(x,0)]$

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- cumulant generating function

$$\mu(\lambda) = \lambda Y - \int_0^T dt \int_{-\infty}^{\infty} dx \frac{\sigma(q)}{2}\left(\partial_x p\right)^2$$

What can be extracted without solving?

- $\mu(\lambda, T)$ is even in λ.

$$\mu(\lambda, T) = \frac{\lambda^2}{2!}\langle X_T^2\rangle_c + \frac{\lambda^4}{4!}\langle X_T^4\rangle_c + \cdots$$

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$$Prob\left(\frac{X}{\sqrt{T}} = \xi\right) \asymp \exp\left[-\sqrt{T}\phi\left(\xi\right)\right]$$

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$$Prob\left(\frac{X}{\sqrt{T}} = \xi\right) \asymp \exp\left[-\sqrt{T}\phi\left(\xi\right)\right]$$

  $\phi(\xi)$ is an even function.
  - At large T, using Taylor expansion of $\phi(\xi)$,

$$Prob(X) \asymp \exp\left[-\frac{X^2}{\mathcal{D}\sqrt{T}}\right]$$

‘

An exact solution

Point particles with hard core repulsion

$$D(\rho) = 1 \qquad \text{and} \qquad \sigma(\rho) = 2\rho$$

least action equations

$$\begin{aligned} \partial_t p + \partial_{xx} p &= -(\partial_x p)^2 \\ \partial_t q - \partial_{xx} q &= -\partial_x (2q\partial_x p) \end{aligned}$$

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boundary condition (annealed)

$$\begin{aligned} p(x,T) &= B\theta(x - Y) \qquad \text{where} \qquad B = \frac{\lambda}{q(Y,T)} \\ q(x,0) &= \rho \exp\left[p(x,0) - B\theta(x)\right] \end{aligned}$$

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treat B as parameter and solve it self-consistently

An exact solution

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how to solve?

Define $P = e^p$ and $Q = qe^{-p}$, then

$$\partial_t P + \partial_{xx} P = 0 \qquad \text{and} \qquad \partial_t Q - \partial_{xx} Q = 0$$

annealed

- Solution

$$
\begin{aligned}
p(x,t) &= \log\left[1+\left(e^{B}-1\right)\frac{1}{2}\,\mathrm{erfc}\left(\frac{Y-x}{\sqrt{4D(T-t)}}\right)\right] \\
q(x,t) &= \rho\left[1+\left(e^{-B}-1\right)\frac{1}{2}\,\mathrm{erfc}\left(\frac{x-Y}{\sqrt{4D(T-t)}}\right)\right] \\
&\quad \left[1+\left(e^{B}-1\right)\frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{4Dt}}\right)\right]
\end{aligned}
$$

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$$
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&\quad\left[1+\left(e^{B}-1\right)\frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{4Dt}}\right)\right]
\end{aligned}
$$

- What is Y?

$$
\frac{Y/\sqrt{4DT}}{\frac{\exp[-Y^2/4DT]}{\sqrt{4\pi}}-\frac{Y}{\sqrt{4DT}}\frac{1}{2}\operatorname{erfc}\left(\frac{Y}{\sqrt{4DT}}\right)} = e^{2B}-1
$$

annealed

- Solution

$$
\begin{aligned}
p(x,t) &= \log\left[1+\left(e^{B}-1\right)\frac{1}{2}\operatorname{erfc}\left(\frac{Y-x}{\sqrt{4D(T-t)}}\right)\right] \\
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\end{aligned}
$$

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$$

using $\int_Y^\infty dx q(x,T) = \int_0^\infty dx q(x,0)$

annealed

- Solution

$$
\begin{aligned}
p(x,t) &= \log\left[1+\left(e^{B}-1\right)\frac{1}{2}\operatorname{erfc}\left(\frac{Y-x}{\sqrt{4D(T-t)}}\right)\right]\\
q(x,t) &= \rho\left[1+\left(e^{-B}-1\right)\frac{1}{2}\operatorname{erfc}\left(\frac{x-Y}{\sqrt{4D(T-t)}}\right)\right]\\
&\quad \left[1+\left(e^{B}-1\right)\frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{4Dt}}\right)\right]
\end{aligned}
$$

- What is Y?

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\frac{Y/\sqrt{4DT}}{\frac{\exp[-Y^2/4DT]}{\sqrt{4\pi}}-\frac{Y}{\sqrt{4DT}}\frac{1}{2}\operatorname{erfc}\left(\frac{Y}{\sqrt{4DT}}\right)} = e^{2B}-1
$$

- Cumulant generating function

$$
\mu(\lambda) = \left[\lambda-\rho\frac{e^{B}-1}{e^{B}+1}\right]Y
$$

determine B

Definition $B = \lambda / q(Y, T)$

$q(Y, T)$ can not be determined self-consistently

determine B

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$$q(x,t) = \rho\left[1+\left(e^{-B}-1\right)\frac{1}{2}\,\mathrm{erfc}\left(\frac{x-Y}{\sqrt{4D(T-t)}}\right)\right]\left[1+\left(e^{B}-1\right)\frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{4Dt}}\right)\right]$$

$q(Y,T)$ can not be determined self-consistently

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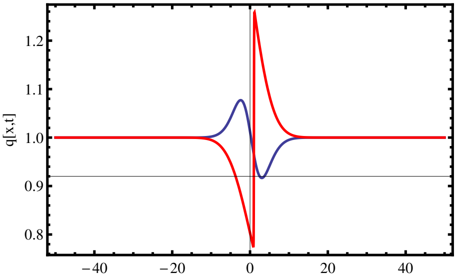
Definition $B = \lambda / q(Y, T)$

$$q(x,t) = \rho \left[1 + \left(e^{-B} - 1\right) \frac{1}{2} \operatorname{erfc}\left(\frac{x - Y}{\sqrt{4D(T-t)}}\right)\right] \left[1 + \left(e^{B} - 1\right) \frac{1}{2} \operatorname{erfc}\left(\frac{x}{\sqrt{4Dt}}\right)\right]$$

$q(x, T)$ is non-analytic at (Y, T)



$q(Y, T)$ can not be determined self-consistently

determine B

Determine B by minimizing the action [Baruch Meerson]

$$\mu_B(\lambda) = \lambda Y(B) + \frac{1 - e^B}{1 + e^B} Y(B).$$

This yields

$$\lambda = \rho(1 - e^{-B}) \left[1 + \left(e^B - 1 \right) \frac{1}{2} \operatorname{erfc} \left(\frac{x}{\sqrt{4DT}} \right) \right]$$

Annealed

Results (parametric form)

$$\mu(\lambda) = \sqrt{4DT}\,\rho\left[\lambda y + \left(e^{B} - 1\right)\alpha(y) + \left(e^{-B} - 1\right)\alpha(-y)\right]$$

$$y = \left(e^{B} - 1\right)\alpha(y) - \left(e^{-B} - 1\right)\alpha(-y)$$

$$\lambda = (1 - e^{-B})\left[1 + \left(e^{B} - 1\right)\frac{1}{2}\,\mathrm{erfc}(y)\right]$$

Results (parametric form)

$$\int_y^\infty \frac{1}{2}\,\mathrm{erfc}(z)\,dz$$

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$$y = \left(e^{B}-1\right)\alpha(y) - \left(e^{-B}-1\right)\alpha(-y)$$

$$\lambda = (1-e^{-B})\left[1 + \left(e^{B}-1\right)\frac{1}{2}\mathrm{erfc}(y)\right]$$

Large deviation function

$$\phi(y) = \rho\left[\sqrt{\alpha(y)} - \sqrt{\alpha(-y)}\right]$$

probability of tracer position

$$P\left[\frac{X_T}{\sqrt{4DT}} = y\right] \asymp e^{-\sqrt{4DT}\phi(y)}$$

Quenched

Results (parametric form)

$$\frac{\mu(\lambda)}{\rho\sqrt{4DT}} = \lambda y + \int_{-\infty}^{\infty} dz \left[\log\left\{ 1 + \left(e^{B} - 1\right) \frac{1}{2}\operatorname{erfc}(-z) \right\} - B e^{B} \frac{\frac{1}{2}\operatorname{erfc}(-z)}{1 + \left(e^{B} - 1\right)\frac{1}{2}\operatorname{erfc}(-z)} \right]$$

$$y = \left(e^{B} - 1\right) \int_{-\infty}^{\infty} dz \frac{\frac{1}{2}\operatorname{erfc}(z)\frac{1}{2}\operatorname{erfc}(-z)}{1 + \left(e^{B} - 1\right)\frac{1}{2}\operatorname{erfc}(-z)}$$

$$\lambda = \lambda(B, Y)$$

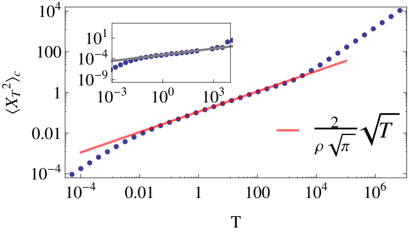
Annealed vs Quenched

- annealed

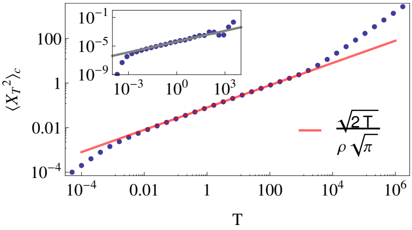
$$\langle X_T^2 \rangle_c = \frac{2}{\rho\sqrt{\pi}}\sqrt{T} \qquad \langle X_T^4 \rangle_c = \frac{6(4-\pi)}{(\rho\sqrt{\pi})^3}\sqrt{T}$$

- quenched

$$\langle X_T^2 \rangle_c = \frac{\sqrt{2}}{\rho\sqrt{\pi}}\sqrt{T} \qquad \langle X_T^4 \rangle_c = \frac{2\sqrt{2}}{\rho^3\sqrt{\pi}}\left[\frac{9}{\pi} arctan\left(\frac{1}{2\sqrt{2}}\right) - 1\right]\sqrt{T}$$



annealed



quenched

what about more general systems ?

Perturbative expansion

$$
\begin{aligned}
q(x,t) &= q_0(x,t) + \lambda q_1(x,t) + \lambda^2 q_2(x,t) + \cdots \\
p(x,t) &= p_0(x,t) + \lambda p_1(x,t) + \lambda^2 p_2(x,t) + \cdots \\
\mu(\lambda) &= \frac{1}{2!}\langle X_T^2\rangle_c \lambda^2 + \frac{1}{4!}\langle X_T^4\rangle_c \lambda^4 + \cdots
\end{aligned}
$$

Quenched case

- zeroth order

$$q_0(x,t) = \rho \qquad \text{and} \qquad p_0(x,t) = 0$$

- first order

$$
\begin{aligned}
\partial_t p_1 &= -D(\rho)\partial_{xx} p_1 \qquad \text{with} \qquad p_1(x,T) = \frac{1}{\rho}\theta(x) \\
\partial_t q_1 &= D(\rho)\partial_{xx} q_1 - \sigma(\rho)\partial_{xx} p_1 \qquad \text{with} \qquad q_1(x,0) = 0
\end{aligned}
$$

- cumulants

$$\frac{1}{2}\langle X_T^2\rangle = \frac{1}{\rho}\int_{-\infty}^{\infty} dx\,[q_1(x,T) - q_1(x,0)] - \frac{\sigma(\rho)}{2}\int_0^T dt \int_{-\infty}^{\infty} dx (\partial_x p_1)^2$$

- Quenched

$$\langle X_T^2 \rangle = \frac{\sigma(\rho)}{\rho^2} \frac{\sqrt{T}}{\sqrt{2\pi D(\rho)}}$$

- Annealed

$$\langle X_T^2 \rangle = \sqrt{2}\frac{\sigma(\rho)}{\rho^2} \frac{\sqrt{T}}{\sqrt{2\pi D(\rho)}}$$

Point particles

$$\frac{\langle X_T^2 \rangle_{ann}}{\sqrt{T}} = \boxed{\frac{2}{\rho}\frac{1}{\sqrt{\pi}}}$$

[Harris,1965]

SEP

$$\frac{\langle X_T^2 \rangle_{ann}}{\sqrt{T}} = \frac{2(1-\rho)}{\rho}\frac{1}{\sqrt{\pi}}$$

colloidal system

$$\frac{\langle X_T^2 \rangle_{ann}}{\sqrt{T}} = \frac{2S}{\rho^2}\sqrt{\frac{D}{\pi}}$$

results

- Quenched

$$\langle X_T^2 \rangle = \frac{\sigma(\rho)}{\rho^2} \frac{\sqrt{T}}{\sqrt{2\pi D(\rho)}}$$

- Annealed

$$\langle X_T^2 \rangle = \sqrt{2} \frac{\sigma(\rho)}{\rho^2} \frac{\sqrt{T}}{\sqrt{2\pi D(\rho)}}$$

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[Arratia,1983]

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colloidal system

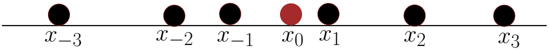
$$\frac{\langle X_T^2 \rangle_{ann}}{\sqrt{T}} = \boxed{\frac{2S}{\rho^2} \sqrt{\frac{D}{\pi}}}$$

[Kollman,2003]

Microscopic calculation

Point particles

Point particles



Diffusion in the sector $x_i < x_{i+1}$

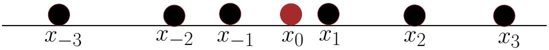
$$\partial_t P_t(\mathbf{X}|\mathbf{Y}) = D \sum_{i=-n}^{n} \partial_{x_i}^2 P_t(\mathbf{X}|\mathbf{Y})$$

reflecting boundary

$$\partial_{x_i} P_t(\mathbf{X}|\mathbf{Y}) = \partial_{x_{i+1}} P_t(\mathbf{X}|\mathbf{Y})$$

along $x_i = x_{i+1}$

Point particles



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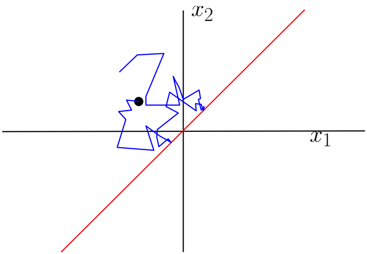
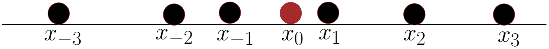


Point particles



Diffusion in the sector $x_i < x_{i+1}$

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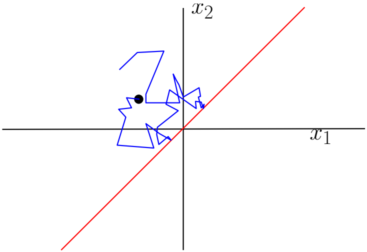


reflecting boundary

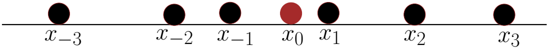
$$\partial_{x_i} P_t(\mathbf{X}|\mathbf{Y}) = \partial_{x_{i+1}} P_t(\mathbf{X}|\mathbf{Y})$$

along $x_i = x_{i+1}$

solution

$$\begin{aligned} P_t(\mathbf{X}|\mathbf{Y}) &= \sum_{\sigma} \prod_{j} g_t(x_j|y_{\sigma(j)}) \\ &= g_t(x_{-n}|y_{-n}) g_t(x_{-n+1}|y_{-n+1}) \cdots g_t(x_n|y_n) \\ &\quad + g_t(x_{-n}|y_{-n+1}) g_t(x_{-n+1}|y_{-n}) \cdots g_t(x_n|y_n) + \ldots \end{aligned}$$

Point particles



Diffusion in the sector $x_i < x_{i+1}$

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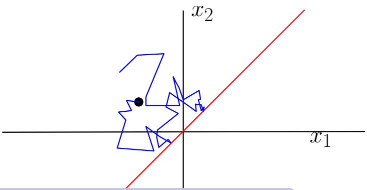

reflecting boundary

$$\partial_{x_i} P_t(\mathbf{X}|\mathbf{Y}) = \partial_{x_{i+1}} P_t(\mathbf{X}|\mathbf{Y})$$

along $x_i = x_{i+1}$

$$\frac{1}{\sqrt{4\pi D t}} \exp\left[\frac{-\left(x_j - y_{\sigma(j)}\right)^2}{4Dt}\right]$$

solution

$$\begin{aligned} P_t(\mathbf{X}|\mathbf{Y}) &= \sum_{\sigma} \prod_j g_t(x_j|y_{\sigma(j)}) \\ &= g_t(x_{-n}|y_{-n}) g_t(x_{-n+1}|y_{-n+1}) \cdots g_t(x_n|y_n) \\ &\quad + g_t(x_{-n}|y_{-n+1}) g_t(x_{-n+1}|y_{-n}) \cdots g_t(x_n|y_n) + \ldots \end{aligned}$$

tracer position

Let central particle is the tracer.

- Probability of tracer position

$$Prob_t(x_0|\mathbf{Y}) = \int_{-\infty}^{x_{-n+1}} dx_{-n} \cdots \int_{-\infty}^{x_0} dx_{-1} \int_{x_0}^{\infty} dx_1 \cdots \int_{x_{n-1}}^{\infty} dx_n P_t(\mathbf{X}|\mathbf{Y})$$

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- using combinatorial identities

$$Prob_t(x_0|\mathbf{Y}) = \sum_{k=-n}^{n} A_k[x_0, \mathbf{Y}, t] g_t(x_0|y_k)$$

$$A_k[x_0, \mathbf{Y}, t] = \sum_{s_{-n}=\pm 1} \cdots \sum_{s_{k-1}=\pm 1} \sum_{s_{k+1}=\pm 1} \cdots \sum_{s_n=\pm 1} \delta_{\sum_{q\neq k} s_q, 0} \prod_{j\neq k} \frac{1}{2} \operatorname{erfc}\left[s_j \frac{(x_0 - y_j)}{\sqrt{4Dt}} \right]$$

Averaging over initial positions

- annealed: Initially particles are distributed randomly in $[-L, L]$.

$$P(x_0) = \frac{n!n!}{L^{2n}} \int_{-L}^{y_{-n+1}} dy_{-n} \ldots \int_{-L}^{0} dy_{-1} \int_{0}^{L} dy_1 \ldots \int_{y_{n-1}}^{L} \sum_{k=-n}^{n} A_k[x_0, \mathbf{Y}, t] g_t(x_0|y_k)$$

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- at large n and L, with $n/L = \rho$

$$P_t(x_0) = \exp\left[-\rho\, R_t(x_0) - \rho R_t(-x_0)\right] I_0\left(2\rho\sqrt{R_t(x_0)R_t(-x_0)}\right) \Bigg[g(x_0|0) + 2\rho\, H_t(x_0)H_t(-x_0) + \rho\left\{H_t(x_0)^2\sqrt{\frac{R_t(x_0)}{R_t(-x_0)}} + H_t(-x_0)^2\sqrt{\frac{R_t(-x_0)}{R_t(x_0)}}\right\} \frac{I_2\left(2\rho\sqrt{R_t(x_0)R_t(-x_0)}\right)}{I_1\left(2\rho\sqrt{R_t(x_0)R_t(-x_0)}\right)}\Bigg]$$

Averaging over initial positions

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$$P(x_0) = \frac{n!n!}{L^{2n}} \int_{-L}^{y_{-n+1}} dy_{-n} \dots \int_{-L}^{0} dy_{-1} \int_{0}^{L} dy_1 \dots \int_{y_{n-1}}^{L} \sum_{k=-n}^{n} A_k[x_0, \mathbf{Y}, t] g_t(x_0|y_k)$$

$\int_{x_0}^{\infty} dz \frac{1}{2} \operatorname{erfc}\left(\frac{z}{\sqrt{4Dt}}\right)$

- at large n and L, with $n/L = \rho$

$$P_t(x_0) = \exp\left[-\rho R_t(x_0) - \rho R_t(-x_0)\right] I_0(2\rho\sqrt{R_t(x_0)R_t(-x_0)}) \Big[g(x_0|0) + 2\rho\, H_t(x_0)H_t(-x_0) + \rho\left\{H_t(x_0)^2\sqrt{\frac{R_t(x_0)}{R_t(-x_0)}} + H_t(-x_0)^2\sqrt{\frac{R_t(-x_0)}{R_t(x_0)}}\right\} \frac{I_2(2\rho\sqrt{R_t(x_0)R_t(-x_0)})}{I_1(2\rho\sqrt{R_t(x_0)R_t(-x_0)})}\Big]$$

Averaging over initial positions

- annealed: Initially particles are distributed randomly in $[-L, L]$.

$$P(x_0) = \frac{n!n!}{L^{2n}} \int_{-L}^{y_{-n+1}} dy_{-n} \dots \int_{-L}^{0} dy_{-1} \int_{0}^{L} dy_1 \dots \int_{y_{n-1}}^{L}$$

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$$+\rho \left\{ H_t(x_0)^2 \sqrt{\frac{R_t(x_0)}{R_t(-x_0)}} + H_t(-x_0)^2 \sqrt{\frac{R_t(-x_0)}{R_t(x_0)}} \right\} \frac{I_2\left(2\rho\sqrt{R_t(x_0) R_t(-x_0)}\right)}{I_1\left(2\rho\sqrt{R_t(x_0) R_t(-x_0)}\right)} \Bigg]$$

$\int_y^\infty dz \frac{1}{2} \operatorname{erfc}(z)$

at large times

$$P\left(\frac{x_0}{\sqrt{4Dt}} = y\right) \asymp \exp\left[-\sqrt{4Dt}\,\rho\,\left\{\sqrt{\alpha(y)} - \sqrt{\alpha(-y)}\right\}^2\right]$$

Same as obtained by macroscopic approach

last slide

- Take home message

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Thank you