

RADIATIVE CORRECTIONS

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Abstract

What one should understand as a central point in the subject of radiative corrections is a matter of debate. Theoretically oriented physicist would say that proof of the field theory renormalizability should be considered as a focusing point in the domain of radiative corrections. On the other hand, one can think of radiative corrections in a very practical sense, that is, as theoretical corrections that must be included in the data analysis. In these lectures we will try to explain radiative corrections in this second practical context. In particular we will try to explain the meaning of the following keywords: “factorization”, “inclusive and exclusive exponentiation”, “soft photon factor”, “leading logarithms”. We will also discuss briefly the definition of the technical and physical precision of theoretical predictions and their relevance to the quality of experimental data and thus, possibly, to everyday experimental work.

The aim of my lectures is to explain the relation between the concepts listed above and the prediction of the perturbative Quantum Field Theory. We will limit ourselves to the Quantum Electrodynamics process $e^+e^- \rightarrow \gamma^* (Z^*) + \text{initial-state bremsstrahlung}$, $\gamma^* (Z^*) \rightarrow \text{anything}$, but some of the results are similar in the case of QCD. We will rely heavily on the explicit example calculations that we present throughout the text. A step-by-step recalculation of the presented material is strongly recommended for understanding these lectures. Some information on how to read them is given in the summary.

Over the years 2004 and 2005 significant changes were introduced. At present LEP era applications are becoming less interesting. Material concerning explanation of relation between matrix element calculations, formalism of fixed body final states, exponentiation and formulation of parton showers using as an intermediate example PHOTOS; algorithm for generation of bremsstrahlung in decays, is slowly expanding, but the new shape is not yet visible. The old outdated parts of 1993 lectures are not removed.

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1 INTRODUCTION

Over the last twenty years the precision and complexity of high-energy e^+e^- experiments has increased significantly. Testing the theory by a direct comparison of measured cross sections and calculated quantities has become only an idealized picture.

The most obvious reason why it can only be an idealization is due to experimental inefficiencies such as the lack of hermeticity of the detectors or existence of the regions of the phase space which are strongly contaminated by the background, to be excluded by means of cuts. It is sometimes very difficult or just impossible to impose such cuts on the theoretical predictions represented in the form of analytical calculations [1]. The use of Monte Carlo simulations thus becomes imperative.

Let us use a symbolic, algebraic-like notation. If $\hat{\mathbf{x}}$ denotes a physical event as it really *happens*, then the response of the detector can be symbolically noted as an action of the operator \mathbf{B} on $\hat{\mathbf{x}}$. The expression $\mathbf{B} \otimes \hat{\mathbf{x}}$ should thus be understood as an electronic response of the detector. Such a signal is then analysed and finally an event $\hat{\mathbf{y}} = \mathbf{A} \otimes \mathbf{B} \otimes \hat{\mathbf{x}}$ is reconstructed – here \mathbf{A} represents the selection and analysis of the data. A difference $\Delta = \hat{\mathbf{y}} - \hat{\mathbf{x}}$ represents an essential ingredient of the systematic error. A study of this error also requires a Monte Carlo for the physics processes under consideration. With the help of the MC one can generate a series of events $\{\mathbf{x}_i\}$ and later, relying on the detailed knowledge of the detector components, generate a series of detector responses $\{\bar{\mathbf{B}} \otimes \mathbf{x}_i\}$. Such a simulation represents a perfect testing environment in which study the properties of \mathbf{A} . For idealized theoretical events \mathbf{x}_i , the difference $\Delta = \mathbf{y}_i - \mathbf{x}_i$ can be explicitly calculated for every generated event. It is obvious that imperfections in our knowledge of \mathbf{B} will introduce a systematic bias in \mathbf{A} . However an inappropriate choice of the theoretical sample $\{\mathbf{x}_i\}$, due to missing topologies of final states and, to some extent, to crude approximations in the theoretical differential distribution, will also indirectly affect the analysis \mathbf{A} and contribute to the total systematic error. For charged particles, due to properties of QED radiative corrections, to obtain physically meaningful results one has to perform summation over final states with a different number (zero to infinity) of additional outgoing real photons.

This is the motivation for studying radiative corrections for somebody active (or planning to be) in the field of high-precision high-energy data analysis. That is also why we will concentrate on real photon emission corrections, which introduce experimentally significant effects. We will not go however into a discussion of the details of Monte Carlo implementations of these calculations.

In fact, it is only in a very special case of QED and/or electroweak perturbative calculation that the question of the theoretical systematic error can be addressed in a fully satisfactory way. In other cases, where hadronic low-energy interactions play an important role, the situation is much less satisfactory.

On the other hand, Monte Carlo is also used in defining selection criteria, experimental cut-offs, etc. In this way the systematic error of the Monte Carlo will also enter the experimental data in an irreversible way.

2 CALCULATIONS FROM FIELD THEORY

For the electroweak interactions one tends to use “QED-subtracted data”, which exhibit solely the properties of hard processes, with QED bremsstrahlung and related detector-dependent effects removed. In such an approach, one finally confronts theory and experiment in the following (indirect) way. On one side there are “data”, which in fact include implicitly some *theoretical* effects due to QED subtraction, and on the other side theoretical predictions, calculated in a subset of the Standard Model *without* QED bremsstrahlung.

At first, the above scenario looks awkward because “theory” is involved in both sides of the Equation “data” = “theory”. The concept of “QED-subtracted data” is however a very useful one because such corrected data are free of all detector-dependent effects. It should be stressed that the dependence of the QED corrections on particular experimental cuts is the major argument for subtracting them from the data. In practice QED subtraction is done simultaneously with removing the acceptance of the detector. On the other hand “QED-subtracted theoretical predictions” are also easier to calculate than the complete results.

The above QED (detector acceptance) subtraction approach brings a new kind of experimental error owing to uncertainty in the theoretical calculations used in the data analysis. The means of calculating this “theoretical” component of the experimental systematic error should thus be provided with the theoretical calculation to be used in the data analysis. Is it possible?

It is well known that QED is a field theory of well-defined perturbation expansion and in principle any physical prediction can be calculated with practically infinite precision, up to, perhaps, the Dyson limit [2]. That is also why, in principle, there exists a standard way of *calculating* the systematic errors of QED predictions as well. The idea is quite simple; one should calculate predictions for a given observable P at different orders of perturbation expansion P_0, P_1, P_2, \dots , and the calculation should be continued until the difference $\Delta_n = P_n - P_{n-1}$ is smaller, by a safe factor (usually 3), than the expected experimental error for the observable P . This analysis has to be performed for every observable and every new set of cuts, which may eventually change the size of the corrections. There are serious, although not fundamental, difficulties in applying the above scheme in practice. If the Born predictions for a given process can be calculated within days, the calculation of $\mathcal{O}(\alpha)$ corrections already may require up to one year. It is only in a very special case of the Z line-shape calculation for LEP [3] that the complete $\mathcal{O}(\alpha^2)$ QED results are available. Even, this is true only for idealized cuts and only for initial-state radiation ¹.

Owing to these practical limitations, it is rather clear that the above scheme of calculating QED systematic errors cannot be applied directly. Having at our disposal only Born and $\mathcal{O}(\alpha)$ results it is rather obvious that the smallness of the difference can be just a simple numerical accident. In fact, the situation is quite often the opposite, and the difference Δ_1 is significantly larger than the experimental error, indicating that the higher orders should be included.

On the other hand, we should not forget that the electromagnetic coupling constant is

¹In addition, starting at $\mathcal{O}(\alpha^2)$, QED corrections cannot be separated from complete electroweak corrections. Genuine $\mathcal{O}(\alpha^2)$ electroweak corrections (which are not yet calculated) can be numerically of the same order as QED $\mathcal{O}(\alpha^2)$ terms. At this level, the procedure of subtracting QED corrections cannot be defined, and complete electroweak corrections and detector effects have to be analysed simultaneously.

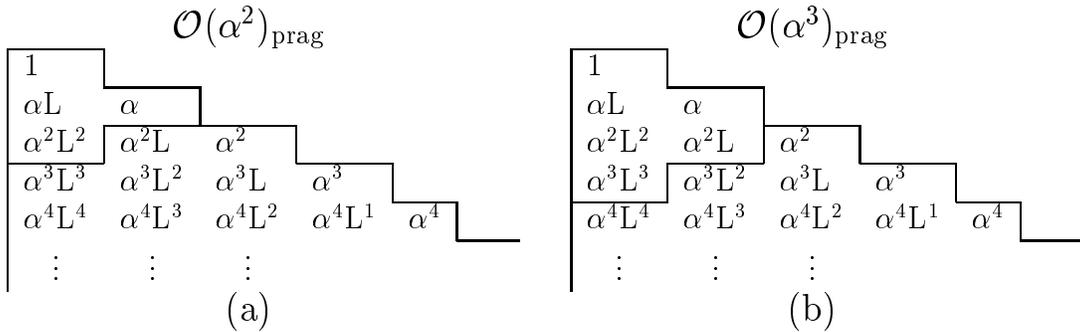


Figure 1: *QED perturbative leading and subleading corrections. The rows represent corrections in consecutive perturbative orders – the first row is the Born contribution. The first column represents the leading logarithmic (LL) approximation and the second column depicts the next-to-leading (NLL) approximation. In the Figure, terms selected for (a) second and (b) third order pragmatic expansion (for photon emission from the electron at LEP energies) are limited with the help of an additional line.*

rather small, of the order of 1% and one would thus expect the $\mathcal{O}(\alpha^2)$ terms to be typically 10^{-4} and therefore completely negligible. Corrections are however larger due to various enhancement factors present in the QED results. The source of these enhancements is well understood and governed by the structure of QED singularities; ultraviolet, infrared and collinear. Because the structure of QED singularities is much simpler than that of the full theory, a multitude of techniques (such as structure functions, exponentiation and running coupling constants) was developed to control these enhanced terms. Using these techniques it is possible either to calculate higher-order leading terms (which are very often *the only* higher-order interesting terms) in a relatively easy way or to improve the convergence of the perturbation expansion, by the appropriate redefinition and/or reordering of the expansion.

It is important to realize that in order to improve the precision of QED predictions, the strict approach of the order-by-order calculation is in most cases not the optimal one. To understand this point it is convenient to consider the contributions to a certain observable (for instance total cross section or asymmetry), of any perturbation order, separated into leading logarithmic terms, next-to-leading logarithmic terms, etc. (see Fig. 1). There L stands for the leading log and α for the coupling constant. If our calculation is limited, let us say to order n , then we may omit all contributions that are smaller than $\alpha^{n+1}L^{n+1}$, that is all terms proportional to $\alpha^i L^{i-j} < \alpha^{n+1}L^{n+1}$, without weakening the total precision of our calculations. We will call it “pragmatic” n th order.

For the typical LEP applications $L \simeq 24$ (electron in the initial state), and is not so much smaller for the other leptons. Inclusion of complete order- α^2 non-leading terms does not improve on the precision, even if we work in “pragmatic” third order (see Fig. 1), because fourth-order (leading logarithmic) terms are expected to be larger².

In discussing the systematic error of QED predictions one should not forget that large computer programs can be prone to simple programming bugs, machine rounding errors and other numerical problems, which we call collectively *technical errors*. They form the

²It should be stressed here that, unlike in QCD where there is always a certain uncertainty due to, e.g., the non-perturbative content of the structure functions, in QED the answers are unique, once the framework of calculation is defined.

technical component of the total precision/error and have to be calculated first, before any attempt at discussing the physical precision can be made. A solution to this problem is to calculate a certain observable using different methods and obtain numerically the same answer. The optimal solution is to calculate the prediction for a certain observable analytically and to compare this result with the Monte Carlo simulation obtained with identical cuts and identical physical input.

The aim of these lectures is to present relatively simple calculations (with only minor and explicitly listed simplifications), which are useful in understanding physics of radiative corrections. Calculations are presented at length so that the experimental physicist is helped to comprehend theoretical techniques. Parts of the text which may disturb the reader at first reading, are written in smaller, footnote-size characters.

3 BASIC DEFINITIONS

In the following, we will recall conventions for spinors normalization, Feynman rules for QED etc. as defined in ref. [4]. Following these conventions we will use the speed of light $c = 1$, and all energies and momenta as well as masses will be given in units of GeV. We assume that the reader will browse quickly through this section first, and later will come back for definitions whenever necessary. Actually there is nothing more than a list of definitions in this section.

3.1 Phase Space, Decay Rate, Cross Section

We will start with a definition of the differential cross section:

$$d\sigma = \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} |\mathcal{M}|^2 \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} (2\pi)^4 \delta^4 \left(p_1 + p_2 - \sum_1^n k_i \right) \times S \quad (1)$$

and the differential decay rate of particle with mass M (and four-momentum P) as:

$$d\omega = \frac{1}{2M} |\mathcal{M}|^2 \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} (2\pi)^4 \delta^4 \left(P - \sum_1^n k_i \right) \times S. \quad (2)$$

In these two formulas one can find (i) a kinematical factor, (ii) a matrix element squared, and (iii) the phase space for the final-state particles. Let us elaborate on these three ingredients of the cross section in more detail:

(i) In the case of the decay, the kinematical factor is just $\frac{1}{2}$ of the inverse decaying particle mass. For the scattering process, the expression is slightly more complicated:

$$\frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0}; \quad (3)$$

here v_1 , v_2 denote velocities of colliding particles and p_1 , p_2 their four-momenta. Zero-th components of four-momenta denote energies.

(ii) All dynamical information (which may be calculated from QED for example) is included in the matrix element \mathcal{M} . As particles of non-zero spin (or carrying colour, for instance) may participate in the reaction, $|\mathcal{M}|^2$ should read as a contraction of the

corresponding density matrices [5] of the initial and final states with spin amplitude \mathcal{M} and its Hermitian conjugate \mathcal{M}^\dagger . If one is not interested in spin-dependent effects, a summation over all possible final-state spin states and an average over initial-state spin states should be performed. The \mathcal{M} is a function of all momenta of incoming and outgoing particles.

(iii) The phase space

$$dLips_n(P) \times S = \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} (2\pi)^4 \delta^4 \left(P - \sum_1^n k_i \right) \times S \quad (4)$$

consists of a four-dimensional δ^4 function enforcing four-momentum conservation and, for each particle in the final-state, a Lorentz-invariant integration element over particle momenta³ $\frac{d^3 k_i}{2k_i^0 (2\pi)^3}$. In addition if there are m_i identical particles of type i in the final state, then the statistical factor reads: $S = \prod_i \frac{1}{m_i!}$. Depending on the convention for the spinor normalization in use, a factor $\frac{m}{p_i^0}$ may need to replace factor $\frac{1}{2k_i}$ for the final-state and initial-state particles. In case of scattering: $P = p_1 + p_2$.

3.2 Dirac Equation, gamma matrices and some of their relations

Let me start with the four-dimensional γ^μ matrices ($\mu = 0, 1, 2, 3$). Their most important relation, from our point of view, is the anti commutation relation:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \mathbf{1}, \quad (7)$$

where $\mathbf{1}$ stands for the four-dimensional unit matrix (it will usually be omitted in our formulas). The Lorentz metric tensor is $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$ and otherwise zero; also, $g^{\mu\nu} = g_{\mu\nu}$. We use the following short-hand notations:

$$\begin{aligned} pq &= p^\mu q_\mu = p^\mu q^\nu g_{\mu\nu} = \sum_{\mu=0}^3 \sum_{\nu=0}^3 p^\mu q^\nu g_{\mu\nu} \\ \not{p} &= p_\mu \gamma^\mu = p^\mu \gamma_\mu = \sum_{\mu=0}^3 \sum_{\nu=0}^3 p^\mu \gamma^\nu g_{\mu\nu}. \end{aligned} \quad (8)$$

It is now straightforward to realize⁴

$$\begin{aligned} \not{p} \not{q} &= - \not{q} \not{p} + 2pq \\ \not{p} \not{p} &= p^2. \end{aligned} \quad (9)$$

³Note that

$$\frac{d^3 k_i}{2k_i^0} = d^4 k_i \delta(k_i^2 - m^2) \Theta(k_i^0). \quad (5)$$

This relation can be easily obtained by simple integration of the δ function over the zero-th component of k , using the following mathematical formula:

$$\int g(x) \delta(f(x) - a) dx = g(x) \frac{1}{|f'(x)|} \Big|_{x=f^{-1}(a)} \quad (6)$$

Convince yourself!

⁴Do it!

The spinors $u(p, s)$ and $v(p, s)$, for particles and anti particles, are four-component complex vector-like objects, but of different Lorentz transformation properties with respect to usual vectors. If their momentum and spin are denoted as p and s , they satisfy the Dirac Equation:

$$\begin{aligned}(\not{p} - m)u(p, s) &= 0 \\ (\not{p} + m)v(p, s) &= 0.\end{aligned}\tag{10}$$

We will use also adjoint spinors

$$\begin{aligned}\bar{u} &= u^\dagger \gamma^0 \\ \bar{v} &= v^\dagger \gamma^0.\end{aligned}\tag{11}$$

For adjoint spinors, the Dirac Equation reads⁵

$$\begin{aligned}\bar{u}(p, s)(\not{p} + m) &= 0 \\ \bar{v}(p, s)(\not{p} - m) &= 0.\end{aligned}\tag{12}$$

The normalization condition for spinors reads as follows

$$\begin{aligned}\bar{u}(p, s)u(p, s) &= 1 \\ \bar{v}(p, s)v(p, s) &= -1.\end{aligned}\tag{13}$$

The following relations (projection operators) will turn out to be very useful in calculating the squares of matrix elements:

$$\begin{aligned}u(p, s)\bar{u}(p, s) &= \left(\frac{\not{p} + m}{2m} \cdot \frac{1 + \gamma_5 \not{s}}{2}\right) \\ v(p, s)\bar{v}(p, s) &= -\left(\frac{\not{p} - m}{2m} \cdot \frac{1 + \gamma_5 \not{s}}{2}\right) \\ \gamma_5 = \gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3.\end{aligned}\tag{14}$$

Summing over the spin simplifies projection operators; they take a form

$$\begin{aligned}\sum_{\pm s} u(p, s)\bar{u}(p, s) &= \left(\frac{\not{p} + m}{2m}\right) \\ \sum_{\pm s} v(p, s)\bar{v}(p, s) &= -\left(\frac{\not{p} - m}{2m}\right).\end{aligned}\tag{15}$$

3.3 Feynman rules

In the following, let us list those of the Feynman rules of QED that will be used in our lectures for calculating \mathcal{M} .

1. For each internal fermion line (i.e. connecting two vertices) carrying momentum p there is a fermion propagator $iS_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}$. See Fig. 2. Note that the fermion line is oriented and p is the momentum carried in the diagram *along* fermion line orientation.

⁵It can be obtained from the previous two Equations.

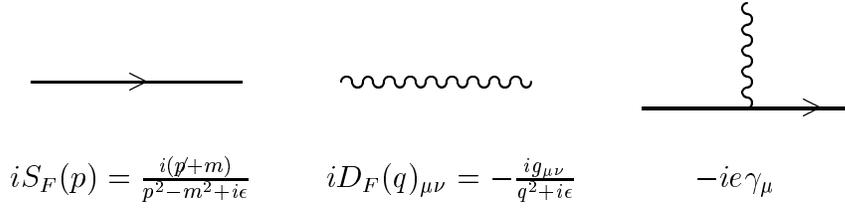


Figure 2: Graphic representation for Feynman rules, respectively for fermion and photon propagators, and for electromagnetic coupling.

2. For each internal photon line there is a photon propagator $iD_F(q)_{\mu\nu} = -\frac{ig_{\mu\nu}}{q^2+i\epsilon}$. See Fig. 2.
3. For the photon coupling to fermion line the vertex is: $-ie\gamma_\mu$. See Fig. 2.
4. For each external fermion line entering a graph there is a factor (scattering particle wave function) $u(p, s)$ or $v(p, s)$ according to whether it enters in the initial or final state. Similarly for the fermion line leaving the diagram in the initial and final state factor: $\bar{u}(p, s)$ or $\bar{v}(p, s)$.
5. For every external photon line, a factor ε_μ (photon polarization) has to be introduced.
6. For each internal momentum l not fixed by momentum conservation at vertices there is a factor $\int \frac{d^4l}{(2\pi)^4}$.
7. For each closed fermion loop there is a factor -1 .

For completeness, let us note that

$$\alpha = \frac{e^2}{4\pi}. \quad (16)$$

3.4 Electron-Muon Scattering $e^+e^- \rightarrow \mu^+\mu^-$ (*)

During lectures in a year 2004, it turned out that at this point it is interesting, and possible to calculate amplitude and later cross section for the process: $e^+e^- \rightarrow \mu^+\mu^-$. This section was written by Bartłomiej Biedroń on the basis on my improvised lecture.

Let us start with the amplitude \mathcal{M} for our process. Following the Feynman's rules for QED

$$\mathcal{M} = \bar{v}(p_2)\gamma^\mu(-ie)u(p_1)\frac{ig_{\mu\nu}}{(p_1+p_2)^2+i\epsilon}\bar{u}(q_2)\gamma^\nu(ie)v(q_1) \quad (17)$$

Averaging over initial (factor $\frac{1}{4}$) and summing over final spins we get

$$\begin{aligned} \mathcal{M}\mathcal{M}^\dagger &= \frac{1}{4} \sum_{s_1, s_2, s'_1, s'_2} \bar{v}(p_2, s_2)\gamma^\mu(-ie)u(p_1, s_1)\frac{ig_{\mu\nu}}{(p_1+p_2)^2+i\epsilon}\bar{u}(q_2, s'_2)\gamma^\nu(ie)v(q_1, s'_1) \\ &\quad \bar{u}(p_1, s_1)\gamma^{\mu'}(ie)v(p_2, s_2)\frac{-ig_{\mu'\nu'}}{(p_1+p_2)^2-i\epsilon}\bar{v}(q_1, s'_1)\gamma^{\nu'}(-ie)u(q_2, s'_2) \end{aligned} \quad (18)$$

At this point all variables are still defined in the frame independent way

$$\begin{aligned} \mathcal{M}\mathcal{M}^\dagger &= \frac{1}{4} \left[\frac{e^2}{(p_1 + p_2)^2} \right]^2 \sum_{spin} \bar{v}(p_2, s_2) \gamma^\mu u(p_1, s_1) \bar{u}(p_1, s_1) \gamma^{\mu'} v(p_2, s_2) \\ &\quad \bar{u}(q_2, s'_2) \gamma_\mu v(q_1, s'_1) \bar{v}(q_1, s'_1) \gamma_{\mu'} u(q_2, s'_2) \end{aligned} \quad (19)$$

Let us use now formula (15), then

$$\begin{aligned} \mathcal{M}\mathcal{M}^\dagger &= \frac{1}{4} \left[\frac{e^2}{(p_1 + p_2)^2} \right]^2 Tr \left[\frac{-(p_2 - m)}{2m} \gamma^\mu \frac{(p_1 + m)}{2m} \gamma^{\mu'} \right] Tr \left[\frac{-(q_2 - m')}{2m'} \gamma_\mu \frac{(q_1 + m')}{2m'} \gamma_{\mu'} \right] \\ &= \frac{1}{4} \left[\frac{e^2}{(p_1 + p_2)^2} \right]^2 \frac{1}{4m^2 4(m')^2} Tr \left[p_1 \gamma^\mu p_2 \gamma^{\mu'} \right] Tr \left[q_1 \gamma_\mu q_2 \gamma_{\mu'} \right] \end{aligned} \quad (20)$$

Next we will use the following identity which can be obtained from the anticommutation property of γ matrices

$$Tr(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(g^{\mu\nu} g^{\rho\sigma} + g^{\mu\sigma} g^{\nu\rho} - g^{\mu\rho} g^{\nu\sigma}) \quad (21)$$

We neglect all masses, that is we will work in ultrarelativistic limit. The square of the invariant amplitude now becomes

$$\begin{aligned} \mathcal{M}\mathcal{M}^\dagger &= \frac{1}{4} \left[\frac{e^2}{(p_1 + p_2)^2} \right]^2 \frac{1}{16m^2(m')^2} 4 \left[(p_2)^\mu (p_1)^{\mu'} + (p_2)^{\mu'} (p_1)^\mu - g^{\mu\mu'} p_1 p_2 \right] \\ &\quad 4 \left[(q_2)_\mu (q_1)_{\mu'} + (q_2)_{\mu'} (q_1)_\mu - g_{\mu\mu'} q_1 q_2 \right] \end{aligned} \quad (22)$$

Using also equation

$$s = (p_1 + p_2)^2 = (q_1 + q_2)^2 \simeq 2p_1 p_2 \simeq 2q_1 q_2 \quad (23)$$

we calculate

$$\begin{aligned} \mathcal{M}\mathcal{M}^\dagger &= \frac{1}{4} \frac{e^4}{s^2 m^2 (m')^2} \left[(p_1)^\mu (p_2)^{\mu'} + (p_1)^{\mu'} (p_2)^\mu - g^{\mu\mu'} \frac{s}{2} \right] \\ &\quad \left[(q_1)_\mu (q_2)_{\mu'} + (q_1)_{\mu'} (q_2)_\mu - g_{\mu\mu'} \frac{s}{2} \right] \end{aligned} \quad (24)$$

Let us choose now, the explicit coordinate frame, where $w = \cos\theta$, $l = \sin\theta$. θ denotes the angle between the axes of incoming and outgoing particles. In this centre-of-mass system, we have

$$\begin{aligned} p_1 &= (E, 0, 0, E) \\ p_2 &= (E, 0, 0, -E) \\ q_1 &= (E, 0, lE, wE) \\ q_2 &= (E, 0, -lE, -wE) \end{aligned} \quad (25)$$

The spin average square of the invariant amplitude can be written as

$$\begin{aligned}
\mathcal{M}\mathcal{M}^\dagger &= \frac{1}{4} \frac{e^4}{s^2} \frac{1}{m^2(m')^2} \left(2p_1q_1p_2q_2 + 2p_1q_2p_2q_1 - sp_1p_2 - sq_1q_2 + s^2 \right) \\
&= \frac{e^4}{s^2} \frac{1}{m^2(m')^2} 2(p_1q_1p_2q_2 + p_1q_2p_2q_1) \\
&= \frac{e^4}{s^2} \frac{1}{m^2(m')^2} 2E^4 [(1-w)(1-w) + (1+w)(1+w)]
\end{aligned} \tag{26}$$

We finally get for the matrix element averaged over the spin

$$\mathcal{M}\mathcal{M}^\dagger = \frac{e^4}{s^4} \frac{1}{m^2(m')^2} E^4 (1 + \cos^2 \theta) \tag{27}$$

$$\mathcal{M}\mathcal{M}^\dagger = \frac{e^4}{s^4} \frac{1}{m^2(m')^2} E^4 (1 + \cos^2 \theta) \tag{28}$$

In the next step we can use phase space parametrization (from the next Section) and definition of the cross section from Section 3.1 (36) Note replacement in this formula (1) due to incoming and outgoing fermionic fields. They are defined in the text, not in the formulas. They give factor $(2m)^2(2m')^2$.

$$\begin{aligned}
\frac{d\sigma}{d \cos \theta d\phi} &= \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \cdot \frac{e^4}{s^2} \frac{1}{m^2(m')^2} E^4 (1 + \cos^2 \theta) \\
&\cdot \frac{1}{(2\pi)^2} \frac{1}{8} \frac{\lambda^{\frac{1}{2}}(M^2, m_1^2, m_2^2)}{M^2} \cdot (2m)^2 (2m')^2
\end{aligned} \tag{29}$$

after simplifications

$$\frac{d\sigma}{d \cos \theta d\phi} = \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \frac{e^4}{s^2} s^2 (1 + \cos^2 \theta) \frac{1}{(2\pi)^2} \frac{1}{8} \frac{\lambda^{\frac{1}{2}}(M^2, m_1^2, m_2^2)}{M^2} \tag{30}$$

or (in ultrarelativistic limit $\lambda^{\frac{1}{2}}(M^2, m_1^2, m_2^2) = M^2$)

$$\frac{d\sigma}{d \cos \theta d\phi} = \frac{1}{|v_1 - v_2|} \frac{1}{s} \frac{e^4}{8(2\pi)^2} (1 + \cos^2 \theta) \tag{31}$$

and finally, if one use definition of α , ultrarelativistic limit and natural units where $v_1 - v_2 = 2c = 2$ we get:

$$\frac{d\sigma}{d \cos \theta d\phi} = \frac{1}{2|v_1 - v_2|} \frac{1}{s} \alpha^2 (1 + \cos^2 \theta) = \frac{\alpha^2}{4} \frac{1}{s} (1 + \cos^2 \theta) \tag{32}$$

After integration over angles:

$$\sigma_{tot} = \frac{\alpha^2}{4} \frac{1}{s} 2\pi \frac{8}{3} = \frac{4\pi}{3} \frac{\alpha^2}{s} \tag{33}$$

We are ready to search for missing factors of 2π etc etc.

(*) *End of part collected by Bartłomiej Biedron*

4 PHASE SPACE

The differential cross section is the product of the phase space and the matrix element. To explain properties of factorization, we shall discuss them first independently and later combine them.

The plan of this chapter is to first present a calculation of the two-body phase space as an example of the method. In the second step we will present the relation between n -body and the $(n + 1)$ -body phase space, initially in a general form and later in a form suitable for exponentiation. Finally, we will obtain the same result in an intuitive, easy but oversimplified way.

5 Two-body phase space

In the case of a two-body final state, the phase space (4) can be written, with the help of (5), in the following form,

$$\begin{aligned}
 dLips_2(P) &= \\
 & \frac{d^3 k_1}{2k_1^0(2\pi)^3} \frac{d^3 k_2}{2k_2^0(2\pi)^3} (2\pi)^4 \delta^4(P - k_1 - k_2) \\
 &= \frac{1}{(2\pi)^2} \frac{d^3 k_1}{2k_1^0} \frac{d^3 k_2}{2k_2^0} \delta^4(P - k_1 - k_2) \\
 &= \frac{1}{(2\pi)^2} d^4 k_1 \delta(k_1^2 - m_1^2) \Theta(k_1^0) d^4 k_2 \delta(k_2^2 - m_2^2) \Theta(k_2^0) \delta^4(P - k_1 - k_2), \quad (34)
 \end{aligned}$$

where m_1, m_2 denote the masses of the two particles in the final state. We integrate four-dimensional δ^4 with the $d^4 k_2$, and in the next step $\delta(k_1^2 - m_2^2)$ with dk_1^0 . We get the following expression:

$$\begin{aligned}
 dLips_2(P) &= \\
 & \frac{1}{(2\pi)^2} d^4 k_1 \delta(k_1^2 - m_1^2) \delta\left((P - k_1)^2 - m_2^2\right) \Theta(k_1^0) \Theta(P^0 - k_1^0) \\
 &= \frac{1}{(2\pi)^2} \frac{d^3 k_1}{2k_1^0} \delta\left((P - k_1)^2 - m_2^2\right). \quad (35)
 \end{aligned}$$

At this moment we are left with the three-dimensional integration and a one-dimensional δ . We represent a three-dimensional volume in spherical coordinates $d^3 k_1 = k^2 dk d \cos \theta d \phi$. Here k denotes a module of the three-vector part of k_1^μ ; $k = \sqrt{(k_1^1)^2 + (k_1^2)^2 + (k_1^3)^2}$ angular variables θ and ϕ can be defined in any coordinate frame, provided that its definition is independent from the four-vector $k_1 - k_2$. In the final step of our calculation, we will first change the variables for Lorentz-dependent ones and later use formula (6):

$$\begin{aligned}
 dLips_2(P) &= \\
 & \frac{1}{(2\pi)^2} d \cos \theta d \phi dk \frac{k^2}{2\sqrt{k^2 + m_1^2}} \delta\left(M^2 - 2M\sqrt{k^2 + m_1^2} + m_1^2 - m_2^2\right) \\
 &= \frac{1}{(2\pi)^2} d \cos \theta d \phi \frac{1}{8} \frac{\lambda^{\frac{1}{2}}(M^2, m_1^2, m_2^2)}{M^2}. \quad (36)
 \end{aligned}$$

Here for the first time we use the λ function

$$\lambda(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc, \quad (37)$$

which is very common in any calculation of the phase space with massive particle kinematics. Finally $M^2 = P^2$.

5.1 $(n + 1) \rightarrow n$ -body phase space

Now, having gained some experience, let us browse through the case of the n -body phase space. We will try to calculate the relation expressing $(n+1)$ -body phase space as a convolution of n -body phase space with single particle variables. By iteration we will get an explicit phase-space parametrization valid for any number of final-state particles.

Let us start again with formula (4) and rewrite it for $(n + 1)$ different particles. We define $q = k_{n+1}$ and $p = \sum_1^n k_i$; $p^2 = M_1^2$, $m_{n+1} = m$. We then have

$$\begin{aligned} dLips_{n+1}(P) &= \\ & \frac{d^3 k_1}{2k_1^0(2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0(2\pi)^3} \frac{d^3 q}{2q^0(2\pi)^3} (2\pi)^4 \delta^4\left(P - \sum_1^n k_i - q\right) \\ &= d^4 p \delta^4(P - p - q) \frac{d^3 q}{2q^0(2\pi)^3} \frac{d^3 k_1}{2k_1^0(2\pi)^3} \cdots \frac{d^3 k_n}{2k_n^0(2\pi)^3} (2\pi)^4 \delta^4\left(p - \sum_1^n k_i\right) \\ &= d^4 p \delta^4(P - p - q) \frac{d^3 q}{2q^0(2\pi)^3} dLips_n(p \rightarrow k_1 \dots k_n). \end{aligned} \quad (38)$$

We have found a relation between $(n + 1)$ -body phase space and n -body phase space. Let us rewrite it in a more convenient way. To this end we introduce another integration over M_1^2 i.e. the mass of p :

$$\begin{aligned} dLips_{n+1}(P) &= \\ & dM_1^2 \left\{ d^4 p \delta(p^2 - M_1^2) \Theta(p^0) \delta^4(P - p - q) \frac{d^3 q}{2q^0(2\pi)^3} \right\} dLips_n(p \rightarrow k_1 \dots k_n) \\ &= dM_1^2 (2\pi)^{-1} \left\{ \frac{d^3 p}{2p^0(2\pi)^3} (2\pi)^4 \delta^4(P - p - q) \frac{d^3 q}{2q^0(2\pi)^3} \right\} dLips_n(p \rightarrow k_1 \dots k_n) \\ &= dM_1^2 (2\pi)^{-1} dLips_2(P \rightarrow p q) \times dLips_n(p \rightarrow k_1 \dots k_n) \\ &= dM_1^2 (2\pi)^{-1} \left\{ \frac{1}{(2\pi)^2} d \cos \theta d\phi \frac{1}{8} \frac{\lambda^{\frac{1}{2}}(M^2, M_1^2, m^2)}{M^2} \right\} \times dLips_n(p \rightarrow k_1 \dots k_n) \\ &= \left[dM_1^2 d \cos \theta d\phi \frac{1}{8(2\pi)^3} \frac{\lambda^{\frac{1}{2}}(M^2, M_1^2, m^2)}{M^2} \right] \times dLips_n(p \rightarrow k_1 \dots k_n). \end{aligned} \quad (39)$$

Let us now recall the two most interesting forms of the $(n + 1)$ -body phase space:

$$\begin{aligned} dLips_{n+1}(P) &= \\ & \frac{dM_1^2}{(2\pi)} dLips_2(P \rightarrow p q) \times dLips_n(p \rightarrow k_1 \dots k_n) \\ &= dM_1^2 \left[d \cos \theta d\phi \frac{1}{8(2\pi)^3} \frac{\lambda^{\frac{1}{2}}(M^2, M_1^2, m^2)}{M^2} \right] \times dLips_n(p \rightarrow k_1 \dots k_n). \end{aligned} \quad (40)$$

The first one exhibits the iterative relation between $n + 1$ -, n - and two-body phase space as in the cascade decay. The second one will be used later in our lectures. Note that angles θ and ϕ define the direction of the p or q in the P rest frame. As in the case of two-body phase space these angles can be defined with respect to any frame, under the sole condition that it is independent from the $(p - q)$ four-vector. It may depend on all other kinematical variables!

It is rather easy to realize that formula (39) can be iterated to give explicit parametrization of n -body phase space using invariant masses M_k of $(n - k)$ particle systems and θ_k, ϕ_k angles defining $(n - k)$ particle orientation in the M_k restframe.

$$dLips_n(P) = \prod_{i=1}^{n-2} \left[dM_i^2 d \cos \theta_i d\phi_i \frac{1}{8(2\pi)^3} \frac{\lambda^{\frac{1}{2}}(M_{i-1}^2, M_i^2, m_{n-i+1}^2)}{M_{i-1}^2} \right] \\ \times \left(\frac{1}{8(2\pi)^3} d \cos \theta_{n-1} d\phi_{n-1} \frac{\lambda^{\frac{1}{2}}(M_{n-2}^2, m_1^2, m_2^2)}{M_{n-2}^2} \right), \quad (41)$$

where

$$M_i^2 = \left(\sum_{l=1}^{n-i} k_l \right)^2, \quad M_0 = M. \quad (42)$$

5.2 Phase Space in form for Exponentiation

Let us now go back to formula (39) and let the photon be an $(n + 1)$ particle. Because of the photon zero mass our formula simplifies and we get

$$dLips_{n+1}(P) = \left[dM_1^2 d \cos \theta d\phi \frac{1}{8(2\pi)^3} \frac{(M^2 - M_1^2)}{M^2} \right] \times dLips_n(p \rightarrow k_1 \dots k_n). \quad (43)$$

Now, we can change variables, and instead of M_1 , use the photon energy k_γ defined in the rest frame of the all $(n + 1)$ particles. From energy-momentum conservation, we find that in this frame the space-like components of momenta p and $q = k_\gamma$ have to have the same value but opposite signs. That is why $p^0 = \sqrt{k_\gamma^2 + M_1^2}$. From the energy conservation we find

$$k_\gamma + \sqrt{k_\gamma^2 + M_1^2} = M, \quad (44)$$

and we can calculate easily

$$k_\gamma = \frac{M^2 - M_1^2}{2M} \\ dM_1^2 = 2M dk_\gamma. \quad (45)$$

Now we get:

$$dLips_{n+1}(P) = \\ \left[2M dk_\gamma \frac{2k_\gamma}{M} d \cos \theta d\phi \frac{1}{8(2\pi)^3} \right] \times dLips_n(p \rightarrow k_1 \dots k_n) \\ = \left[k_\gamma dk_\gamma d \cos \theta d\phi \frac{1}{2(2\pi)^3} \right] \times dLips_n(p \rightarrow k_1 \dots k_n), \quad (46)$$

If we have more than one photon, let us say l photons and n other particles, then the factor in brackets will become iterated (we can repeat the reasoning presented before l -times to convince ourselves) and the statistical factor $S = \frac{1}{l!}$ will have to be introduced. We will obtain:

$$dLips_{n+l}(P) \times S = \frac{1}{l!} \prod_{i=1}^l \left[k_{\gamma_i} dk_{\gamma_i} d \cos \theta_i d\phi_i \frac{1}{2(2\pi)^3} \right] \times dLips_n(p \rightarrow k_1 \dots k_n). \quad (47)$$

One can see that, in the limit of all photons having small energies, one starts to obtain something like the l -th element of exponent expansion. Note that $p = p(P, k_{\gamma_1}, \dots, k_{\gamma_l})$.

This property, which we will use later in explaining how the exponentiation can be obtained, is not as easy to apply as it seems here. The reason is very simple: for each photon, the energy k_i and the angular variables θ_i, ϕ_i are defined in a different frame separated by a boost. Nevertheless the calculation presented here is performed without any approximations.

5.3 Do it easily and fast!

A question is in order now: Is it really necessary to perform such a complicated reasoning to obtain the soft photon limit of formula (47)? If we assume that the momenta of all photons are negligibly small with respect to those of all other particles, and we therefore drop them from the arguments of the δ^4 ; function we can rewrite formula (4) in the following form:

$$\begin{aligned} dLips_{n+l}(P) \times S &= \\ & \frac{1}{l!} \frac{d^3 k_{\gamma_1}}{2k_{\gamma_1}^0 (2\pi)^3} \dots \frac{d^3 k_{\gamma_l}}{2k_{\gamma_l}^0 (2\pi)^3} \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \dots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} (2\pi)^4 \delta^4 \left(P - \sum_1^n k_i - \sum_1^l k_{\gamma_i} \right) \\ &= \frac{1}{l!} \frac{1}{2(2\pi)^3} d \cos \theta_{\gamma_1} d\phi_{\gamma_1} k_{\gamma_1}^0 dk_{\gamma_1}^0 \dots \frac{1}{2(2\pi)^3} d \cos \theta_{\gamma_l} d\phi_{\gamma_l} k_{\gamma_l}^0 dk_{\gamma_l}^0 \\ & \times \frac{d^3 k_1}{2k_1^0 (2\pi)^3} \dots \frac{d^3 k_n}{2k_n^0 (2\pi)^3} (2\pi)^4 \delta^4 \left(P - \sum_1^n k_i \right). \end{aligned} \quad (48)$$

Here we have used again the spherical coordinates $d^3 k_\gamma = (k_\gamma^0)^2 d \cos \theta d\phi$.

Even though we have obtained intuitively the same result as in the previous chapter, its quality is much lower. In particular the obtained formula would not be valid beyond the soft photon limit.

6 FACTORIZATION OF THE REAL SOFT PHOTON

Now, having prepared the phase space, we will turn our attention to the matrix element. In this chapter we will concentrate on one of the classes of diagrams, that is real bremsstrahlung. We recommend that the reader should glance through any Born level calculation of any cross section from Feynman rules⁶ before recalculating this chapter. We will show that in the soft

⁶For instance your own notes from physics classes or from the exercises to Prof. Bilenky lectures.

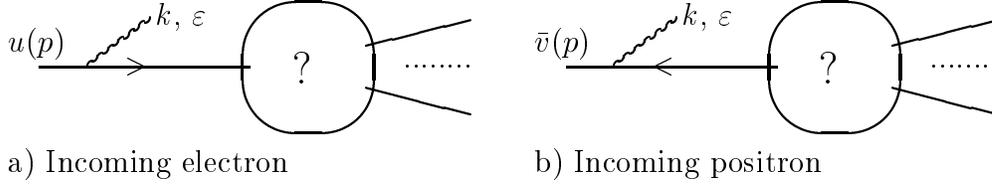


Figure 3: *Feynman diagrams for photon emission in initial state respectively from electron and positron. Dots represent all other fields entering amplitude (initial or final). Note that in case of positron arrow points in opposite direction, even though it is also initial state particle.*

photon limit this matrix element can be represented as a product of the lower-order matrix element times the *soft photon factor*, which turns out to be independent from the properties of the particular Born process under consideration.

6.1 Bremsstrahlung from the incoming electron

Let us start with the amplitude where bremsstrahlung occurs from the incoming electron. Using the Feynman rules collected in section 3, we find that the amplitude for our process (see Fig. 3) reads:

$$\begin{aligned} \mathcal{M} &= \dots iS_F(p-k)(-ie\gamma_\mu)u(p,s)\varepsilon^\mu \\ &= \dots \frac{\not{p}-\not{k}+m}{(p-k)^2-m^2+i\epsilon}e\not{\varepsilon}u(p,s). \end{aligned} \quad (49)$$

In this formula ε^μ denotes the photon polarization vector. We can simplify formula (49) further as $(p-k)^2 = p^2 - 2pk + k^2 = m^2 - 2pk$, and get

$$\mathcal{M} = \dots \frac{\not{p}-\not{k}+m}{-2pk}e\not{\varepsilon}u(p,s). \quad (50)$$

Note that we have omitted the infinitesimally small $i\epsilon$ term, which is important only for the virtual corrections. At the next step we will neglect \not{k} , because it is small (in the soft photon limit) with respect to other terms in the numerator of the propagator; later, we will commute \not{p} and $\not{\varepsilon}$ using formula (9) and the Dirac Equation (10):

$$\begin{aligned} \mathcal{M} &= \frac{-e}{2pk} \dots (\not{p}+m)\not{\varepsilon}u(p,s) \\ &= \frac{-e}{2pk} \dots (2p\varepsilon + \not{\varepsilon}(-\not{p}+m))u(p,s) \\ &= \frac{-e}{2pk} \dots (2p\varepsilon)u(p,s) \\ &= e\frac{-\varepsilon p}{pk} \dots u(p,s) \\ &= -e\frac{\varepsilon p}{pk}\mathcal{M}_B, \end{aligned} \quad (51)$$

where, assuming that the Born level amplitude was only weakly dependent on the difference between p and $p - k$, we have replaced $\dots u(p, s)$ by \mathcal{M}_B . We can write finally:

$$\mathcal{M} = -e \frac{\varepsilon p}{pk} \mathcal{M}_B + \mathcal{O}(k). \quad (52)$$

The phenomenological consequence of this last formula is important. We have obtained that the bremsstrahlung matrix element in the soft photon limit, can be represented as a product of the Born amplitude multiplied by the universal soft photon factor (including the electric charge).

This result can be refined substantially. In particular the terms we neglected, which we write to be at most proportional to the photon energy, deserve much more attention than given to them here!

6.2 Bremsstrahlung from the incoming positron

The calculation is in this case nearly identical. The difference is that the fermion line has the opposite orientation and that instead of a u spinor and the Dirac Equation we will have a \bar{v} spinor and the Dirac Equation in the form (12) (see also Fig. 3):

$$\begin{aligned} \mathcal{M} &= \bar{v}(p, s) (-ie\gamma_\mu) iS_F(-(p-k)) \dots \varepsilon^\mu \\ &= \bar{v}(p, s) e \not{\varepsilon} \frac{-(\not{p} - \not{k}) + m}{(p-k)^2 - m^2 + i\epsilon} \dots \\ &= \frac{e}{2pk} \bar{v}(p, s) \not{\varepsilon} ((\not{p} - \not{k}) - m) \dots \\ &= \frac{e}{2pk} \bar{v}(p, s) (2\varepsilon p - (\not{p} + m) \not{\varepsilon}) \dots \\ &= e \frac{\varepsilon p}{pk} v(p, s) \dots \\ &= e \frac{\varepsilon p}{pk} \mathcal{M}_B. \end{aligned} \quad (53)$$

Let us note that if the process we study is, for instance, from colliding e^+e^- beams we can simply add the contributions from the emission from electron and positron, and obtain

$$\mathcal{M}_R = (-e) \left(\frac{\varepsilon p_2}{p_2 k} - \frac{\varepsilon p_1}{p_1 k} \right) \mathcal{M}_B = \mathcal{M}_B \times \tilde{S}_\varepsilon(p_1, p_2, k). \quad (54)$$

6.3 Bremsstrahlung from the outgoing electron and positron

We suggest that the reader reproduce the calculation for this case. If this turns out to be too difficult, we recommend going back to the previous two sections.

6.4 Double bremsstrahlung

Let us have a short look at the case of double bremsstrahlung from the incoming electron. There are two diagrams (Fig. 4) that contribute in this case and, using the Feynman

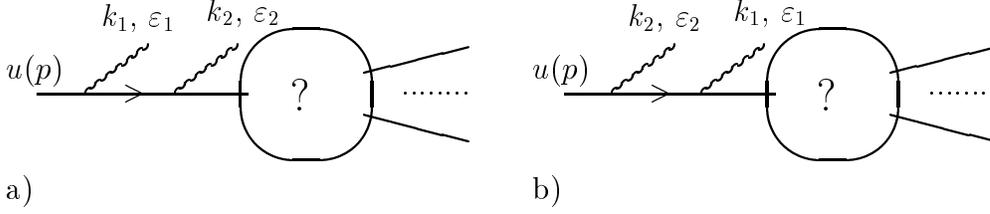


Figure 4: Feynman diagrams for double photon emission in the initial state from electron. Dots represent all other fields entering amplitude (initial or final).

diagrams from section 3, their contributions can be written as follows:

$$\begin{aligned}\mathcal{M}_1 &= \dots iS_F(p - k_1 - k_2)(-ie\gamma_\mu)iS_F(p - k_1)(-ie\gamma_\nu)u(p, s) \varepsilon_2^\mu \varepsilon_1^\nu \\ \mathcal{M}_2 &= \dots iS_F(p - k_1 - k_2)(-ie\gamma_\mu)iS_F(p - k_2)(-ie\gamma_\nu)u(p, s) \varepsilon_1^\mu \varepsilon_2^\nu.\end{aligned}\quad (55)$$

For the time being, we will concentrate on \mathcal{M}_1 . After a short manipulation we get, as in the previous cases:

$$\begin{aligned}\mathcal{M}_1 &= e^2 \frac{-1}{2k_1 p} \frac{-1}{2k_1 p + 2k_2 p - 2k_1 k_2} \dots (\not{p} - \not{k}_1 - \not{k}_2 + m) \not{\varepsilon}_2 (\not{p} - \not{k}_1 + m) \not{\varepsilon}_1 u(p, s) \\ &= e^2 \frac{-1}{2k_1 p} \frac{-1}{2k_1 p + 2k_2 p - 2k_1 k_2} \dots (\not{p} + m) \not{\varepsilon}_2 (\not{p} + m) \not{\varepsilon}_1 u(p, s).\end{aligned}\quad (56)$$

Again we commute $\not{\varepsilon}_1$ and \not{p} to obtain, with the help of the Dirac Equation:

$$\mathcal{M}_1 = e^2 \frac{-1}{2k_1 p} \frac{-1}{2k_1 p + 2k_2 p - 2k_1 k_2} \dots (\not{p} + m) \not{\varepsilon}_2 u(p, s) \quad 2\varepsilon_1 p. \quad (57)$$

After performing the same trick again for $\not{\varepsilon}_2$, we finally obtain:

$$\mathcal{M}_1 = e^2 \frac{-1}{2k_1 p} \frac{-1}{2k_1 p + 2k_2 p - 2k_1 k_2} \dots u(p, s) \quad 2\varepsilon_1 p \quad 2\varepsilon_2 p. \quad (58)$$

After some simple reorganization, and neglecting the $k_1 k_2$ term, which is small with respect to $k_1 p + k_2 p$, we get

$$\mathcal{M}_1 = e^2 \frac{-\varepsilon_1 p}{k_1 p} \frac{-\varepsilon_2 p}{k_1 p + k_2 p} \dots u(p, s) = e \frac{-\varepsilon_1 p}{k_1 p} e \frac{-\varepsilon_2 p}{k_1 p + k_2 p} \mathcal{M}_B. \quad (59)$$

Summing $\mathcal{M}_1, \mathcal{M}_2$ contributions we obtain

$$\begin{aligned}\mathcal{M} = \mathcal{M}_1 + \mathcal{M}_2 &= \left(e \frac{-\varepsilon_1 p}{k_1 p} e \frac{-\varepsilon_2 p}{k_1 p + k_2 p} + e \frac{-\varepsilon_2 p}{k_2 p} e \frac{-\varepsilon_1 p}{k_1 p + k_2 p} \right) \mathcal{M}_B \\ &= \left(e \frac{-\varepsilon_1 p}{k_1 p} \right) \left(e \frac{-\varepsilon_2 p}{k_2 p} \right) \mathcal{M}_B.\end{aligned}\quad (60)$$

Similarly, if there were to be photon emission from two different fermion lines, we would also get just the Born spin amplitude times the corresponding soft photon factors (we recommend the reader to do this calculation):

$$\mathcal{M} = \left(e \left(\frac{\varepsilon_1 p_2}{k_1 p_2} - \frac{\varepsilon_1 p_1}{k_1 p_1} \right) \right) \left(e \left(\frac{\varepsilon_2 p_2}{k_2 p_2} - \frac{\varepsilon_2 p_1}{k_2 p_1} \right) \right) \mathcal{M}_B. \quad (61)$$

In complete analogy, one can obtain⁷ for l photons:

$$\mathcal{M} = \mathcal{M}_B \prod_{i=1}^l \left(e \left(\frac{\varepsilon_i p_2}{k_i p_2} - \frac{\varepsilon_i p_1}{k_i p_1} \right) \right). \quad (62)$$

7 REAL PHOTON EXPONENT

In this step we will combine our formula for the l soft photons matrix element (62) with the corresponding phase space $dLips_{n+l}$ (47), thus obtaining the expression for the production of n particles accompanied by l soft photons (we will sum over their polarization states ε); bremsstrahlung in the initial state:

$$\begin{aligned} & |\mathcal{M}|^2 dLips_{n+l}(P) \times S \\ &= \frac{1}{l!} \times \prod_{i=1}^l \left[k_i dk_i d \cos \theta_i d\phi_i \frac{1}{2(2\pi)^3} \right] dLips_n(P \rightarrow q_1 \dots q_n) \\ &\times |\mathcal{M}_B|^2 \prod_{i=1}^l \left(e^2 \sum_{\varepsilon_i} \left(\frac{\varepsilon_i p_2}{k_i p_2} - \frac{\varepsilon_i p_1}{k_i p_1} \right)^2 \right) \\ &= \frac{1}{l!} \prod_{i=1}^l \left[k_i dk_i d \cos \theta_i d\phi_i \frac{1}{2(2\pi)^3} \sum_{\varepsilon_i} \left(e^2 \left(\frac{\varepsilon_i p_2}{k_i p_2} - \frac{\varepsilon_i p_1}{k_i p_1} \right)^2 \right) \right] \\ &\times |\mathcal{M}_B|^2 dLips_n(P \rightarrow q_1 \dots q_n). \end{aligned} \quad (63)$$

Due to our approximations, this expression is valid only for the soft photons of energies that are small with respect to any dynamical scale in the process. It is possible to write it in a more elaborate way, including all those omitted by us, in the phase space, terms due to photons effects on four-momentum conservation [6, 7, 8] (see also section 10). Then, differential distributions can be corrected, order by order in perturbation expansion, by appropriate replacement of soft photon factors with bremsstrahlung matrix element. Unfortunately, in this case the formalism becomes less friendly for intuition, and it requires much more time to be understood.

Now we can obtain an exponent for the first time. If we sum the cross sections for the configurations with 0, 1, 2, ..., photons we obtain:

$$\begin{aligned} & d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) \\ &= \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \\ &\times \sum_{l=0}^{\infty} \frac{1}{l!} \prod_{i=1}^l \left[k_i dk_i d \cos \theta_i d\phi_i \frac{1}{2(2\pi)^3} \sum_{\varepsilon_i} e^2 \left(\frac{\varepsilon_i p_2}{k_i p_2} - \frac{\varepsilon_i p_1}{k_i p_1} \right)^2 \right]. \end{aligned} \quad (64)$$

Finally, in short-hand notation:

$$d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) =$$

⁷By mathematical induction. Another point is here to recall. Namely if we divide phase space into sections, wher for example $k_i p_1 \ll k_{i+1} p_1$, then in such korners of phase-space only one of diagrams like \mathcal{M}_1 dominate. The factorization property would not require then to sum all diagrams. Such simplification is not necessary for QED, but is at the heart of design of many parton shower algorithms for QCD, it is known there under name of p_T ordering. Several variants are in use.

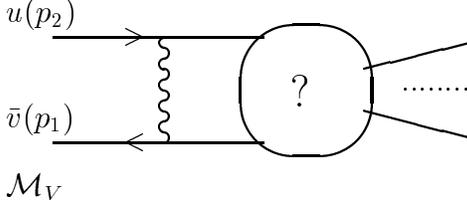


Figure 5: *Feynman diagram for the vertex-like correction in initial state in e^+ , e^- collision. Dots represent all final state fields.*

$$\frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp \left[k dk d \cos \theta d \phi \frac{1}{2(2\pi)^3} \sum_{\varepsilon} e^2 \left(\frac{\varepsilon p_2}{k p_2} - \frac{\varepsilon p_1}{k p_1} \right)^2 \right]. \quad (65)$$

It is an important and nearly complete result of our presentation of exclusive exponentiation. Unfortunately we face *infrared catastrophe*. As can be seen with the help of explicit integration of the soft photon factors over the phase space (see below), our result leads to an infinitely large prediction for the total cross section. It is unphysical. We will discuss how to resolve this later. Here, let us introduce the fictitious photon mass λ and in this way replace the $k_i dk_i$ factors of integration over the photon energy by better behaving in the soft limit term: $k_i^2 dk_i / \sqrt{k_i^2 + \lambda^2}$. This is called a *regularization* procedure.

We will use the same trick, in fact for the same purpose, in discussing the virtual corrections in the next chapters.

8 FACTORIZATION OF THE VIRTUAL SOFT PHOTON

Now, we will study the factorization of the virtual corrections. First, we will take a vertex-type amplitude \mathcal{M}_V (see Fig. 5, see also e.g. [9]) for e^+e^- scattering. Using the Feynman rules defined in section 3, it can be written as:

$$\begin{aligned} \mathcal{M}_V &= \int \frac{d^4 k}{(2\pi)^4} \left(-i \frac{g_{\mu\nu}}{k^2 + i\epsilon} \right) \\ &\quad \bar{v}(p_1) (-ie\gamma^\mu) i \frac{-(\not{p}_1 + \not{k}) + m}{(p_1 + k)^2 - m^2 + i\epsilon} \dots i \frac{(\not{p}_2 + \not{k}) + m}{(p_2 + k)^2 - m^2 + i\epsilon} (-ie\gamma^\nu) u(p_2) \\ &= -i \frac{e^2}{(2\pi)^4} \int d^4 k \frac{1}{k^2 + i\epsilon} \bar{v}(p_1) \gamma^\mu \frac{-(\not{p}_1 + \not{k}) + m}{2p_1 k + k^2 + i\epsilon} \dots \frac{(\not{p}_2 + \not{k}) + m}{2p_2 k + k^2 + i\epsilon} \gamma_\mu u(p_2). \quad (66) \end{aligned}$$

Since we are interested, as in the real bremsstrahlung case, only in the contribution of the soft photons $k^\mu \simeq 0$ region, we will neglect the \not{k} terms in the numerators of the propagators and assume that the remaining part of the diagram, noted by us as ..., does not depend on the virtual photon momentum as well.

In this way we are free also of the ultraviolet infinity, which, depending on the particular form of the Born interaction, may also be present in our diagram.

In the first step of our calculation, we will commute γ^μ matrices with \not{p}_1 and \not{p}_2 ; $\gamma^\mu \not{p}_1 = 2p_1^\mu - \not{p}_1 \gamma^\mu$, $\gamma_\mu \not{p}_2 = 2(p_2)_\mu - \not{p}_2 \gamma_\mu$ and later use Dirac Equation (10,12):

$$\begin{aligned}
\mathcal{M}_V &= \\
& -i \frac{e^2}{(2\pi)^4} \int d^4k \frac{1}{(k^2 + i\epsilon)} \frac{1}{(2p_1k + k^2 + i\epsilon)} \frac{1}{(2p_2k + k^2 + i\epsilon)} \\
& \bar{v}(p_1) \gamma^\mu (-\not{p}_1 + m) \dots (\not{p}_2 + m) \gamma_\mu u(p_2) \\
&= -i \frac{e^2}{(2\pi)^4} \int d^4k \frac{1}{(k^2 + i\epsilon)} \frac{1}{(2p_1k + k^2 + i\epsilon)} \frac{1}{(2p_2k + k^2 + i\epsilon)} \\
& \bar{v}(p_1) \left(-2(p_1)^\mu + (\not{p}_1 + m) \gamma^\mu \right) \dots \left(2(p_2)_\mu + \gamma_\mu (-\not{p}_2 + m) \right) u(p_2) \\
&= -i \frac{e^2}{(2\pi)^4} \int d^4k \frac{1}{(k^2 + i\epsilon)} \frac{1}{(2p_1k + k^2 + i\epsilon)} \frac{1}{(2p_2k + k^2 + i\epsilon)} \\
& \bar{v}(p_1) (-2)p_1^\mu \dots 2(p_2)_\mu u(p_2) \\
&= -i \frac{e^2}{(2\pi)^4} \int d^4k \frac{-4p_1p_2}{(k^2 + i\epsilon)(2p_1k + k^2 + i\epsilon)(2p_2k + k^2 + i\epsilon)} \bar{v}(p_1) \dots u(p_2) \\
&= i \frac{2\alpha}{(2\pi)^3} \int d^4k \frac{4p_1p_2}{(k^2 + i\epsilon)(2p_1k + k^2 + i\epsilon)(2p_2k + k^2 + i\epsilon)} \mathcal{M}_B. \tag{67}
\end{aligned}$$

The important result of our calculation is that the contribution from the soft photons in our diagram can be contained in a factor B' , which is independent of the Born level amplitude

$$\begin{aligned}
\mathcal{M} &= \mathcal{M}_B \times B' \\
B' &= 2\alpha \int \frac{d^4k}{(k^2 + i\epsilon)} \frac{i}{(2\pi)^3} \frac{4p_1p_2}{(2p_1k + k^2 + i\epsilon)(2p_2k + k^2 + i\epsilon)}. \tag{68}
\end{aligned}$$

It is interesting for us, that the real part of our function B' is infinitely negative! We can *regularize* it, as in the real photon case, using the photon mass λ and replacing $\frac{d^4k}{(k^2 + i\epsilon)}$ by $\frac{d^4k}{(k^2 - \lambda^2 + i\epsilon)}$.

In Yennie–Frautchi–Suura theory [6], instead of our function B' one introduces the gauge-invariant

$$B = 2\alpha \Re \int \frac{d^4k}{(k^2 + i\epsilon)} \frac{i}{(2\pi)^3} \left(\frac{2p_1 - k}{2p_1k + k^2 + i\epsilon} - \frac{2p_2 - k}{2p_2k + k^2 + i\epsilon} \right)^2, \tag{69}$$

which differs from our result only by terms non-leading in powers of k .

It would be instructive to recalculate our correction in the case of final-state bremsstrahlung and interference (additional photon line connecting charged lines from the initial and final states), we encourage the reader to repeat the calculation in these cases as well.

9 VIRTUAL PHOTON EXPONENT

In the previous section we have shown that the amplitude \mathcal{M}_V of first-order vertex-like correction in the soft photon approximation equals $\mathcal{M}_B \times B'$ (or B) (68). Here we should sketch how the soft photon exponent is forming. We think, however, that it is

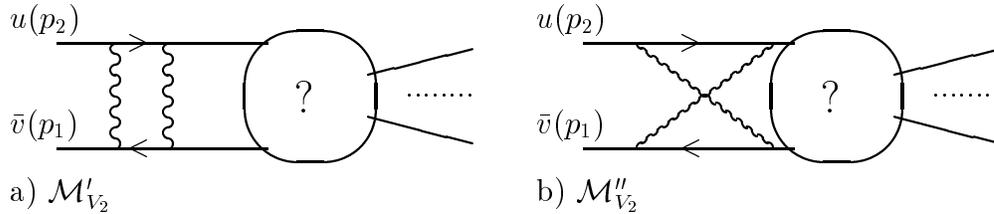


Figure 6: *Feynman diagrams for the vertex-like correction in initial state in e^+ , e^- collision, second order. Dots represent all final state fields.*

too complicated and not sufficiently explanatory to perform this discussion. We will just quote the result: the second-order vertex amplitude (see Fig. 6) \mathcal{M}_{V_2} reads:

$$\mathcal{M}_{V_2} = \mathcal{M}_B \times \frac{1}{2!} B^2, \quad (70)$$

and similarly for the n -th order

$$\mathcal{M}_{V_n} = \mathcal{M}_B \times \frac{1}{n!} B^n. \quad (71)$$

Summing all these soft photon vertex-like corrections we again obtain an exponent

$$\mathcal{M}_{V_{exp}} = \mathcal{M}_B \times \sum_{n=0}^{\infty} \frac{1}{n!} B^n = \mathcal{M}_B \times \exp(B). \quad (72)$$

As we have already mentioned in the previous section, B is negative-infinite if the regulator (photon mass λ) goes to zero. This is quite the opposite to the effect of the real soft photon effect described in the section 6.

There following technical points complicate the picture of how the virtual photon exponent forms. We must understand where $\frac{1}{n!}$ comes from. Finally one may want to understand how ultraviolet subtractions mix the soft photon structure of the calculations. We refer the motivated reader to [6, 10].

Let us point out that mixed, real-bremsstrahlung vertex-like amplitudes (see Fig. 7) lead to the following result in the soft photon limit:

$$\mathcal{M}_{V_1 R_1} = \mathcal{M}_B \times B \times \tilde{S}_\varepsilon(p_1, p_2, k) \quad (73)$$

and, if we sum over all diagrams for j virtual photon lines and l real, we get (again in the soft photon limit!)

$$\mathcal{M}_{V_j R_l} = \mathcal{M}_B \times \frac{1}{j!} B^j \times \prod_{m=1}^l \tilde{S}_{\varepsilon_m}(p_1, p_2, k_m). \quad (74)$$

Again, as in the case of higher-order vertex corrections, we omit proof or any other discussion on why this factorization holds. We address the determined reader to [6].

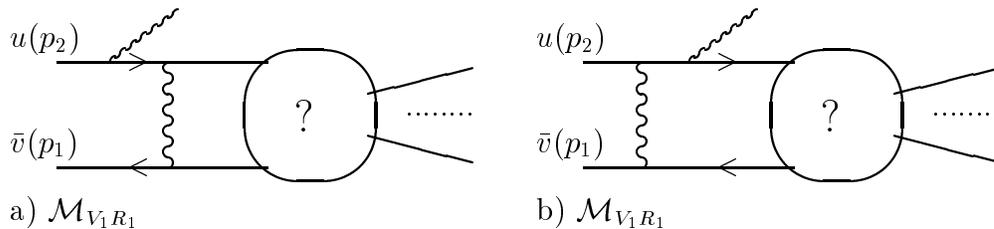


Figure 7: *Feynman diagrams for the mixed real-bremsstrahlung vertex-like correction in initial state in e^+ , e^- collision, second order, emission from electron (diagrams for emission from positron are similar but dropped here). Dots represent all final state fields.*

10 EXCLUSIVE EXPONENTIATION, ZEROTH ORDER

In some sense, this section is the keystone of our lectures. We will combine here the virtual photon exponent developed in section 8 with the real photon exponent developed in section 6. Now, we are finally ready to obtain an exclusive exponentiation formula: To this end we substitute \mathcal{M}_B in formula (65) with $\mathcal{M}_{V_{exp}}$ of (72) obtaining:

$$d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) = \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \times \\ |\mathcal{M}_B \exp(B)|^2 dLips_n(p \rightarrow q_1 \dots q_n) \exp \left[k dk d \cos \theta d\phi \frac{\alpha}{(2\pi)^2} \sum_{\epsilon} \left(\frac{\epsilon p_2}{k p_2} - \frac{\epsilon p_1}{k p_1} \right)^2 \right]. \quad (75)$$

This expression would be badly defined (of the $0 \cdot \infty$ type) if the photon mass λ – regulator of the infrared singularity – was put to zero. Let us keep it thus non-zero for a while.

A discussion on the physics is here in place. If one notices that final states with additional, extremely soft photons are indistinguishable, by any method, from the ones where they are absent, we can say that they are physically identical and, as a principle, do not request theory to bring meaningful answers to non-physical questions requiring separation of these states. (The theory of coherent states is an appropriate framework in which to discuss this problem in a mathematically exhaustive way). Let us assume that k_{min} is an energy of the photon, which is well below any experimental accessibility of our detectors. We can divide our photons into two groups, of energy larger and smaller than k_{min} ; the latter we may safely omit from our kinematical consideration and integrate over their directions⁸:

$$d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) =$$

⁸We simplify again: we do not discuss the situation when the energy of every omitted from kinematical considerations photon would happen to be smaller than k_{min} , but their sum significantly larger. This obstacle can be easily overcome in exact treatment [8], v_{min} can be introduced as a boundary set on the sum of energies of all photons with neglected kinematics.

$$\begin{aligned}
& \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \times |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(2B) \\
& \times \exp \left[kdk \Theta(k_{min} - k) d \cos \theta d\phi \frac{\alpha}{(2\pi)^2} \sum_{\varepsilon} \left(\frac{\varepsilon p_2}{k p_2} - \frac{\varepsilon p_1}{k p_1} \right)^2 \right] \\
& \times \exp \left[kdk \Theta(k - k_{min}) d \cos \theta d\phi \frac{\alpha}{(2\pi)^2} \sum_{\varepsilon} \left(\frac{\varepsilon p_2}{k p_2} - \frac{\varepsilon p_1}{k p_1} \right)^2 \right]. \tag{76}
\end{aligned}$$

We can now define

$$Y = 2B + \left[kdk \Theta(k_{min} - k) d \cos \theta d\phi \frac{\alpha}{(2\pi)^2} \sum_{\varepsilon} \left(\frac{\varepsilon p_2}{k p_2} - \frac{\varepsilon p_1}{k p_1} \right)^2 \right]; \tag{77}$$

after standard (but not so short) calculation, we find, that in the $\lambda \rightarrow 0$ limit:

$$\begin{aligned}
Y &= \gamma \ln \frac{2k_{min}}{\sqrt{2p_1 p_2}} + \delta_{YFS} \\
\delta_{YFS} &= \frac{1}{4} \gamma + \frac{\alpha}{\pi} \left(-\frac{1}{2} + \frac{\pi^2}{3} \right) \\
\gamma &= 2 \frac{\alpha}{\pi} \left(\ln \frac{2p_1 p_2}{m_e^2} - 1 \right). \tag{78}
\end{aligned}$$

The exact form of δ_{YFS} is beyond the pedagogical level of our calculation, in the next step, we will keep it or drop it depending on the quality of the other terms. With this, we can write our final result for the zeroth order exponentiated exclusive cross section:

$$\begin{aligned}
& d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) = \\
& \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \times |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(Y) \\
& \exp \left[kdk \Theta(k - k_{min}) d \cos \theta d\phi \frac{\alpha}{(2\pi)^2} \sum_{\varepsilon} \left(\frac{\varepsilon p_2}{k p_2} - \frac{\varepsilon p_1}{k p_1} \right)^2 \right] \tag{79}
\end{aligned}$$

or in an explicit way:

$$\begin{aligned}
& d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) = \\
& \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(Y) \\
& \sum_{i=0}^{\infty} \frac{1}{i!} \prod_{l=1}^i \left[k_l dk_l \Theta(k_l - k_{min}) d \cos \theta_l d\phi_l \frac{\alpha}{(2\pi)^2} \sum_{\varepsilon_l} \left(\frac{\varepsilon_l p_2}{k_l p_2} - \frac{\varepsilon_l p_1}{k_l p_1} \right)^2 \right]. \tag{80}
\end{aligned}$$

Note that the deficiencies in this formula are due to a neglect of the phase-space constraint on the photon energies. Let us limit the integration over photon energy from above, by hand, by introducing the maximum photon energy cut⁹ $\Theta(k_{max} - k_l)$, $k_{max} = \frac{1}{2} \sqrt{2p_1 p_2}$. We get our exponentiation result:

$$\begin{aligned}
& d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) = \\
& \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \times |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(Y) \\
& \sum_{i=0}^{\infty} \frac{1}{i!} \prod_{l=1}^i \left[k_l dk_l \Theta(k_l - k_{min}) \Theta(k_{max} - k_l) d \cos \theta_l d\phi_l \frac{\alpha}{(2\pi)^2} \sum_{\varepsilon_l} \left(\frac{\varepsilon_l p_2}{k_l p_2} - \frac{\varepsilon_l p_1}{k_l p_1} \right)^2 \right]. \tag{81}
\end{aligned}$$

⁹We can get this limit, e.g. from the first-order calculation.

Note that kinematical variables of every additional photon accompanying our q_1, \dots, q_n final state is explicitly included and not integrated. The cross section is fully differential and therefore we call it exclusively exponentiated.

11 EXCLUSIVE EXPONENTIATION AND PERTURBATION EXPANSION

We will start this section with a long passage in smaller font-size, on the exclusive exponentiation formula, equivalent to our formula (81), but written for the t -channel process. We suggest to the reader that he drop this passage at first reading. In fact it is not even connected to the rest of our lectures through the conventions of notations, but it contains a complete presentation of the $\mathcal{O}(\alpha)$ exponentiated cross section (multi photon bremsstrahlung both in the initial and final state) for the process $e^+e^- \rightarrow e^+e^-$ at small angles. In fact it is chapter 2.1 of Ref. [11].

Note, however, that there is a passage at the end of this section that need not be omitted at first reading.

11.1 Realistic example

The complete *master formula* for the $\mathcal{O}(\alpha^1)$ exponentiated total cross section for the process $e^+(p_1) + e^-(q_1) \rightarrow e^+(p_2) + e^-(q_2) + n\gamma(k_i) + n'\gamma(k'_j)$ as actually implemented in the BHLUMI 2.01 Monte Carlo program is the same as in Ref. [14] and it reads as follows¹⁰:

$$\begin{aligned} \sigma = & \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{1}{n!} \frac{1}{n'!} \int \frac{d^3 q_2}{q_2^0} \frac{d^3 p_2}{p_2^0} \delta^{(4)} \left(p_1 + q_1 - p_2 - q_2 - \sum_{i=1}^n k_i - \sum_{j=1}^{n'} k'_j \right) \\ & \exp \left(Y(\Omega_1, p_1, p_2) + Y(\Omega_2, q_1, q_2) \right) \\ & \int \prod_{i=1}^n \frac{d^3 k_i}{k_i^0} \tilde{S}(p_1, p_2, k_i) (1 - \Theta(\Omega_1; k_i)) \int \prod_{j=1}^{n'} \frac{d^3 k'_j}{k'^0_j} \tilde{S}(q_1, q_2, k'_j) (1 - \Theta(\Omega_2; k'_j)) \\ & \left(\bar{\beta}_0^{(1)}(Q, p_1, p_2, q_1, q_2) + \sum_{i=1}^n \bar{\beta}_1^{(1)}(Q, p_1, p_2, q_1, q_2, k_i) / \tilde{S}(p_1, p_2, k_i) \right. \\ & \left. + \sum_{j=1}^{n'} \bar{\beta}_{1'}^{(1)}(Q, p_1, p_2, q_1, q_2, k'_j) / \tilde{S}(q_1, q_2, k'_j) \right) \Xi_{\text{MC}}(p_i, q_i, k_l, k'_m), \end{aligned} \quad (82)$$

where $\tilde{S}(p_1, p_2, k) = -(\alpha/4\pi^2)((p_1/kp_1) - (p_2/kp_2))^2$ is the real photon infra-red factor and

$$\begin{aligned} Y(\Omega, p_1, p_2) &= 2\alpha \tilde{B}(\Omega, p_1, p_2) + 2\alpha \Re B(p_1, p_2) \\ &= -2\alpha \frac{1}{8\pi^2} \int \frac{d^3 k}{k^0} \Theta(\Omega; k) \left(\frac{p_1}{kp_1} - \frac{p_2}{kp_2} \right)^2 \\ &\quad + 2\alpha \Re \int \frac{d^4 k}{k^2} \frac{i}{(2\pi)^3} \left(\frac{2p_1 - k}{2kp_1 - k^2} - \frac{2p_2 - k}{2kp_2 - k^2} \right)^2 \end{aligned} \quad (83)$$

is the standard Yennie–Frautschi–Suura form factor [6]. It is infra-red-finite and $\Theta(\Omega; k) = 1$ for $k \in \Omega$ and 0 for $k \notin \Omega$. The infra-red Ω region includes the $k = 0$ infra-red point and its definition may implicitly

¹⁰Note that taking only $n + n' = 0, 1$ and expanding the form factor $\exp(Y(\Omega_1) + Y(\Omega_2))$ one recovers the ordinary non-exponentiated $\mathcal{O}(\alpha^1)$ expression for the differential cross sections. For instance, defining $\Omega_{1,2}$ by $k^0 < \varepsilon\sqrt{s}/2$ in the laboratory frame one recovers exactly Eq. (1) of Ref. [13].

involve the dependence on the fermion four-momenta p_i and q_i [6]. None of the physically sensible results depends on the choice of Ω ! The Ω domain is typically defined through the $k^0 < E_{\min}$ condition in a certain reference frame. (In fact, the program features two types of Ω but only one of them is in use, see later in this section.) We shall define $\Omega_{1,2}$ and give the corresponding explicit formula for the form factors later, while describing the Monte Carlo algorithm.

The perturbative $\mathcal{O}(\alpha)$ QED matrix element is located in the $\bar{\beta}$'s, which are¹¹

$$\bar{\beta}_0^{(1)}(Q, p_1, p_2, q_1, q_2) = \bar{\beta}_0^{(0)}(Q, p_1, p_2, q_1, q_2)(1 + 2\delta_0 + \delta_\gamma + \delta_Z), \quad (84)$$

$$\delta_0 = 2\Re F_1(Q^2) - 2\Re B(Q^2) = \frac{1}{2}\beta_t, \quad \beta_t = 2\frac{\alpha}{\pi} \left(\ln \frac{|Q^2|}{m_e^2} - 1 \right), \quad (85)$$

$$\bar{\beta}_0^{(0)}(Q, p_1, p_2, q_1, q_2) = \frac{2\alpha_r(t)^2}{s} \frac{(s^2 + u^2 + s_1^2 + u_1^2)}{4t_p t_q}, \quad (86)$$

$$\begin{aligned} \bar{\beta}_1^{(1)}(Q, p_1, p_2, q_1, q_2, k_i) &= \frac{\alpha_r(t)^2}{2s} \frac{\alpha}{4\pi^2} D_1^{(1)}(Q, p_1, p_2, q_1, q_2, k_i) \\ &\quad - \tilde{S}(p_1, p_2, k) \bar{\beta}_0^{(0)}(Q, p_1, p_2, q_1, q_2), \end{aligned} \quad (87)$$

$$\begin{aligned} \bar{\beta}_1^{(1)}(Q, p_1, p_2, q_1, q_2, k'_j) &= \frac{\alpha_r(t)^2}{2s} \frac{\alpha}{4\pi^2} D_{1'}^{(1)}(Q, p_1, p_2, q_1, q_2, k'_j) \\ &\quad - \tilde{S}(q_1, q_2, k'_j) \bar{\beta}_0^{(0)}(Q, p_1, p_2, q_1, q_2), \end{aligned} \quad (88)$$

$$\begin{aligned} D_1^{(1)}(Q, p_1, p_2, q_1, q_2, k) &= \\ &= \frac{1}{(kp_1)(kp_2)} \left\{ \frac{s^2 + u_1^2}{|t_q|} \left(1 - \frac{2m_e^2}{|t_q|} \frac{kp_1}{kp_2} \right) + \frac{s_1^2 + u^2}{|t_q|} \left(1 - \frac{2m_e^2}{|t_q|} \frac{kp_2}{kp_1} \right) \right\}, \end{aligned} \quad (89)$$

$$\begin{aligned} D_{1'}^{(1)}(Q, p_1, p_2, q_1, q_2, k) &= \\ &= \frac{1}{(kq_1)(kq_2)} \left\{ \frac{s^2 + u_1^2}{|t_p|} \left(1 - \frac{2m_e^2}{|t_p|} \frac{kq_1}{kq_2} \right) + \frac{s_1^2 + u^2}{|t_p|} \left(1 - \frac{2m_e^2}{|t_p|} \frac{kq_2}{kq_1} \right) \right\}, \end{aligned} \quad (90)$$

$$\begin{aligned} t = Q^2 &= \left(p_2 + \sum_{i=1}^n k_i - p_1 \right)^2, \quad t_p = -2p_1 p_2, \quad t_q = -2q_1 q_2, \\ s &= 2p_1 q_1, \quad s_1 = 2p_2 q_2, \quad u = -2p_1 q_2, \quad u_1 = -2q_1 p_2. \end{aligned}$$

We implement vacuum polarization through the QED running coupling constant $\alpha_r(t) = \alpha/|1 + \Pi(t)|$ at the proper $Q^2 = t$ scale. This takes into account the vacuum polarization correction in the $\mathcal{O}(\alpha^2 L^2)$, as was pointed out in Ref. [15]. The correction $\delta_\gamma = t/s$ is due to s -channel γ exchange and the correction δ_Z represents here¹² the interference of the t -channel photon with the s -channel Z :

$$\delta_Z = \frac{\alpha_r(s)}{\alpha_r(t)} \left(\frac{t}{s} \right) \frac{2s^2}{s^2 + (t+s)^2} \left(1 + \frac{t}{s} \right)^3 (v^2 + a^2) \Re \left(\frac{s}{s - M^2 + is\Gamma/M} \right), \quad (91)$$

where $a = -1/(4 \sin \theta_W \cos \theta_W)$, $v = a(1 - 4 \sin^2 \theta_W)$, M and Γ are the usual coupling constants, mass and width of Z . We use $\sin^2 \theta_W = 0.2306$, $M = 91.161$ GeV and $\Gamma = 2.534$ GeV, and these values are already precise enough for the purpose of luminosity measurement. In the above two corrections we keep terms that are necessary for the precision $< 10^{-4}$ for angles $\vartheta < 10^\circ$.

The main difference in the above QED matrix element with respect to BHLUMI 1.xx is the neglect of up-down interference. This contribution was found in Ref. [13] to be very small in small-angle Bhabha's, for $\vartheta < 6^\circ$ it is generally below 0.02%. In any case, for the purpose of the discussion of the physical error the OLDBIS sub-generator will provide the value of this contribution for any cut or acceptance.

¹¹Note that in the analogous formula in Ref. [14] the expression for $\tilde{\beta}_1^{(1)}$ was distorted and the factor 2 in front of δ_0 was omitted. The formula in the program was always correct, so this does not have any consequences for the numerical results in this paper.

¹²For the present-day (1993) precision, it is necessary to include also QED corrections to δ_Z .

Dropping up–down interference allows us to consider bremsstrahlung from upper e^+ and lower e^- fermion lines independently, and to simplify the multiphoton bremsstrahlung matrix element considerably. In the process of writing the $\mathcal{O}(\alpha)$ multiphoton matrix element in the YFS exponentiation it is necessary to extend (extrapolate) the single bremsstrahlung matrix element beyond the three-body phase space.¹³ Instead of doing it by means of manipulating four-momenta arguments in the corresponding expressions, as in Refs. [8, 12] (the so-called reduction procedure), we rather extrapolate the single bremsstrahlung matrix element expressed in terms of Mandelstam variables, see Eq. (82). This method gives almost the same numerical result, while it leads to more compact and explicit expressions, which are faster and numerically more stable in the computer evaluation. It should be stressed, however, that reinstalling up–down interference in the present program is possible and it would be rather straightforward – the basic Monte Carlo algorithm is already prepared for this, see later in this section. We did not do it because in the small-angle Bhabha we regard up–down interference as an unnecessary complication!

The function

$$\Xi_{\text{MC}}(p_i, q_i, k_j) = \theta(|t| - |t_{\min}|)\theta(|t_{\max}| - |t|) \quad (92)$$

defines phase space for events generated in the Monte Carlo run. The user’s own experimental trigger $\Xi_{\text{exper.}}$ is imposed later by the usual rejection method, see section 4.4 (of Ref. [11]) for discussion of the practical choice of $t_{\min, \max}$. Cross sections and distributions obtained with $\Xi_{\text{exper.}}$ do not and should not depend on the particular values of $t_{\min, \max}$. Note also that transfer t has physical meaning only if up–down interference is neglected and/or in the leading logarithm approximation. Otherwise it is an intermediate parameter in the Monte Carlo generation, being a complicated function of the photon momenta, dependent on details of the Monte Carlo generation algorithm.

11.2 n -photon probability

Let us now turn back to our simple formula (81). It tells us that the process $p_1, p_2 \rightarrow q_1, \dots, q_n$, because of the soft photon structure of QED has to be calculated and discussed simultaneously with other processes, where in addition to q_1, \dots, q_n there is an undefined number of soft photons, but of differential distribution, at the zero level of approximation, completely independent from the particular hard process under consideration. In addition, this differential distribution can be improved order by order in perturbation expansion! Note that higher-order corrections will not introduce new kinematical configurations (i.e. more complicated phase space), but only improve distributions.

We will calculate here, what the probability is of having n photons accompanying our final state. In this way we will make the first step into the direction of inclusive exponentiation. As we can see in (81), every photon distribution is independent from the other ones, it will thus be rather simple to integrate over its angles:

$$\int S_k k dk d \cos \theta d \phi = \int_{k_{\min}}^{k_{\max}} k dk \int d \cos \theta \int d \phi \left[\frac{\alpha}{(2\pi)^2} \sum_{\varepsilon} \left(\frac{\varepsilon p_2}{k p_2} - \frac{\varepsilon p_1}{k p_1} \right)^2 \right]. \quad (93)$$

In the centre-of-mass system, we have

$$\begin{aligned} p_1^\mu &= E(1, 0, 0, \beta) \\ p_2^\mu &= E(1, 0, 0, -\beta) \\ k^\mu &= k(1, \sin \theta \sin \phi, \sin \theta \cos \phi, \cos \theta) \end{aligned}$$

¹³This extrapolation is inherent in any kind of exponentiation and due to the fact that infra-red singularities were subtracted and summed up to infinite order; see Refs. [6, 16] for more comments.

$$\begin{aligned}
\varepsilon_1 &= (0, \cos \phi, -\sin \phi, 0) \\
\varepsilon_2 &= (0, \cos \theta \sin \phi, \cos \theta \cos \phi, -\sin \theta) \\
\beta &= \sqrt{1 - m_e^2/E^2}
\end{aligned} \tag{94}$$

and we find

$$\begin{aligned}
\int S_k k dk d \cos \theta d \phi &= \frac{\alpha}{4\pi^2} \int \frac{dk}{k} \frac{4 \sin^2 \theta}{(1 - \beta^2 \cos^2 \theta)^2} d \cos \theta d \phi \\
&= \frac{\alpha}{4\pi^2} \times \ln \frac{k_{max}}{k_{min}} \times \left(\ln \frac{2p_1 p_2}{m_e^2} - 1 \right) \times 2\pi \\
&= \ln \frac{\sqrt{2p_1 p_2}}{k_{min}} \frac{2\alpha}{\pi} \left(\ln \frac{2p_1 p_2}{m_e^2} - 1 \right) \\
&= \gamma \ln \frac{\sqrt{2p_1 p_2}}{k_{min}},
\end{aligned} \tag{95}$$

where we have used the definition of γ in (78). We introduce this into (81) and obtain

$$\begin{aligned}
&d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) = \\
&\frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(Y) \sum_{l=0}^{\infty} \frac{1}{l!} \left(\gamma \ln \frac{\sqrt{2p_1 p_2}}{k_{min}} \right)^l.
\end{aligned} \tag{96}$$

We can now neglect δ_{YFS} in Y (it is not proportional to the logarithm of the photon energy and is thus negligible in our approximation) and get $Y \simeq \gamma \ln \frac{\sqrt{2p_1 p_2}}{k_{min}}$. This leads to a Poissonian distribution in the number of photons accompanying our final state, of energy bigger than k_{min} ,

$$\begin{aligned}
P_n(k_{min}) &= \exp\left(-\gamma \ln \frac{\sqrt{2p_1 p_2}}{k_{min}}\right) \frac{1}{n!} \left(\gamma \ln \frac{\sqrt{2p_1 p_2}}{k_{min}}\right)^n \\
\sum_{n=0}^{\infty} P_n &= 1.
\end{aligned} \tag{97}$$

As expected, the average multiplicity of the photons depends on k_{min} and tends logarithmically to infinity with $k_{min} \rightarrow 0$.

Finally let us point to different games with the phase space limits here. We were ignoring phase space constraints in calculation of $P_n(k_{min})$ and each of the photon phase space limits was left unaffected by the presence of the other ones. See figures in draft of new Section 14

12 INCLUSIVE EXPONENTIATION

With the help of formula (97) and P_0 , we can answer the question of what is the cross section for our process $p_1, p_2 \rightarrow q_1, \dots, q_n$, and photons

where the energy carried away by photons is not larger than $v\sqrt{2p_1 p_2} = k_{min}$ (again we cross the weak point of our pedagogical approximation by replacing the minimal energy of one photon k_{min} by the limit v on sum of the energies of all photons; we also introduce back our δ_{YFS} factor). We obtain

$$\begin{aligned}
& d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, \text{ and photons}) = \\
& \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(\delta_{YFS}) \exp\left(-\gamma \ln \frac{1}{v}\right) \\
& = \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(\delta_{YFS}) v^\gamma. \tag{98}
\end{aligned}$$

It is now a question of simple differentiation to get the differential cross section in the final state and the scaled electromagnetic energy v :

$$\begin{aligned}
& \frac{d\sigma(p_1, p_2 \rightarrow q_1, \dots, q_n, v)}{dv} = \\
& \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(\delta_{YFS}) \frac{d}{dv} \left\{ \exp\left(\gamma \ln v\right) \right\} \\
& = \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \times |\mathcal{M}_B|^2 dLips_n(p \rightarrow q_1 \dots q_n) \times \exp(\delta_{YFS}) \gamma v^{\gamma-1}. \tag{99}
\end{aligned}$$

If we are interested in the total cross section, we may integrate over $dLips_n(p \rightarrow q_1 \dots q_n)$. (We take not only the phase space at a reduced centre-of-mass energy due to photons, but also the matrix element \mathcal{M}_B ; this can be understood by inspection of the exact matrix element calculation presented later in this section or by the leading logarithm factorization of the next section.) We get ($s = 2p_1 p_2$):

$$\sigma_{exp}^{tot} \simeq \exp(\delta_{YFS}) \int_0^1 dv \gamma v^{\gamma-1} \sigma_B^{tot}(s(1-v)). \tag{100}$$

In our calculation (mainly due to our having neglected the difference in the definition of k_{min} and v_{min}) we have missed the normalization constant. In fact our formula should read:

$$\sigma_{exp}^{tot} = \frac{e^{-c\gamma}}{\Gamma(1+\gamma)} \exp(\delta_{YFS}) \int_0^1 dv \gamma v^{\gamma-1} \sigma_B^{tot}(s(1-v)), \tag{101}$$

where $c = 0.57721\dots$ is the Euler constant.

The form of this result is very similar to the result of the complete $\mathcal{O}(\alpha)$ calculation (see e.g. [9]), where we get

$$\begin{aligned}
\sigma_{tot}^{\mathcal{O}(\alpha)} &= \int_0^1 dv \rho(v) \sigma_B^{tot}(s(1-v)) \\
\rho^{\mathcal{O}(\alpha)}(v) &= \delta(v) \left\{ 1 + \gamma \ln v_0 + \frac{3}{4}\gamma + \frac{\alpha}{\pi} \left(-\frac{1}{2} + \frac{\pi^2}{3}\right) \right\} \\
&\quad + \Theta(v - v_0) \gamma \frac{1}{v} \left(1 - v \left(1 - \frac{1}{2}v\right)\right). \tag{102}
\end{aligned}$$

We will now rewrite $\rho^{\mathcal{O}(\alpha)}(v)$ in a form suitable for comparison with our exponentiated formula (101)

$$\begin{aligned}
\rho^{\mathcal{O}(\alpha)}(v) &= \delta(v) \left\{ 1 + \gamma \ln v_0 + \delta_{YFS} + \delta_S \right\} + \Theta(v - v_0) \gamma \frac{1}{v} \left(1 + \delta_H(v)\right) \\
\delta_s &= \frac{1}{2}\gamma \\
\delta_H(v) &= -v \left(1 - \frac{1}{2}v\right). \tag{103}
\end{aligned}$$

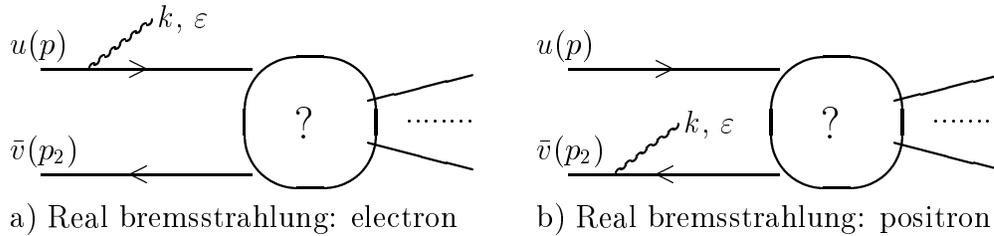


Figure 8: *Feynman diagrams for photon emission in initial state respectively from electron and positron. Case of e^+e^- collision.*

Now we can realize that the $\mathcal{O}(\alpha)$ expansion¹⁴ of the v spectrum from formula (101) coincides with this result, except in the δ_S and δ_H terms. It is useful to realize that our function $\rho^{\mathcal{O}(\alpha)}(v)$ is right at first order, whereas the exponentiated spectrum has some (at least v^γ for small v) terms right for higher orders in α . We may interpolate between formulae (101) and (102) and get:

$$\begin{aligned}\sigma_{tot}^{exp(\alpha)} &= \int_0^1 dv \rho(v) \sigma_B^{tot}(s(1-v)) \\ \rho^{exp(\alpha)}(v) &= \frac{e^{-c\gamma}}{\Gamma(1+\gamma)} e^{\delta_{YFS}} \gamma v^{\gamma-1} (1 + \delta_S + \delta_H(v)).\end{aligned}\quad (104)$$

This recipe of inclusive exponentiation is not the only one; there is generally a certain freedom of choice as to how to obtain formula (101) and, related with it, an uncontrollable uncertainty of the result. First of all, integrating over soft photon phase space we lost control over the photons configuration. This would of course be cured up to the $\mathcal{O}(\alpha)$ with the help of interpolation to the exact result calculated at this order (even including effects of cuts). As one could see, we were able to keep in our reasoning the differential distribution over final-state phase space $dLips_n(p)$ to the end. This shows that it would be straightforward to extend our exponentiation procedure to other, less inclusive observables, such as asymmetries etc. However, one should have in mind that the less inclusive the observables the larger are uncontrolled uncertainties.

As a final remark let us point out that a consistency check of any inclusive exponentiation formula is an expansion to fixed order and a comparison with the exact analytical result calculated at this order (or leading-log terms to fixed order with the leading-log results).

13 FACTORISATION OF THE LEADING LOG KERNEL

As one could see from our previous discussion on exponentiation, the typical size of coefficient enhancing the size of radiative correction was $\gamma \simeq \frac{\alpha}{\pi} \ln \frac{2p_1 p_2}{m_e^2}$. The origin of this logarithm (see the calculation in formula (95)) is an integration over an angle θ between

¹⁴Before the expansion, we have to introduce v_0 into (99); to this end we integrate this spectrum up to v_0 and multiply the result by $\delta(v)$.

a fermion and a photon directions ¹⁵. In the following, we will try to extract such leading terms from the diagrams of Figs. 8 and 5.

13.1 Real bremsstrahlung

To begin with, we will take the contribution of the photon emission from the electron (Fig. 8a) only. Here, contrary to the soft photon factor calculation, we will start from the calculation of $\sum_s \sum_\varepsilon \mathcal{M} \mathcal{M}^\dagger$ and, from the beginning, we will assume that m is negligibly small with respect to $p_1 - p_2$. In the following we will omit the subscript 1 in p_1 . From the Feynman rules of section 3 (see also formula (49)) we get

$$\sum_s \sum_\varepsilon \mathcal{M} \mathcal{M}^\dagger = \sum_s \sum_\varepsilon \dots \frac{\not{p} - \not{k} + m}{(p - k)^2 - m^2} e \not{\varepsilon} u(p, s) \bar{u}(p, s) \not{\varepsilon}^* e \frac{\not{p} - \not{k} + m}{(p - k)^2 - m^2} \dots \quad (105)$$

Neglecting the electron mass terms and using the projection operators (15), we obtain:

$$\sum_s \sum_\varepsilon \mathcal{M} \mathcal{M}^\dagger = \frac{e^2}{4(kp)^2} \frac{1}{2m} \sum_\varepsilon \dots (\not{p} - \not{k}) \not{\varepsilon} \not{p} \not{\varepsilon}^* (\not{p} - \not{k}) \dots \quad (106)$$

and later, if we choose ε to be real,

$$\begin{aligned} & \sum_s \sum_\varepsilon \mathcal{M} \mathcal{M}^\dagger = \\ & \frac{e^2}{4(kp)^2} \frac{1}{2m} \sum_\varepsilon \dots (\not{p} - \not{k}) \left(2\varepsilon p \not{\varepsilon} - \not{p} \not{\varepsilon} \not{\varepsilon} \right) (\not{p} - \not{k}) \dots \\ & = \frac{e^2}{4(kp)^2} \frac{1}{2m} \sum_\varepsilon \dots \left((\not{p} - \not{k}) 2\varepsilon p \not{\varepsilon} (\not{p} - \not{k}) + (\not{p} - \not{k}) \not{p} (\not{p} - \not{k}) \right) \dots \\ & = \frac{e^2}{4(kp)^2} \frac{1}{2m} \sum_\varepsilon \dots \left(2\varepsilon p (\not{p} - \not{k}) \not{\varepsilon} (\not{p} - \not{k}) + (\not{p} - \not{k}) \not{p} (\not{p} - \not{k}) \right) \dots \\ & = \frac{e^2}{4(kp)^2} \frac{1}{2m} \sum_\varepsilon \dots \left((2\varepsilon p)^2 (\not{p} - \not{k}) - 2\varepsilon p \not{\varepsilon} (\not{p} - \not{k}) (\not{p} - \not{k}) + \not{k} \not{p} \not{k} \right) \dots \\ & = \frac{e^2}{4(kp)^2} \frac{1}{2m} \sum_\varepsilon \dots \left((2\varepsilon p)^2 (\not{p} - \not{k}) + 4\varepsilon p \not{p} \not{\varepsilon} + 2kp \not{k} \right) \dots \end{aligned} \quad (107)$$

We will now estimate which of these terms may bring contributions of the logarithmic type. For this purpose we will use a parametrization of the phase space (46). For convenience we introduce the following short-hand notation:

$$\begin{aligned} c_\theta &= \cos \theta, & s_\theta &= \sin \theta, \\ c_\phi &= \cos \phi, & s_\phi &= \sin \phi. \end{aligned} \quad (108)$$

¹⁵I will omit here another class of leading-logarithm corrections due to photon vacuum polarization correction or, in other words, due to the evolution of the electromagnetic coupling constant from the m_e energy scale to $\sqrt{2p_1 p_2}$. Note that, for the inclusive quantities such as total cross sections and QED corrections in the final state, the corrections we will discuss here cancel out completely. This cancellation, which is also present in QCD, enables concept of the structure-function Evolution Equations.

In parametrizing the phase space, we will replace k by the dimensionless variable x . Now we can write

$$\begin{aligned}
p &= E(1, 0, 0, \beta) \\
\beta &= \sqrt{1 - \frac{m_e^2}{E^2}} \\
k &= Ex(1, s_\theta s_\phi, s_\theta c_\phi, c_\theta) \\
\varepsilon_1 &= (0, c_\phi, -s_\phi, 0) \\
\varepsilon_2 &= (0, c_\theta s_\phi, c_\theta c_\phi, -s_\theta).
\end{aligned} \tag{109}$$

With this notation we find, again neglecting mass terms wherever possible:

$$\begin{aligned}
kp &= E^2 x(1 - \beta c_\theta) \\
\varepsilon_1 p &= 0 \\
\varepsilon_2 p &= Es_\theta.
\end{aligned} \tag{110}$$

By inspecting formula (107), we may notice that the term $\varepsilon_2 p \not{p} k \sim s_\theta(1 - \beta c_\theta)$ and thus overcomes the $\frac{1}{(1 - \beta c_\theta)^2}$ singularity in the denominators of the fermion propagators $(kp)^2$, and that it does not contribute to the logarithmic term, so that we may drop it. We realize that, to the logarithmic (γ) terms contribute only those configurations where the directions of k and p nearly coincide, at our approximation level; we will thus write $\not{k} = x \not{p}$. We get

$$\begin{aligned}
&\sum_s \sum_\varepsilon \mathcal{M} \mathcal{M}^\dagger|_{LL} \\
&= \sum_\varepsilon \frac{e^2}{4(kp)^2} \frac{1}{2m} \dots \left((2\varepsilon p)^2 (1 - x) \not{p} + 2kp x \not{p} \right) \dots \\
&= \sum_\varepsilon \frac{e^2}{4(kp)^2} \frac{1}{2m} \left((2\varepsilon p)^2 (1 - x) + 2kp x \right) \frac{1}{1 - x} \dots \not{p} (1 - x) \dots \\
&= \sum_\varepsilon \frac{1}{1 - x} \frac{e^2}{4(kp)^2} \left((2\varepsilon p)^2 (1 - x) + 2kp x \right) \sum_s \mathcal{M}_B(p - k) \mathcal{M}_B^\dagger(p - k).
\end{aligned} \tag{111}$$

Our result tells us that the part of bremsstrahlung matrix element that leads to the leading logarithms can be written as a product of the Born amplitude squared, with incoming fermion momentum $p - k$ times a universal function $\frac{1}{1-x}F$:

$$F(E, x, \theta) = \sum_\varepsilon \frac{e^2}{4(kp)^2} \left((2\varepsilon p)^2 (1 - x) + 2kp x \right). \tag{112}$$

We will now rewrite F in x , θ and ϕ , introduce our matrix element squared into phase space (46), and integrate over angles.

$$\begin{aligned}
F(E, x, \theta, \phi) &= \frac{e^2}{4E^4 x^2 (1 - \beta c_\theta)^2} \left(4(1 - x)E^2 s_\theta^2 + 4(1 - \beta c_\theta)x^2 E^2 \right) \\
&= \frac{e^2}{E^2 x^2 (1 - \beta c_\theta)^2} \left((1 - x)s_\theta^2 + (1 - c_\theta)x^2 \right).
\end{aligned} \tag{113}$$

Let us now write the complete formula for the cross section:

$$d\sigma = \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \sum_s \sum_\varepsilon \mathcal{M} \mathcal{M}^\dagger \left[kdkdc_\theta d\phi \frac{1}{2(2\pi)^3} \right] \times dLips_n(p_1 - k, p_2 \rightarrow q_1 \dots q_n). \quad (114)$$

Substituting our expression for the matrix element squared (111), we get

$$\begin{aligned} d\sigma &= \frac{1}{|v_1 - v_2|} \frac{1}{2p_1^0} \frac{1}{2p_2^0} \frac{1}{1-x} \sum_s \mathcal{M}_B(p-k) \mathcal{M}_B^\dagger(p-k) \\ &\quad \frac{e^2}{E^2 x^2 (1-\beta c_\theta)^2} \left((1-x)s_\theta^2 + (1-c_\theta)x^2 \right) \left[kdkdc_\theta d\phi \frac{1}{2(2\pi)^3} \right] \\ &\quad \times dLips_n(p_1 - k, p_2 \rightarrow q_1 \dots q_n) \end{aligned} \quad (115)$$

or using relations $p_1^0(1-x) = p_1^0 - k^0$:

$$\begin{aligned} d\sigma &= \frac{1}{|v_1 - v_2|} \frac{1}{2(p_1^0 - k^0)} \frac{1}{2p_2^0} \sum_s \mathcal{M}_B(p-k) \mathcal{M}_B^\dagger(p-k) \times dLips_n(p_1 - k, p_2 \rightarrow q_1 \dots q_n) \\ &\quad \frac{e^2}{E^2 x^2 (1-\beta c_\theta)^2} \left((1-x)s_\theta^2 + (1-c_\theta)x^2 \right) \left[kdkdc_\theta d\phi \frac{1}{2(2\pi)^3} \right] \\ &= d\sigma_{born}(p_1 - k, p_2 \rightarrow q_1 \dots q_n) \\ &\quad x dx dc_\theta d\phi \frac{1}{2(2\pi)^3} \frac{e^2}{x^2 (1-\beta c_\theta)^2} \left((1-x)s_\theta^2 + (1-c_\theta)x^2 \right). \end{aligned} \quad (116)$$

After integration over ϕ and θ , and neglecting the non-logarithmic components of integration over dc_θ , this leads to

$$\begin{aligned} d\sigma &= d\sigma_{born}(p_1 - k, p_2 \rightarrow q_1 \dots q_n) \\ &\quad \frac{e^2}{2(2\pi)^2} \frac{dx}{x} \frac{1}{(1-\beta c_\theta)^2} \left((1-x)(1-c_\theta)(1+c_\theta) + (1-c_\theta)x^2 \right) dc_\theta \\ &\simeq d\sigma_B(p_1 - k, p_2 \rightarrow q_1 \dots q_n) \cdot \frac{e^2}{2(2\pi)^2} \frac{dx}{x} \left((1-x)2 \ln \frac{E^2}{m_e^2} + x^2 \ln \frac{E^2}{m_e^2} \right) \\ &= d\sigma_B(p_1 - k, p_2 \rightarrow q_1 \dots q_n) \cdot \frac{e^2}{2(2\pi)^2} \frac{dx}{x} \left((1-x)2 + x^2 \right) \ln \frac{E^2}{m_e^2} \\ &= d\sigma_B(p_1 - k, p_2 \rightarrow q_1 \dots q_n) \cdot \frac{dx}{x} \frac{1}{2} \left(1 + (1-x)^2 \right) \frac{\alpha}{\pi} \ln \frac{E^2}{m_e^2}. \end{aligned} \quad (117)$$

This is the basic formula of the leading logs. We see that we can describe the emission of the photons from the electron in the initial state as a convolution of the well-defined kernel $1 + (1-x)^2$ with the Born level cross section, where instead of first beam, we take effectively its four-momentum reduced by the fraction x carried away by the photon. The case of the initial-state positron is identical, owing to the neglect of the mass terms in our approximations.

It is a matter of exercise to convince oneself that the interference between the amplitudes of Figs. 8a and 8b does not give any terms proportional to $\ln E^2/m_e^2$.

13.2 Inclusion of virtual corrections

In principle we could repeat here a calculation similar to that in section 7 for the vertex correction amplitude of Fig. 5, but this time in the leading-log approximation. I do not

think however that it is sufficiently instructive to justify its presentation here. I leave it as an exercise for the very dedicated reader. Let us only mention that in this calculation there are two regions in d^4k integration over virtual photon four-momentum. They correspond respectively to the photon direction close to p_1 and p_2 . We can separate (in leading logs) the vertex correction into two parts. We get (see also formula (102) from which we take virtual and soft correction as “half” of the complete $\mathcal{O}(\alpha)$ result) the complete result, including virtual corrections:

$$\begin{aligned}
d\sigma(p_1, p_2 \rightarrow q_1 \dots q_n, x) &= d\sigma_{born}(p_1 - k, p_2 \rightarrow q_1 \dots q_n) \cdot dx f(x) \\
f(x) &= \delta(x) + P(x) \\
P(x) &= \delta(x) \left(\frac{\alpha}{\pi} \ln \frac{E^2}{m_e^2} \ln x_0 + \frac{3}{4} \frac{\alpha}{\pi} \ln \frac{E^2}{m_e^2} \right) \\
&+ \Theta(x - x_0) \frac{\alpha}{\pi} \ln \frac{E^2}{m_e^2} \frac{1}{x} \frac{1}{2} \left(1 + (1 - x)^2 \right). \quad (118)
\end{aligned}$$

The result we obtain for the photon emission from the positron is identical. It is interesting to note that $\int_0^1 f(x) dx = 1$. This is a consequence of the Kinoshita–Lee–Nauenberg theorem [17].

13.3 ITERATION: LL TO ALL ORDERS

Terms which produce leading logarithms in QED factorize also in higher orders. One can find that,

$$\begin{aligned}
f^\infty(x) &= \delta(x) + P(x) + \frac{1}{2!} \{P \otimes P\}(x) + \frac{1}{3!} \{P \otimes P \otimes P\}(x) + \dots, \\
\{P \otimes P\}(x) &= \int_0^1 dx_1 \int_0^1 dx_2 \delta(x_1 + x_2 - x_1 x_2 - x) P(x_1) P(x_2). \quad (119)
\end{aligned}$$

We omit here the discussion of how to obtain this result. We would like to mention that it turns out not to be so easy to improve this picture order by order in perturbation theory. The practical applications are limited to the next-to-leading-log approximation.

14 EXPONENTIATION AND LANGUAGE OF TANGENT SPACES

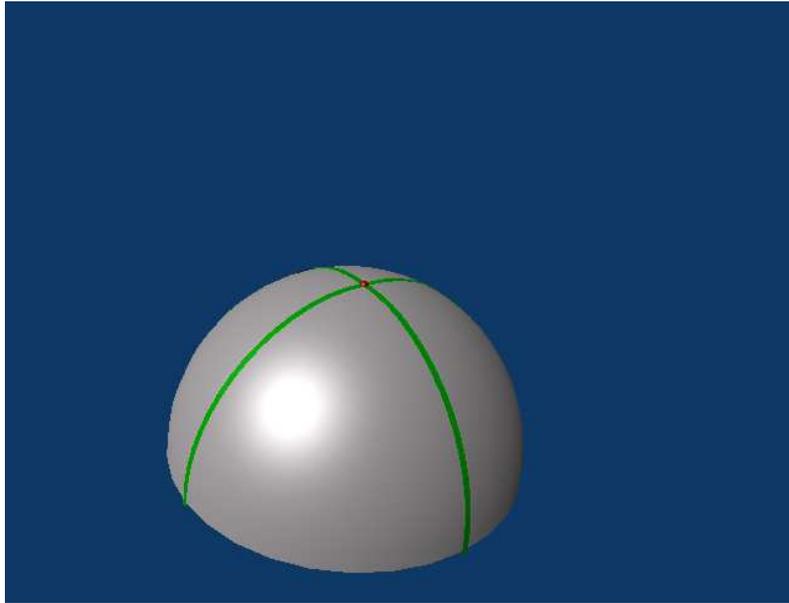


Figure 9: *Symbolic presentation of manifolds:*
 1) for *n*-body phase space (red point)
 2) for *n*-body plus extra particle (green lines)
 3) for *n*-body plus two extra particles (hemisphere)

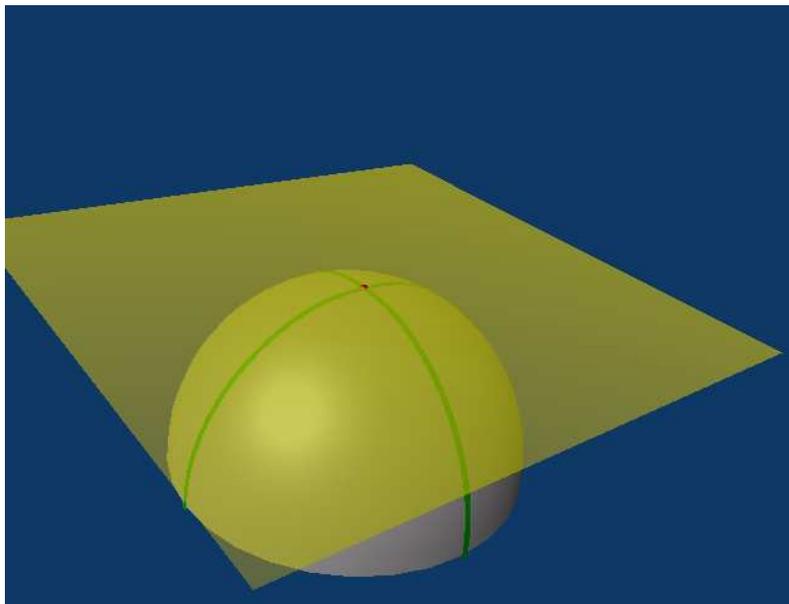


Figure 10: *Symbolic presentation of manifolds:*
 1) for *n*-body phase space (red point)
 2) for *n*-body plus extra particle (green lines)
 3) for *n*-body plus two extra particles (hemisphere)
 plus tangent space at infrared point

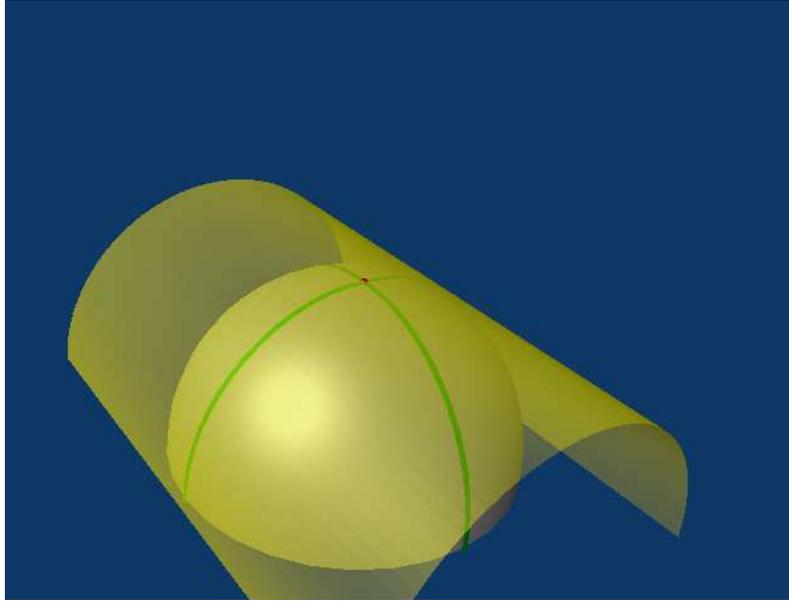


Figure 11: *Symbolic presentation of manifolds:*

- 1) *for n -body phase space (red point)*
- 2) *for n -body plus extra particle (green lines)*
- 3) *for n -body plus two extra particles (hemisphere) plus tangent space for second extra particle*

15 SUMMARY, INVITATION TO FURTHER READING

As can be seen from Fig. 21, these lectures are divided into four parts. The first one includes the introduction, gives the motivation for the subject, etc, and it is not necessary to understand the rest of the lectures. The main goal of these lectures was to present those aspects of radiative corrections that are of direct phenomenological consequence for the high energy experiments, with special emphasis on effects that because of the cuts cannot be separated from data analysis. This leads to our discussion of exponentiation and leading-logarithms approximation. The infrared structure of QED was presented. In particular we have visualized that *to obtain physically meaningful results one has to perform summation over final states with a different number (zero to infinity) of additional outgoing real photons.*

We have completely neglected those parts of radiative correction discussions (such as: regularization, renormalization, renormalization group equation, running coupling constant, etc.), which are usually presented in detail in any field theory textbooks. Our lectures are complementary in this respect to Prof. Bilenky's lectures.

We precede our discussion by a detailed presentation of notations (this usually is an appendix in field theory lectures) and of phase-space parametrization. This latter is especially important as we want to stress that all our techniques are well rooted in the Relativistic Quantum Mechanics and perturbation expansion of QED and, as such,

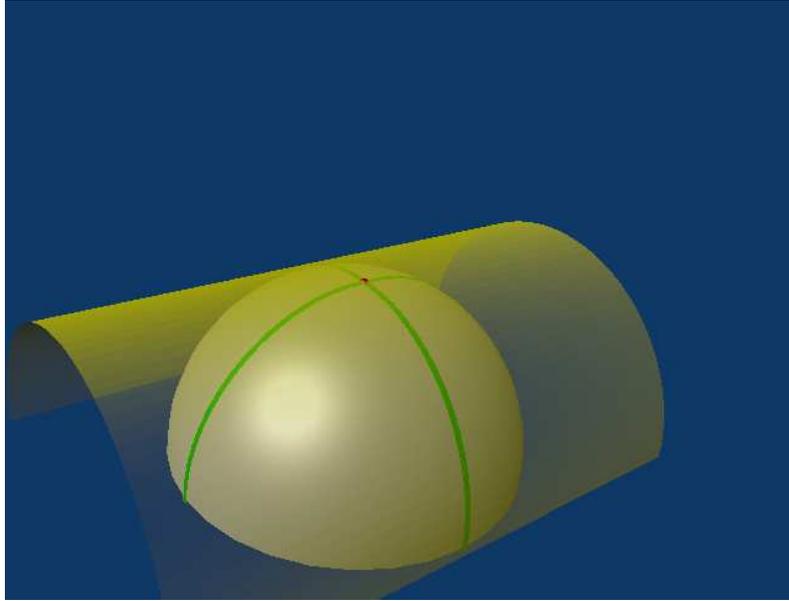


Figure 12: *Symbolic presentation of manifolds:*
 1) for n -body phase space (red point)
 2) for n -body plus extra particle (green lines)
 3) for n -body plus two extra particles (hemisphere)
 alternatively tangent space is for first extra particle

provide a well-defined method of discussing uncertainties coming with the results.

Finally, we have decided to skip the examples section. An excellent source of application examples, based on LEP I physics can be taken in Ref. [18]. We recommend the reader to scan through the Proceedings of “Physics at ...” workshop of the experiments of her/his interest.

In notes such as these, correctness and consistency of all formulas is very important for the readers. If misprints are to be detected later on, a postscript file **ZAKOPANE PS A1** (to be taken) of updated notes will be stored on the WASM disk at CERNVM. If you detect the misprints yourself, please send a message to WASM at CERNVM.

ACKNOWLEDGEMENTS

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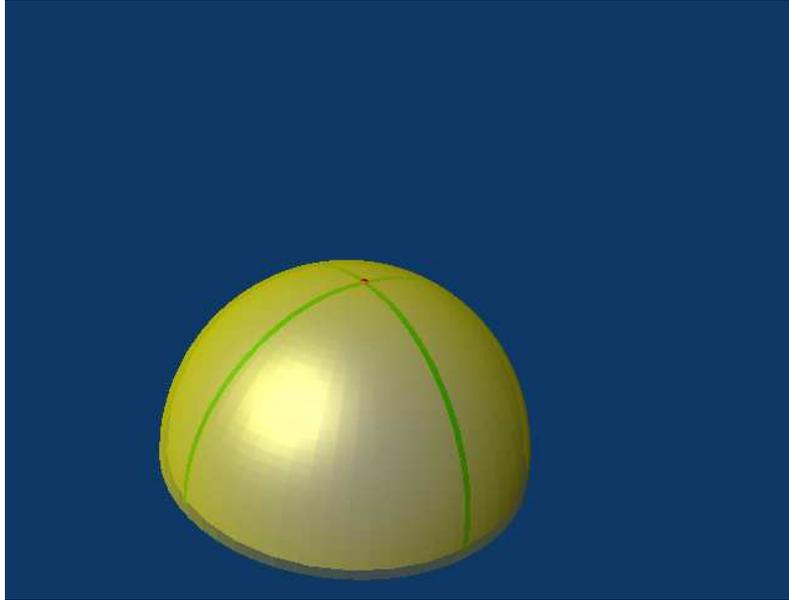


Figure 13: *Symbolic presentation of manifolds:*

1) *for n -body phase space (red point)*

2) *for n -body plus extra particle (green lines)*

3) *for n -body plus two extra particles (hemisphere)*

after two steps tangent space coincide with the manifolds for n - ($n+1$)- and ($n+2$)-phase-spaces

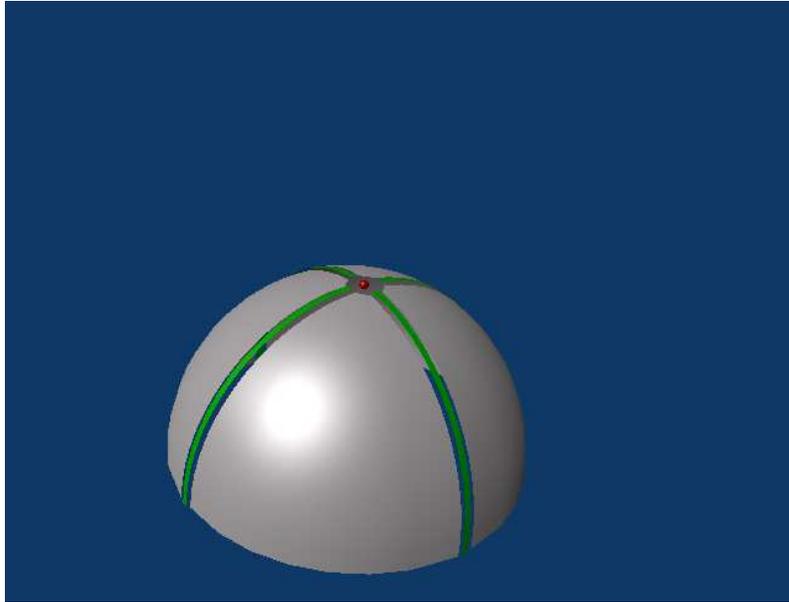


Figure 14: *Symbolic presentation of manifolds:*

1) *for n -body phase space (red point)*

2) *for n -body plus extra particle (green lines)*

3) *for n -body plus two extra particles (hemisphere)*

Regulators are needed to make this picture realizable. Minimum energies (or something introduced)

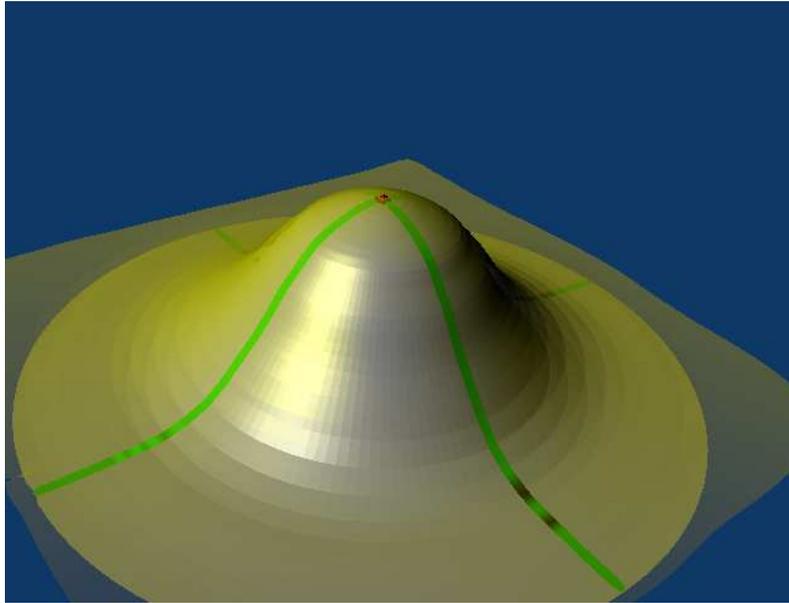


Figure 15: *Symbolic presentation of manifolds:*

1) *for n -body phase space (red point)*

2) *for n -body plus extra particle (green lines)*

3) *for n -body plus two extra particles (hemisphere)*

Do not get attached to sphere, equally well it can be rotated gaussian and tangent spaces can be deformed as well. This can be useful, case of ISR exponentiation for example. From principle point of view it is just question of measure redefinition. In one system we have hemi-sphere in another it looks like a bell. But in reality it is the same. Also note that these plots are in at least 8 dimensions ...

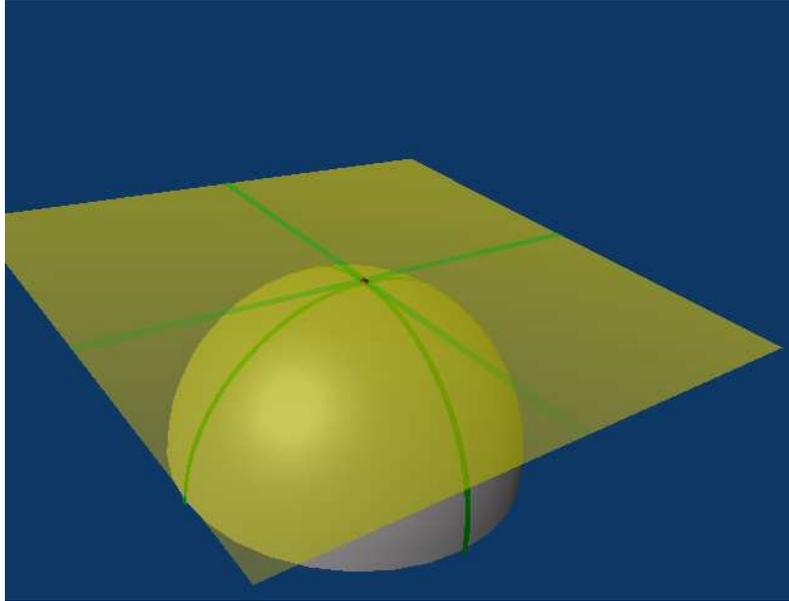


Figure 16: *Symbolic presentation of manifolds:*

1) *for n -body phase space (red point)*

2) *for n -body plus extra particle (green lines)*

3) *for n -body plus two extra particles (hemisphere)*

Tangent space is flat, lines along which dominant contribution differ, but nonetheless replacing main manifold with square can give a picture where formula 99 could be calculated. By looking at differences in shapes even size of the dominant missing term could be obtained.

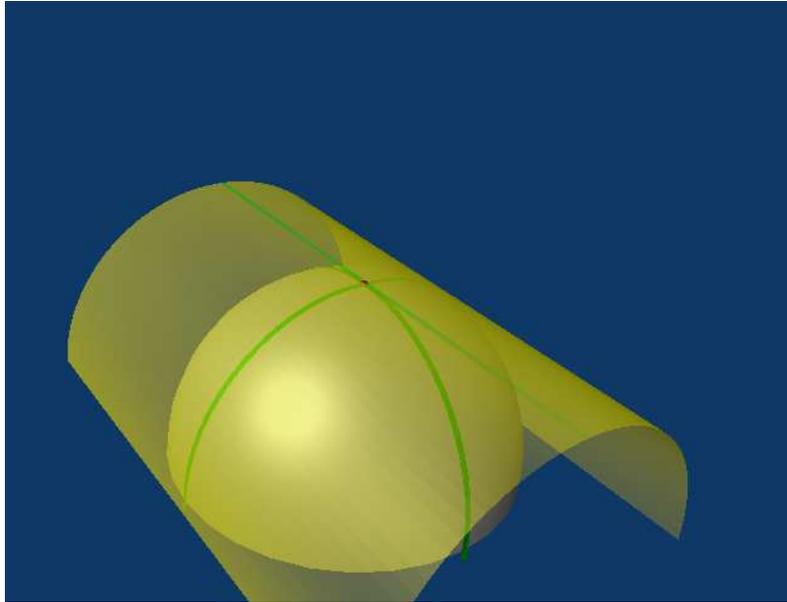


Figure 17: *Symbolic presentation of manifolds:*

1) *for n -body phase space (red point)*

2) *for n -body plus extra particle (green lines)*

3) *for n -body plus two extra particles (hemisphere)*

Tangent space is cylindric, one of the lines along which dominant contribution differ between tangent space and principal manifold, Replacing main manifold with square can give a picture where formula 99 could be improved, but only slightly better.

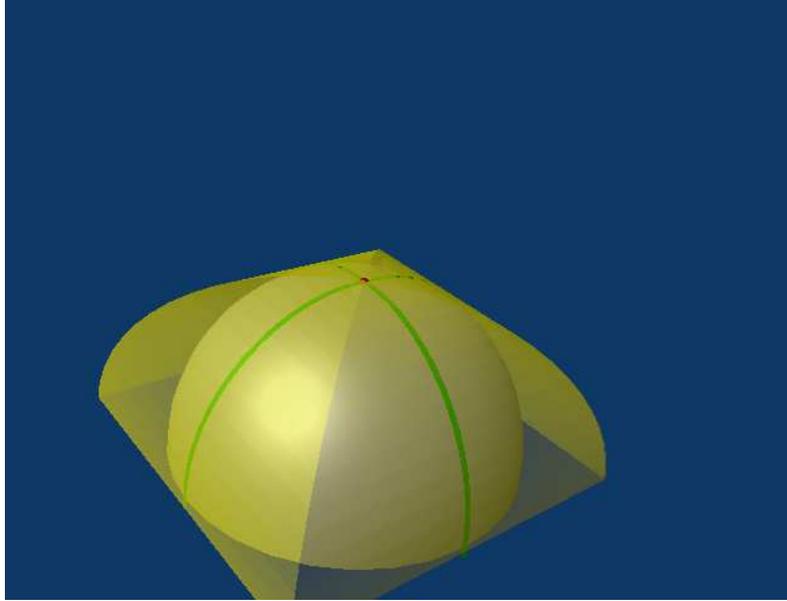


Figure 18: *Symbolic presentation of manifolds:*

- 1) for n -body phase space (red point)
- 2) for n -body plus extra particle (green lines)
- 3) for n -body plus two extra particles (hemisphere)

Tangent space is intersection of two cylinders now, both lines along which dominant contribution forms coincide in tangent space and principal manifold, Replacing main manifold with that figure can give a picture where formula 99 can be replaced by the one with complete LL content. Price paid is discontinuity in tangent space, extra benefit is that tangent space can be understood as sum of two identical pieces. That is principle of both p_T -like orderings and also exponentiation with β_1 terms included only.

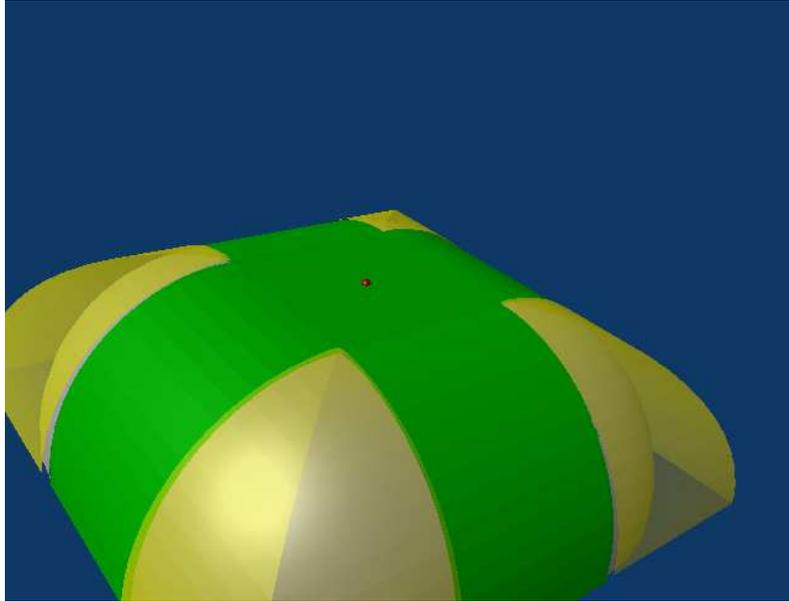


Figure 19: *Symbolic presentation of manifolds:*

- 1) *for n -body phase space (red point)*
- 2) *for n -body plus extra particle (green lines)*
- 3) *for n -body plus two extra particles (hemisphere)*

Tangent space is intersection of two cylinders now, both lines along which dominant contribution forms coincide in tangent space and principal manifold, Replacing main manifold with that figure can give a picture where formula 99 can be replaced by the one with complete LL content. Price paid is discontinuity in tangent space, extra benefit is that tangent space can be understood as sum of two identical pieces. That is principle of both p_T -like orderings and also exponentiation with β_1 terms included only. The thin green lines of previous plot can easily be understood as broad zone, if we think the size of manifold elements not in a language of phase-space volume, but rather include matrix element as well.

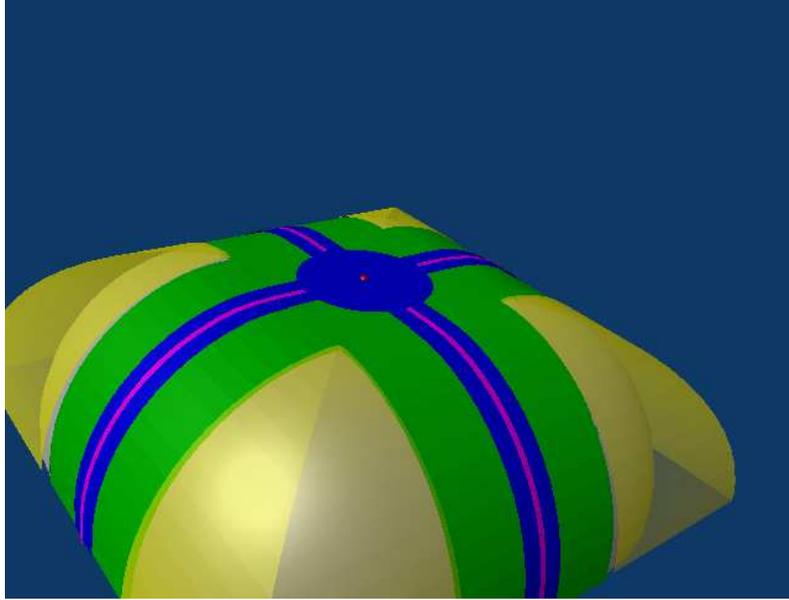


Figure 20: *Symbolic presentation of manifolds:*

- 1) *for n -body phase space (red point)*
- 2) *for n -body plus extra particle (green lines)*
- 3) *for n -body plus two extra particles (hemisphere)*

Tangent space is intersection of two cylinders now, both lines along which dominant contribution forms coincide in tangent space and principal manifold, Replacing main manifold with that figure can give a picture where formula 99 can be replaced by the one with complete LL content. Price paid is discontinuity in tangent space, extra benefit is that tangent space can be understood as sum of two identical pieces. That is principle of both p_T -like orderings and also exponentiation with β_1 terms included only. The thin green lines of previous plot can easily be understood as broad zone, if we think the size of manifold elements not in a language of phase-space volume, but rather include matrix element as well. In contrary to what is done in fig 19 we explicitly cut out those zones of phase spaces where cancellations are explicitly performed. But physicswise present figure and figs 18,19 represent exactly the same physical configuration. In case of perturbative part of QCD predictions they can not be thinner than what is incorporated into non-perturbative zone of condensates hadronization, structure function.

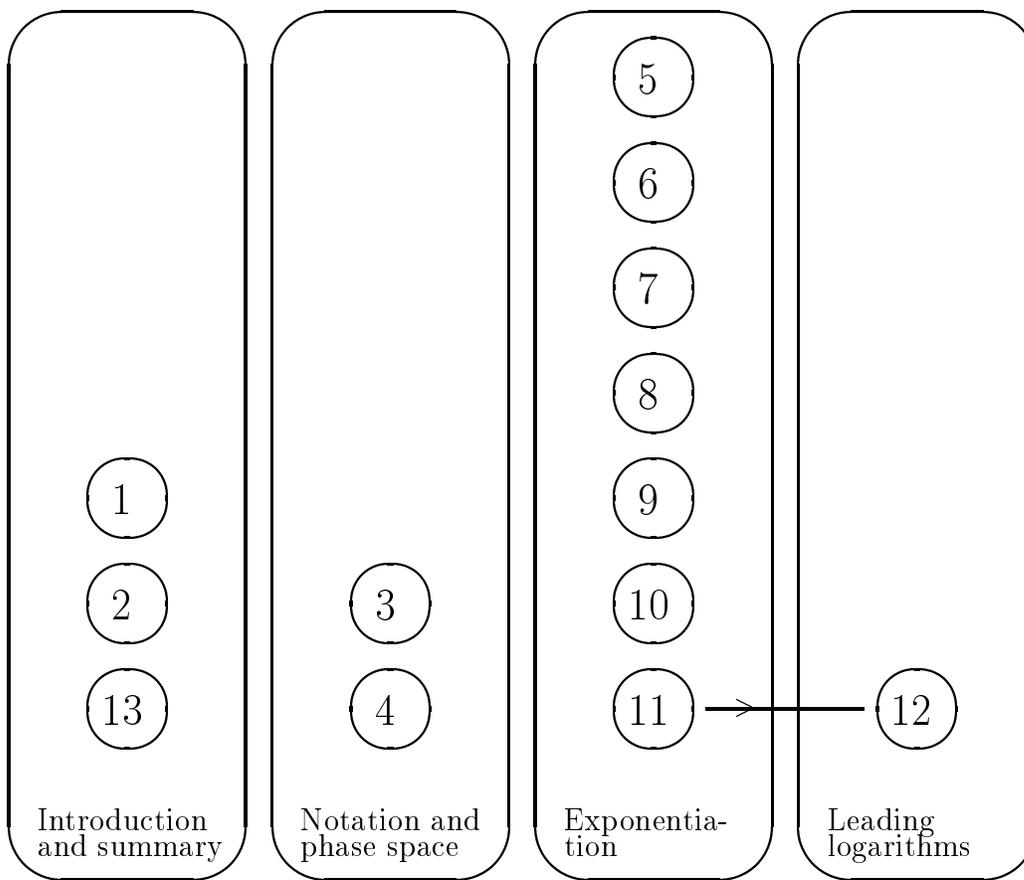


Figure 21: *Relation between old sections. Does not reply to those written after year 2003*

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