# Simplified differential equations approach for NNLO calculations


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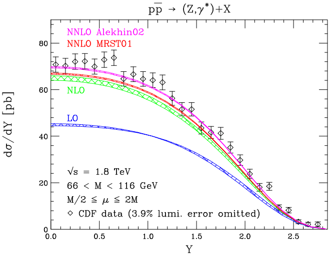
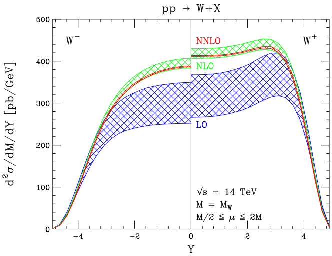
# QCD beyond LO

(N)NLO needed in order to properly interpret the data at the LHC



From Feynman Diagrams to recursive equations: taming the $n$!

- 1999 `HELAC`: The first code to calculate recursively tree-order amplitudes for (practically) arbitrary number of particles

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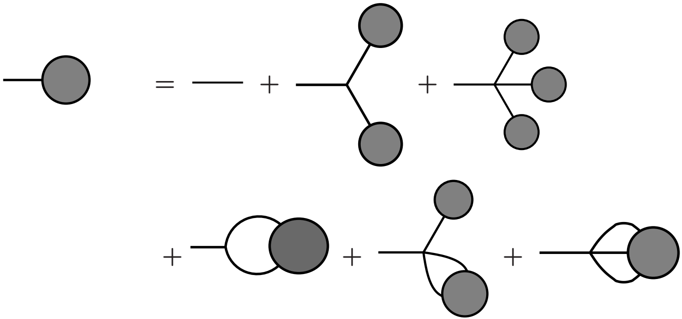
A. Kanaki and C. G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306 [arXiv:hep-ph/0002082].

F. A. Berends and W. T. Giele, Nucl. Phys. B **306** (1988) 759.

F. Caravaglios and M. Moretti, Phys. Lett. B **358** (1995) 332.




Unfortunately not so much on the second line !

What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m |M_m^{(0)}|^2 J_m(\Phi) \\
&+ \int_m d\Phi_m 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

$J_m(\Phi)$ jet function: Infrared safeness $J_{m+1} \to J_m$

# Perturbative QCD at NLO

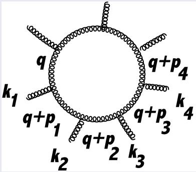
What do we need for an NLO calculation ?

$$p_1, p_2 \to p_3, \ldots, p_{m+2}$$

$$\begin{aligned}
\sigma_{NLO} &= \int_m d\Phi_m^{D=4}\left(|M_m^{(0)}|^2 + 2Re(M_m^{(0)*} M_m^{(CT)}(\epsilon_{UV}))\right) J_m(\Phi) \\
&+ \int_m d\Phi_m^{D=4} 2Re(M_m^{(0)*} M_m^{(1)}(\epsilon_{UV}, \epsilon_{IR})) J_m(\Phi) \\
&+ \int_{m+1} d\Phi_{m+1}^{D=4-2\epsilon_{IR}} |M_{m+1}^{(0)}|^2 J_{m+1}(\Phi)
\end{aligned}$$

IR and UV divergencies, Four-Dimensional-Helicity scheme; scale dependence $\mu_R$
QCD factorization$-\mu_F$ Collinear counter-terms when PDF are involved

Any $m$-point one-loop amplitude can be written as




$$\int d^D q \, A(\bar{q}) = \int d^D q \, \frac{\bar{N}(\bar{q})}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}}$$

A bar denotes objects living in $n = \; 4 + \epsilon$ dimensions

$$\bar{D}_i = (\bar{q} + p_i)^2 - m_i^2$$

$$\bar{q}^2 = q^2 + \tilde{q}^2$$

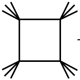
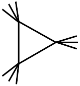
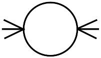
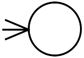
$$\bar{D}_i = D_i + \tilde{q}^2$$

# The one loop paradigm

basis of scalar integrals:

G. Passarino and M. J. G. Veltman, Nucl. Phys. B **160** (1979) 151.

Z. Bern, L. J. Dixon, D. C. Dunbar and D. A. Kosower, Nucl. Phys. B **425** (1994) 217 [arXiv:hep-ph/9403226].

$\mathcal{A} = \sum d_{i_1 i_2 i_3 i_4}$  $+ \sum c_{i_1 i_2 i_3}$  $+ \sum b_{i_1 i_2}$  $+ \sum a_{i_1}$  $+ R$

$a, b, c, d \rightarrow$ cut-constructible part $\qquad R \rightarrow$ rational terms

$$\mathcal{A} \quad = \sum_{I \subset \{0,1,\cdots,m-1\}} \int \frac{\mu^{(4-d)} d^d q}{(2\pi)^d} \frac{\bar{N}_I(\bar{q})}{\prod_{i \in I} \bar{D}_i(\bar{q})}$$

# The old "master" formula

$$
\begin{aligned}
\int A &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3) \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) C_0(i_0 i_1 i_2) \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) B_0(i_0 i_1) \\
&+ \sum_{i_0}^{m-1} a(i_0) A_0(i_0) \\
&+ \text{rational terms}
\end{aligned}
$$

$D_0, C_0, B_0, A_0$, scalar one-loop integrals: 't Hooft and Veltman
`QCDLOOP` Ellis & Zanderighi ; `OneLOop` A. van Hameren

$$
\begin{aligned}
\int \frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} d(i_0 i_1 i_2 i_3) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} c(i_0 i_1 i_2) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} b(i_0 i_1) \int \frac{1}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} a(i_0) \int \frac{1}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

Remove the integration !

# THE NEW "MASTER" FORMULA

$$
\begin{aligned}
\frac{N(q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \frac{d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \frac{c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2)}{\bar{D}_{i0} \bar{D}_{i1} \bar{D}_{i2}} \\
&+ \sum_{i_0<i_1}^{m-1} \frac{b(i_0 i_1) + \tilde{b}(q; i_0 i_1)}{\bar{D}_{i0} \bar{D}_{i1}} \\
&+ \sum_{i_0}^{m-1} \frac{a(i_0) + \tilde{a}(q; i_0)}{\bar{D}_{i0}} \\
&+ \text{rational terms}
\end{aligned}
$$

# OPP "MASTER" FORMULA

Equation in a from "solvable" à la "unitarity"; not the only way

General expression for the 4-dim $N(q)$ at the integrand level in terms of $D_i$

$$
\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} D_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[ c(i_0 i_1 i_2) + \tilde{c}(q; i_0 i_1 i_2) \right] \prod_{i \neq i_0, i_1, i_2}^{m-1} D_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[ b(i_0 i_1) + \tilde{b}(q; i_0 i_1) \right] \prod_{i \neq i_0, i_1}^{m-1} D_i \\
&+ \sum_{i_0}^{m-1} \left[ a(i_0) + \tilde{a}(q; i_0) \right] \prod_{i \neq i_0}^{m-1} D_i
\end{aligned}
$$

# A next to simple example

- Not only tensor integrals need reduction!

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}}$$

$$1 = \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} \sum \left[ d(i_0 i_1 i_2 i_3) + \tilde{d}(q; i_0 i_1 i_2 i_3) \right] D_{i_4} D_{i_5} \dots D_{i_{m-1}}$$

$$\int \frac{1}{D_0 D_1 D_2 D_3 \dots D_{m-1}} = \sum d(i_0 i_1 i_2 i_3) D_0(i_0 i_1 i_2 i_3)$$

$$d(i_0 i_1 i_2 i_3) = \frac{1}{2} \left( \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^+)} + \prod_{j \neq i_0, i_1, i_2, i_3} \frac{1}{D_j(q^-)} \right)$$

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Numerically treat $D = 4 - 2\epsilon$, means $4 \oplus 1$

**Expand in D-dimensions ?**

$$\bar{D}_i = D_i + \tilde{q}^2$$

$$\begin{aligned}
N(q) &= \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1, i_2, i_3}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2)\right] \prod^{m-1} \bar{D}_i + \tilde{P}(q) \prod^{m-1} \bar{D}_i
\end{aligned}$$

# Rational Terms

Numerically treat $D = 4 - 2\epsilon$, means $4 \oplus 1$

Expand in D-dimensions ?

$$
\begin{aligned}
N(q) \quad = \quad & \sum_{i_0<i_1<i_2<i_3}^{m-1} \left[d(i_0 i_1 i_2 i_3; \tilde{q}^2) + \tilde{d}(q; i_0 i_1 i_2 i_3; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2,i_3}^{m-1} \bar{D}_i \\
+ \quad & \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1,i_2}^{m-1} \bar{D}_i \\
+ \quad & \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2)\right] \prod_{i\neq i_0,i_1}^{m-1} \bar{D}_i \\
+ \quad & \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2)\right] \prod_{i\neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

$$m_i^2 \to m_i^2 - \tilde{q}^2$$

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&+ \sum_{i_0<i_1<i_2}^{m-1} \left[c(i_0 i_1 i_2; \tilde{q}^2) + \tilde{c}(q; i_0 i_1 i_2; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1, i_2}^{m-1} \bar{D}_i \\
&+ \sum_{i_0<i_1}^{m-1} \left[b(i_0 i_1; \tilde{q}^2) + \tilde{b}(q; i_0 i_1; \tilde{q}^2)\right] \prod_{i \neq i_0, i_1}^{m-1} \bar{D}_i \\
&+ \sum_{i_0}^{m-1} \left[a(i_0; \tilde{q}^2) + \tilde{a}(q; i_0; \tilde{q}^2)\right] \prod_{i \neq i_0}^{m-1} \bar{D}_i + \tilde{P}(q) \prod_{i}^{m-1} \bar{D}_i
\end{aligned}
$$

$$m_i^2 \to m_i^2 - \tilde{q}^2$$

# Rational Terms

In practice, once the 4-dimensional coefficients have been determined, one can redo the fits for different values of $\tilde{q}^2$, in order to determine $b^{(2)}(ij)$, $c^{(2)}(ijk)$ and $d^{(2m-4)}$.

$$
\begin{aligned}
\mathrm{R}_1 &= -\frac{i}{96\pi^2} d^{(2m-4)} - \frac{i}{32\pi^2} \sum_{i_0<i_1<i_2}^{m-1} c^{(2)}(i_0 i_1 i_2) \\
&- \frac{i}{32\pi^2} \sum_{i_0<i_1}^{m-1} b^{(2)}(i_0 i_1) \left( m_{i_0}^2 + m_{i_1}^2 - \frac{(p_{i_0} - p_{i_1})^2}{3} \right) .
\end{aligned}
$$

G. Ossola, C. G. Papadopoulos and R. Pittau,arXiv:0802.1876 [hep-ph]

# Rational Terms - $R_2$

A different source of Rational Terms, called $R_2$, can also be generated from the $\epsilon$-dimensional part of $N(q)$

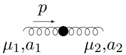
$$\bar{N}(\bar{q}) = N(q) + \tilde{N}(\tilde{q}^2, \epsilon; q)$$

$$\mathrm{R}_2 \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\frac{\tilde{N}(\tilde{q}^2, \epsilon; q)}{\bar{D}_0 \bar{D}_1 \cdots \bar{D}_{m-1}} \equiv \frac{1}{(2\pi)^4} \int d^n\,\bar{q}\,\mathcal{R}_2$$

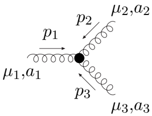
$$\begin{aligned} \bar{q} &= q + \tilde{q}\,, \\ \bar{\gamma}_{\bar{\mu}} &= \gamma_\mu + \tilde{\gamma}_{\tilde{\mu}}\,, \\ \bar{g}^{\bar{\mu}\bar{\nu}} &= g^{\mu\nu} + \tilde{g}^{\tilde{\mu}\tilde{\nu}}\,. \end{aligned}$$

New vertices/particles or GKMZ-approach

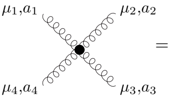
Contribution from $d-$dimensional parts in numerators:




$$= \frac{ig^2 N_{col}}{48\pi^2}\,\delta_{a_1 a_2}\left[\frac{p^2}{2} g_{\mu_1\mu_2} + \lambda_{HV}\left(g_{\mu_1\mu_2}p^2 - p_{\mu_1}p_{\mu_2}\right) + \frac{N_f}{N_{col}}\left(p^2 - 6\,m_q^2\right) g_{\mu_1\mu_2}\right]$$




$$= -\frac{g^3 N_{col}}{48\pi^2}\left(\frac{7}{4} + \lambda_{HV} + 2\frac{N_f}{N_{col}}\right) f^{a_1 a_2 a_3}\, V_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$$

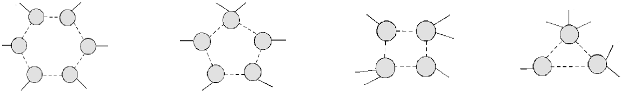



$$= -\frac{ig^4 N_{col}}{96\pi^2}\sum_{P(234)}\Bigg\{\Bigg[\frac{\delta_{a_1a_2}\delta_{a_3a_4} + \delta_{a_1a_3}\delta_{a_4a_2} + \delta_{a_1a_4}\delta_{a_2a_3}}{N_{col}} + 4\,Tr(t^{a_1}t^{a_3}t^{a_2}t^{a_4} + t^{a_1}t^{a_4}t^{a_2}t^{a_3})\,(3+\lambda_{HV}) - Tr(\{t^{a_1}t^{a_2}\}\{t^{a_3}t^{a_4}\})\,(5+2\lambda_{HV})\Bigg] g_{\mu_1\mu_2}g_{\mu_3\mu_4} + 12\frac{N_f}{N_{col}}Tr(t^{a_1}t^{a_2}t^{a_3}t^{a_4})\left(\frac{5}{3}g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_2\mu_3}g_{\mu_1\mu_4}\right)\Bigg\}$$

# The one-loop calculation in a nutshell

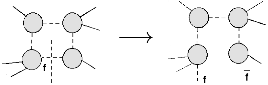
The computation of $pp(p\bar{p}) \to e^+\nu_e\mu^-\bar{\nu}_\mu b\bar{b}$ involves up to six-point functions.
The most generic integrand has therefore the form

$$\mathcal{A}(q) = \sum \underbrace{\frac{N_i^{(6)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_5}}} + \underbrace{\frac{N_i^{(5)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_4}}} + \underbrace{\frac{N_i^{(4)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\cdots\bar{D}_{i_3}}} + \underbrace{\frac{N_i^{(3)}(q)}{\bar{D}_{i_0}\bar{D}_{i_1}\bar{D}_{i_2}}} + \cdots$$



In order to apply the OPP reduction, HELAC evaluates numerically the numerators $N_i^6(q), N_i^5(q), ....$ with the values of the loop momentum $q$ provided by CutTools

- generates all inequivalent partitions of 6,5,4,3... blobs attached to the loop, and check all possible flavours (and colours) that can be consistently running inside
- hard-cuts the loop ($q$ is fixed) to get a $n+2$ *tree-like* process

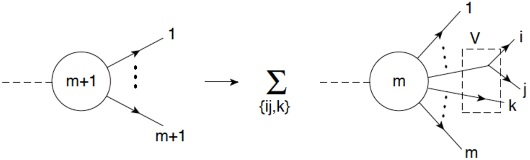



The $R_2$ contributions (rational terms) are calculated in the same way as the tree-order amplitude, taking into account *extra vertices*

Real corrections: $D \to 4$ dimensions (Catani & Seymour)

$$\int_{m+1} d\sigma^R + \int_m d\sigma^V$$

$$\sigma^{NLO} = \int_{m+1} \left[ \left(d\sigma^R\right)_{\epsilon=0} - \left(d\sigma^A\right)_{\epsilon=0} \right] + \int_m \left[ d\sigma^V + \int_1 d\sigma^A \right]_{\epsilon=0}$$

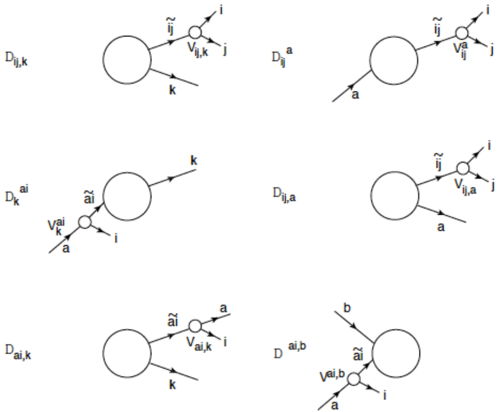



$$\tilde{p}_k^\mu = \frac{1}{1-y_{ij,k}} p_k^\mu \quad , \quad \tilde{p}_{ij}^\mu = p_i^\mu + p_j^\mu - \frac{y_{ij,k}}{1-y_{ij,k}} p_k^\mu$$

$$d\phi(p_i,p_j,p_k;Q) = \frac{d^d p_i}{(2\pi)^{d-1}} \delta_+(p_i^2) \frac{d^d p_j}{(2\pi)^{d-1}} \delta_+(p_j^2) \frac{d^d p_k}{(2\pi)^{d-1}} \delta_+(p_k^2) \; (2\pi)^d \delta^{(d)}(Q-p_i-p_j-p_k)$$

$$d\phi(p_i,p_j,p_k;Q) = d\phi(\tilde{p}_{ij},\tilde{p}_k;Q) \; [dp_i(\tilde{p}_{ij},\tilde{p}_k)]$$

## Dipoles in real life



## Dipoles in real life: the formulae

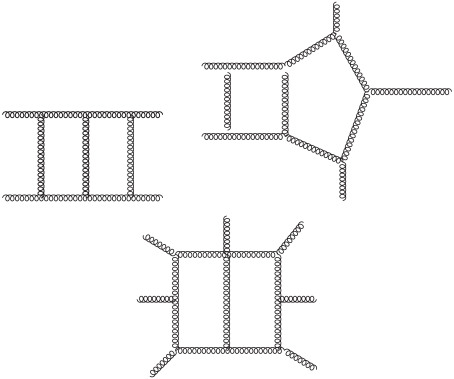
$$
\begin{aligned}
d\sigma^A &= \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1,...,p_{m+1};Q)\, \frac{1}{S_{\{m+1\}}} \\
&\cdot \sum_{\substack{\text{pairs}\\ \text{i,j}}} \sum_{k\neq i,j} \mathcal{D}_{ij,k}(p_1,...,p_{m+1})\, F_J^{(m)}(p_1,..\widetilde{p}_{ij},\widetilde{p}_k,..,p_{m+1})
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{D}_{ij,k}\,(p_1,...,p_{m+1}) = -\frac{1}{2p_i\cdot p_j} \\
&\cdot \quad {}_m\!< 1,..,\widetilde{ij},..,\widetilde{k},..,m+1|\, \frac{\boldsymbol{T}_k\cdot\boldsymbol{T}_{ij}}{\boldsymbol{T}_{ij}^2}\, \boldsymbol{V}_{ij,k}\, |1,..,\widetilde{ij},..,\widetilde{k},..,m+1>_m
\end{aligned}
$$

$$
\begin{aligned}
d\sigma^R - d\sigma^A &= \mathcal{N}_{in} \sum_{\{m+1\}} d\phi_{m+1}(p_1,...,p_{m+1};Q)\, \frac{1}{S_{\{m+1\}}} \\
&\cdot \Big\{|\mathcal{M}_{m+1}(p_1,...,p_{m+1})|^2\, F_J^{(m+1)}(p_1,...,p_{m+1}) \\
&- \sum_{\substack{\text{pairs}\\ \text{i,j}}} \sum_{k\neq i,j} \mathcal{D}_{ij,k}(p_1,...,p_{m+1})\, F_J^{(m)}(p_1,..\widetilde{p}_{ij},\widetilde{p}_k,..,p_{m+1})\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&\int_{m+1} d\sigma^A = -\int_m \mathcal{N}_{in} \sum_{\{m\}} d\phi_m(p_1,..,p_m;Q)\, \frac{1}{S_{\{m\}}}\, F_J^{(m)}(p_1,...,p_m) \\
&\cdot \sum_i \sum_{k\neq i} |\mathcal{M}_m^{i,k}(p_1,...,p_m)|^2\, \frac{\alpha_S}{2\pi} \frac{1}{\Gamma(1-\epsilon)} \left(\frac{4\pi\mu^2}{2p_i\cdot p_k}\right)^{\epsilon} \frac{1}{\boldsymbol{T}_i^2}\, \mathcal{V}_i(\epsilon)\ ,
\end{aligned}
$$

# Perturbative QCD at NNLO



Repeat the one-loop "success story" ?

# Reduction at the integrand level

Over the last few years very important activity to extend unitarity and integrand level reduction ideas beyond one loop

J. Gluza, K. Kajda and D. A. Kosower, "Towards a Basis for Planar Two-Loop Integrals," Phys. Rev. D **83** (2011) 045012 [arXiv:1009.0472 [hep-th]].

D. A. Kosower and K. J. Larsen, "Maximal Unitarity at Two Loops," Phys. Rev. D **85** (2012) 045017 [arXiv:1108.1180 [hep-th]].

P. Mastrolia and G. Ossola, "On the Integrand-Reduction Method for Two-Loop Scattering Amplitudes," JHEP **1111** (2011) 014 [arXiv:1107.6041 [hep-ph]].

S. Badger, H. Frellesvig and Y. Zhang, "Hepta-Cuts of Two-Loop Scattering Amplitudes," JHEP **1204** (2012) 055 [arXiv:1202.2019 [hep-ph]].

Y. Zhang, "Integrand-Level Reduction of Loop Amplitudes by Computational Algebraic Geometry Methods," JHEP **1209** (2012) 042 [arXiv:1205.5707 [hep-ph]].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Integrand-Reduction for Two-Loop Scattering Amplitudes through Multivariate Polynomial Division," arXiv:1209.4319 [hep-ph].

P. Mastrolia, E. Mirabella, G. Ossola and T. Peraro, "Multiloop Integrand Reduction for Dimensionally Regulated Amplitudes," arXiv:1307.5832 [hep-ph].

- Write the "OPP-type" equation at two loops

$$\frac{N\left(l_1, l_2; \{p_i\}\right)}{D_1 D_2 \ldots D_n} = \sum_{m=1}^{\min(n,8)} \sum_{S_{m;n}} \frac{\Delta_{i_1 i_2 \ldots i_m}\left(l_1, l_2; \{p_i\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}}$$

$S_{m;n}$ stands for all subsets of $m$ indices out of the $n$ ones

# Multivariate Division and Groebner Basis

D. Cox, J. Little, D. O'Shea *Ideals, Varieties and Algorithms*

- Given any set of polynomials $\pi_i$, the ideal $I$, $f = \sum_i \pi_i h_i$, we can define a unique Groebner basis up to ordering $< g_1, \ldots, g_s >$

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• Perform the division of an arbitrary polynomial $N$

$$N = h_1 g_1 + \ldots + h_n g_s + v$$

• Express back $g_i$ in terms of $D_i$

$$N = \tilde{h}_1 D_1 + \ldots + \tilde{h}_n D_n + v$$

The simplest case: $n \to n-1$ reduction

The general strategy consists in finding polynomials $\Pi_j \equiv \Pi_j(l_1, l_2)$

$$\sum_{j=1}^{n_1} \Pi_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} \Pi_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} \Pi_j D(l_2 + p_j) = 1 \quad .$$

Is this possible at all ?

$$\sum_{j=1}^{n_1} \Pi_j D(l_1 + p_j) + \sum_{j=n_1+1}^{n_1+n_2} \Pi_j D(l_1 + l_2 + p_j) + \sum_{j=n_1+n_2+1}^{n} \Pi_j D(l_2 + p_j) = 1 \quad .$$

## Hilbert's Nullstellensatz theorem

Hilbert's Nullstellensatz (German for "theorem of zeros," or more literally, "zero-locus-theorem" see Satz) is a theorem which establishes a fundamental relationship between geometry and algebra. This relationship is the basis of algebraic geometry, an important branch of mathematics. It relates algebraic sets to ideals in polynomial rings over algebraically closed fields. This relationship was discovered by David Hilbert who proved Nullstellensatz and several other important related theorems named after him (like Hilbert's basis theorem).

$$1 = g_1 f_1 + \cdots + g_s f_s \quad g_i, f_i \in k[x_1, \ldots, x_n]$$

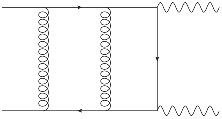
Janos Kollar, J. Amer. Math. Soc., Vol. 1, No. 4. (Oct., 1988), pp 963-975

$$\deg g_i f_i \leq \max\{3, d\}^n \quad d = \max \deg f_i \quad 3^8 = 6561$$

M. Sombra, Adv. in Appl. Math. 22 (1999), 271-295

$$\deg g_i f_i \leq 2^{n+1} \quad 2^9 = 512$$

As an example I reduced a two-loop 7-propagator graph contributing to $q\bar{q} \to \gamma^*\gamma^*$



with $l_1^\mu = \sum_{i=1}^3 z_i v_i^\mu + z_4 \eta^\mu$, with $z_i = l_1 \cdot p_i, i = 1 \ldots, 3$ ($l_2$, with $w_i$ replacing $z_i$).

$$n1 := 155$$

$$1, 1, \text{" ---- "}, \frac{4347392}{81} - \frac{2891776}{243}\sqrt{2}\sqrt{3} + \frac{425984}{27} z3 - \frac{134144}{27}\sqrt{3}\sqrt{2}\, z3 + \frac{1921024}{27} w3 - \frac{1358848}{81}\sqrt{3}\sqrt{2}\, w3 + \frac{66560}{3} w4\, z4 - \frac{20480}{3} w4\, z4\sqrt{2}\sqrt{3} + \frac{16384}{3} z3\, w4\, z4$$
$$+ \frac{524288}{81} w4^2\, w3 + \frac{131072}{243}\sqrt{3}\, w4^2\sqrt{2}\, w3 - \frac{32768}{27}\sqrt{3}\sqrt{2}\, z3\, w4^2 - \frac{16384}{27} w4^2\, z3 + \frac{702464}{81} w4^2 - \frac{53248}{243} w4^2\sqrt{2}\sqrt{3} + 4608\, z4^2, \text{"------"}, 16$$

$$2, dd_3\, dd_5\, dd_1, \text{" ---- "}, \frac{3136}{15}\sqrt{2}\sqrt{3} + \frac{2048}{27} w4^2\sqrt{2}\sqrt{3} + \frac{8192}{9} w4^2, \text{"------"}, 3$$

$$3, dd_3\, dd_8\, dd_5\, dd_4, \text{" ---- "}, \frac{1970176}{5625} w3 - \frac{2363392}{1875} w2 + \frac{575488}{16875}\sqrt{3}\sqrt{2}\, w3 - \frac{956416}{1875}\sqrt{2}\sqrt{3} - \frac{618496}{5625}\sqrt{2}\sqrt{3}\, w2 - \frac{1407232}{625}, \text{"------"}, 6$$

$$4, dd_3\, dd_8\, dd_5\, dd_9, \text{" ---- "}, -\frac{2048}{9} w3 + \frac{2048}{3} w2 - \frac{512}{27}\sqrt{3}\sqrt{2}\, w3 + \frac{512}{9}\sqrt{2}\sqrt{3}\, w2 + \frac{512}{3}\sqrt{2}\sqrt{3} + 512, \text{"------"}, 6$$

$$5, dd_6\, dd_8\, dd_4\, dd_3, \text{" ---- "}, -\frac{309248}{16875}\sqrt{3}\sqrt{2}\, w3 + \frac{323584}{5625}\sqrt{2}\sqrt{3}\, w2 - \frac{1181696}{5625} w3 + \frac{1312768}{1875} w2 + \frac{1167232}{1875} + \frac{956416}{5625}\sqrt{2}\sqrt{3}, \text{"------"}, 6$$

$$6, dd_5\, dd_9\, dd_6\, dd_1, \text{" ---- "}, -\frac{7168}{27}\sqrt{2}\sqrt{3} + \frac{24320}{9}, \text{"------"}, 2$$

$$7, dd_5\, dd_9\, dd_6\, dd_2, \text{" ---- "}, \frac{2176}{9}\sqrt{2}\sqrt{3} - \frac{8192}{3} + \frac{2560}{81}\sqrt{3}\sqrt{2}\, z3 + \frac{10240}{27} z3, \text{"------"}, 4$$

$$8, dd_5\, dd_9\, dd_6\, dd_3, \text{" ---- "}, 256, \text{"------"}, 1$$

$$9, dd_5\, dd_9\, dd_8\, dd_2, \text{" ---- "}, 1536 + 128\sqrt{2}\sqrt{3} - \frac{1024}{3} w2 - \frac{256}{9}\sqrt{2}\sqrt{3}\, w2 + \frac{1024}{9} w3 + \frac{256}{27}\sqrt{3}\sqrt{2}\, w3, \text{"------"}, 6$$

$$10, dd_5\, dd_9\, dd_8\, dd_4, \text{" ---- "}, -\frac{135808}{5625}\sqrt{2}\sqrt{3} - \frac{418816}{1875}, \text{"------"}, 2$$

$$11, dd_6\, dd_9\, dd_4\, dd_3, \text{" ---- "}, -\frac{157312}{16875}\sqrt{2}\sqrt{3} - \frac{615424}{5625}, \text{"------"}, 2$$

$$12, dd_6\, dd_9\, dd_4\, dd_8, \text{" ---- "}, \frac{418816}{1875} + \frac{135808}{5625}\sqrt{2}\sqrt{3}, \text{"------"}, 2$$

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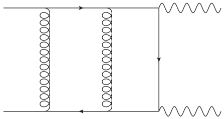

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$$\frac{\Pi\left(\{z_i\}, \{w_j\}\right)}{D_{i_1} D_{i_2} \ldots D_{i_m}} \to \text{spurious} \oplus \text{nonscalar integrals}$$

- IBPI to Master Integrals

S. Badger, talk in Amplitudes 2013

- Rational terms

$$l_1 \to l_1 + l_1^{(2\varepsilon)}, \;\; l_2 \to l_2 + l_2^{(2\varepsilon)}, \;\; l_{1,2} \cdot l_{1,2}^{(2\varepsilon)} = 0$$

$$\left(l_1^{(2\varepsilon)}\right)^2 = \mu_{11}, \quad \left(l_2^{(2\varepsilon)}\right)^2 = \mu_{22}, \quad l_1^{(2\varepsilon)} \cdot l_2^{(2\varepsilon)} = \mu_{12}$$

$$\left\{l_1^{(4)}, l_2^{(4)}\right\} \to \left\{l_1^{(4)}, l_2^{(4)}, \mu_{11}, \mu_{22}, \mu_{12}\right\}$$

Welcome: $I = \sqrt{I}$ prime ideals

- $R_2$ terms

# Master Integrals: The current approach

- $m$ independent momenta $l$ loops, $N = l(l+1)/2 + lm$ scalar products
- basis composed by $D_1 \dots D_N$, allows to express all scalar products $D_i = (\{k,l\} + p_i)^2 - M_i^2$
- $F[a_1, \dots, a_N]$

$$\int d^d k d^d l \;\; \frac{\partial}{\partial \{k^\mu, l^\mu\}} \left( \frac{\{k^\mu, l^\mu, v^\mu\}}{D_1^{a_1} \dots D_N^{a_N}} \right) = 0$$

- IBP Laporta: `FIRE`, `AIR`, `Reduze` reduce these to MI
- MI computed, Feynman parameters, Mellin-Barnes, Differential Equations


Z. Bern, L. J. Dixon and D. A. Kosower, Phys. Lett. B **302** (1993) 299.

T. Gehrmann and E. Remiddi, Nucl. Phys. B **580** (2000) 485 [hep-ph/9912329].

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# DIFFERENTIAL EQUATIONS APPROACH

- Library of MI à la one-loop
- Iterated Integrals    K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

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$$G\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} G\left(a_{n-1}, \ldots, a_1, t\right)$$

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- Library of MI à la one-loop
- Iterated Integrals K. T. Chen, Iterated path integrals, Bull. Amer. Math. Soc. 83 (1977) 831
- Polylogarithms, Symbol algebra

A. B. Goncharov, M. Spradlin, C. Vergu and A. Volovich, Phys. Rev. Lett. **105** (2010) 151605.

C. Duhr, H. Gangl and J. R. Rhodes, JHEP **1210** (2012) 075 [arXiv:1110.0458 [math-ph]].

$$G\left(a_n, \ldots, a_1, x\right) = \int_0^x dt \frac{1}{t - a_n} G\left(a_{n-1}, \ldots, a_1, t\right)$$

with the special cases, $G(x) = 1$ and

$$G\left(\underbrace{0, \ldots 0}_{n}, x\right) = \frac{1}{n!} \log^n(x)$$

# Differential Equations Approach

- Let us consider a simple example

$$\int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_0 D_1 \ldots D_{n-1}}$$

with $D_i = (k + p_0 + \ldots + p_i)^2$ and take for convenience $p_0 = 0$. It can be considered as a function of the external momenta $p_i$.

- It belongs to the topology defined by

$$G_{a_1 \ldots a_n} = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{D_0^{a_1} D_1^{a_2} \ldots D_{n-1}^{a_n}}$$

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The integral is a function of external momenta, so one can set-up differential equations by differentiating with respect to these

$$p_j^\mu \frac{\partial}{\partial p_i^\mu} G[a_1, \ldots, a_n] \to \sum G[a_1', \ldots, a_n']$$

$$p_1^\mu \frac{\partial}{\partial p_1^\mu} (k+p_1)^2 = 2\,(k+p_1)\cdot p_1 = (k+p_1)^2 + p_1^2 - k^2$$

- Find the proper parametrization; Bring the system of equations in a form suitable to express the MI in terms of GPs
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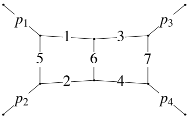
J. M. Henn, K. Melnikov and V. A. Smirnov, arXiv:1402.7078 [hep-ph].




$$S=(q_1+q_2)^2=(q_3+q_4)^2,\quad T=(q_1-q_3)^2=(q_2-q_4)^2,\quad U=(q_1-q_4)^2=(q_2-q_3)^2;$$

$$\frac{S}{M_3^2}=(1+x)(1+xy),\quad \frac{T}{M_3^2}=-xz,\quad \frac{M_4^2}{M_3^2}=x^2y.$$

$$d\,\vec{f}(x,y,z;\epsilon)=\epsilon\,d\,\tilde{A}(x,y,z)\,\vec{f}(x,y,z;\epsilon)$$

$$\tilde{A}=\sum_{i=1}^{15}\tilde{A}_{\alpha_i}\,\log(\alpha_i)$$

$$\alpha=\{x,y,z,1+x,1-y,1-z,1+xy,z-y,1+y(1+x)-z,xy+z,$$
$$1+x(1+y-z),1+xz,1+y-z,z+x(z-y)+xyz,z-y+yz+xyz\}.$$

# The Simplified Differential Equations Approach

C. G. Papadopoulos, arXiv:1401.6057 [hep-ph].

Making the whole procedure systematic (algorithmic) and straightforwardly expressible in terms of GPs.

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$$G_{11\ldots1}(x) = \int \frac{d^d k}{i\pi^{d/2}} \frac{1}{(k^2)\,(k + x\, p_1)^2\,(k + p_1 + p_2)^2 \ldots (k + p_1 + p_2 + \ldots + p_n)^2}$$

- Now the integral becomes a function of $x$, which allows to define a differential equation with respect to $x$, schematically given by

$$\frac{\partial}{\partial x} G_{11\ldots1}(x) = -\frac{1}{x} G_{11\ldots1}(x) + x p_1^2 G_{12\ldots1} + \frac{1}{x} G_{02\ldots1}$$

- and using IBPI we obtain

$$\begin{aligned} m_1 x G_{121} + \frac{1}{x} G_{021} \quad &= \left(\frac{1}{x-1} + \frac{1}{x - m_3/m_1}\right)\left(\frac{d-4}{2}\right) G_{111} \\ &+ \frac{d-3}{m_1 - m_3}\left(\frac{1}{x-1} - \frac{1}{x - m_3/m_1}\right)\left(\frac{G_{101} - G_{110}}{x}\right) \end{aligned}$$

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- The integrating factor $M$ is given by

$$M = x\,(1-x)^{\frac{4-d}{2}}\,(-m_3+m_1x)^{\frac{4-d}{2}}$$

- and the DE takes the form, $d = 4-2\varepsilon$,

$$\frac{\partial}{\partial x} MG_{111} = c_\Gamma \frac{1}{\varepsilon}\,(1-x)^{-1+\varepsilon}\,(-m_3+m_1x)^{-1+\varepsilon}\left(\left(-m_1x^2\right)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)$$

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# THE SIMPLIFIED DIFFERENTIAL EQUATIONS APPROACH

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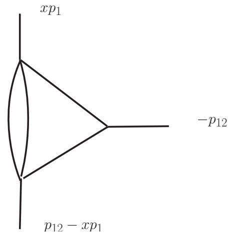
# The Simplified Differential Equations Approach

$$G_{111} = \frac{c_\Gamma}{(m_1 - m_3)x}\mathcal{I}$$

$$\begin{aligned}
\mathcal{I} = {} & \frac{-(-m_1)^{-\varepsilon} + (-m_3)^{-\varepsilon} + \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) x_1}{\varepsilon^2} \\
& + \frac{\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) x_1 G\left(\frac{m_3}{m_1}, 1\right) - \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1}, 1\right) - G\left(\frac{m_3}{m_1}, x\right)\right)}{\varepsilon} \\
& + \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\left(G\left(\frac{m_3}{m_1}, 1\right) G\left(\frac{m_3}{m_1}, x\right) - G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right) - G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, x\right)\right) + x_1 \Big(-2G(0,1,x)(-m_1)^{-\varepsilon} \\
& \quad + 2G\left(0, \frac{m_3}{m_1}, x\right)(-m_1)^{-\varepsilon} + 2G\left(\frac{m_3}{m_1}, 1, x\right)(-m_1)^{-\varepsilon} + G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right)(-m_1)^{-\varepsilon} - G\left(\frac{m_3}{m_1}, x\right)\log(1-x)(-m_1)^{-\varepsilon} \\
& \quad - 2G\left(\frac{m_3}{m_1}, x\right)\log(x)(-m_1)^{-\varepsilon} + 2\log(1-x)\log(x)(-m_1)^{-\varepsilon} - 2(-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1}, 1, x\right) - (-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right) \\
& \quad - \left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) G\left(\frac{m_3}{m_1}, 1\right)\left(G\left(\frac{m_3}{m_1}, x\right) - \log(1-x)\right) + (-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1}, x\right)\log(1-x)\Big) \\
& + \varepsilon \Bigg(\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right)\Big(G\left(\frac{m_3}{m_1}, x\right) G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right) - G\left(\frac{m_3}{m_1}, 1\right) G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, x\right) - G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right) \\
& \quad + G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, \frac{m_3}{m_1}, x\right)\Big) + \frac{1}{2} x_1 \Big(\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) G\left(\frac{m_3}{m_1}, 1\right)\left(\log^2(1-x) + 2G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, x\right)\right) \\
& \quad + G\left(\frac{m_3}{m_1}, x\right)\left(4\log^2(x) - 2\left((-m_1)^{-\varepsilon} - (-m_3)^{-\varepsilon}\right) G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right) + 2\left((-m_3)^{-\varepsilon} - (-m_1)^{-\varepsilon}\right) G\left(\frac{m_3}{m_1}, 1\right)\log(1-x)\right) \\
& \quad + 2\Big(G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right)(-m_1)^{-\varepsilon} + G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right)\log(1-x)(-m_1)^{-\varepsilon} - 2\log(1-x)\log^2(x) - 4G(0,0,1,x) \\
& \quad + 4G\left(0, 0, \frac{m_3}{m_1}, x\right) - 2G(0,1,1,x) + 4G\left(0, \frac{m_3}{m_1}, 1, x\right) - 2G\left(0, \frac{m_3}{m_1}, \frac{m_3}{m_1}, x\right) + 2G\left(\frac{m_3}{m_1}, 0, 1, x\right) - 2G\left(\frac{m_3}{m_1}, 0, \frac{m_3}{m_1}, x\right) \\
& \quad - (-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right) - (-m_3)^{-\varepsilon} G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, 1\right)\log(1-x) + \log^2(1-x)\log(x) \\
& \quad + 4G(0,1,x)\log(x) - 4G\left(0, \frac{m_3}{m_1}, x\right)\log(x) - 4G\left(\frac{m_3}{m_1}, 1, x\right)\log(x) + 2G\left(\frac{m_3}{m_1}, \frac{m_3}{m_1}, x\right)\log(x)\Big)\Big)\Bigg)
\end{aligned}$$

# The Simplified Differential Equations Approach

The two-loop 3-mass triangle



# The Simplified Differential Equations Approach

We are interested in $G_{0101011}$. The DE involves also the MI $G_{0201011}$, so we have a system of two coupled DE, as follows:

$$\frac{\partial}{\partial x}\left(M_{0101011}\,G_{0101011}\right) = \frac{A_3(2-3\varepsilon)(1-x)^{-2\varepsilon}x^{\varepsilon-1}(m_1x-m_3)^{-2\varepsilon}}{2\varepsilon(2\varepsilon-1)} + \frac{m_1\varepsilon(1-x)^{-2\varepsilon}(m_1x-m_3)^{-2\varepsilon}}{2\varepsilon-1}g(x)$$

$$\frac{\partial}{\partial x}\left(M_{0201011}\,G_{0201011}\right) = \frac{A_3(3\varepsilon-2)(3\varepsilon-1)(-m_1)^{-2\varepsilon}(1-x)^{2\varepsilon-1}x^{-3\varepsilon}(m_1x-m_3)^{2\varepsilon-1}}{2\varepsilon^2} + (2\varepsilon-1)(3\varepsilon-1)(1-x)^{2\varepsilon-1}(m_1x-m_3)^{2\varepsilon-1}f(x)$$

where $f(x) \equiv M_{0101011}\,G_{0101011}$ and $g(x) \equiv M_{0201011}\,G_{0201011}$, $M_{0201011} = (1-x)^{2\varepsilon}x^{\varepsilon+1}(m_1x-m_3)^{2\varepsilon}$ and $M_{0101011} = x^{\varepsilon}$

# The Simplified Differential Equations Approach

The singularity structure of the right-hand side is now richer. Singularities at $x = 0$ are all proportional to $x^{-1-2\varepsilon}$ and $x^{-1-\varepsilon}$ and can easily be integrated by the following decomposition

$$\int_0^x dt\ t^{-1-2\varepsilon} F(t) = F(0) \int_0^x dt\ t^{-1-2\varepsilon} + \int_0^x dt\ \frac{F(t)-F(0)}{t} t^{-2\varepsilon}$$
$$= F(0) \frac{x^{-2\varepsilon}}{(-2\varepsilon)} + \int_0^x dt\ \frac{F(t)-F(0)}{t} \left(1 - 2\varepsilon \log(t) + 2\varepsilon^2 \log^2(t) + \ldots\right)$$

# The Simplified Differential Equations Approach

- One-loop up to 5-point at order $\epsilon$: 6 scales, GP-weight 4 (look forward for pentaboxes)
- Two-loop triangles and 4-point MI
- Working/finishing double boxes with two external off-shell legs (more than 100 MI) $\rightarrow$ P12 P13 P23 N12 N13 N34(*) topologies completed and tested!
- Completing the list of all MI with arbitrary off-shell legs ($m = 0$).

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# NNLO IN THE FUTURE

- The NNLO automation to come
  - In a few years the new "wish list" should be completed
    $pp \to t\bar{t}$, $pp \to W^+W^-$, $pp \to W/Z + nj$, $pp \to H + nj$, $\ldots$
  - Virtual amplitudes: Reduction at the integrand level $\oplus$ IBP

    $\rightarrow$ Master Integrals
  - Virtual-Real — A.van Hameren, OneLoop MI
  - Real-Real — STRIPPER, M. Czakon, Phys. Lett. B **693** (2010) 259 [arXiv:1005.0274 [hep-ph]].

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