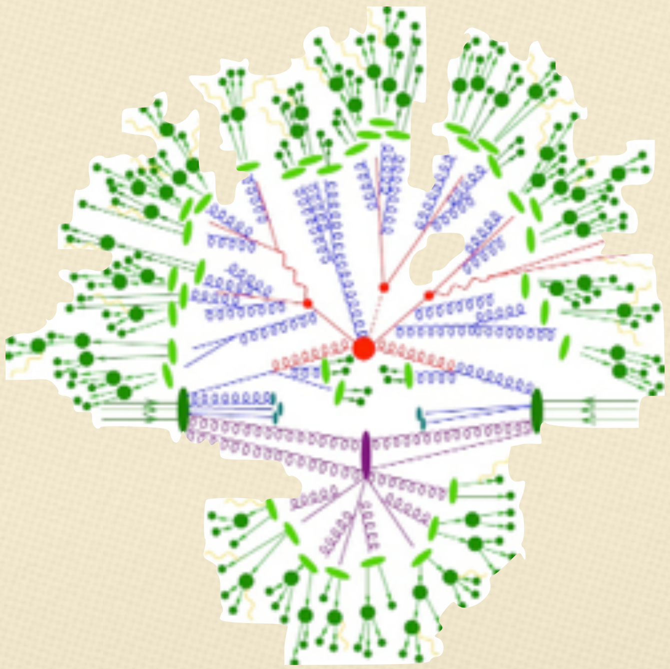
$$\frac{1}{4}\mathrm{Tr}[G^2] - \bar{\psi}(\not{D} - m)\psi$$ ☞



QCD@LHC

pQCD+Resummation

Eric Laenen

GGI Workshop

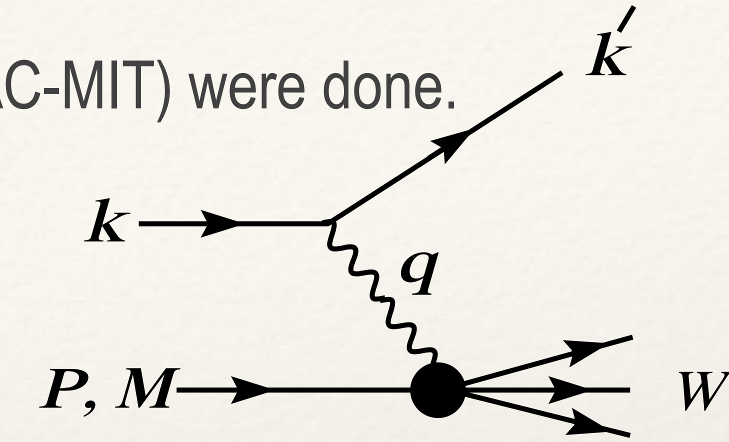
Prospects and precision at the Large Hadron Collider at 14 TeV

Outline

- pQCD
  - QCD Lagrangian, parton model.
  - Renormalization, asymptotic freedom
  - Parton distribution functions
  - Event shapes in e+e- cross sections, IR and collinear divergences, KLN theorem.
  - Drell-Yan: NLO calculation, factorization.
  - IR analysis: pinch surfaces, Landau equations
- Resummation
  - Basics of resummation. Eikonal approximation, webs.
  - Resummation, dQCD and SCET approach
  - Some applications: Heavy quark production, Higgs production, at finite order and resummation

Probing the proton

- In the late sixties, early seventies, deep-inelastic scattering experiments (SLAC-MIT) were done.
- Relation of cross section to “inelastic form factors” of proton F1, F2, F3:

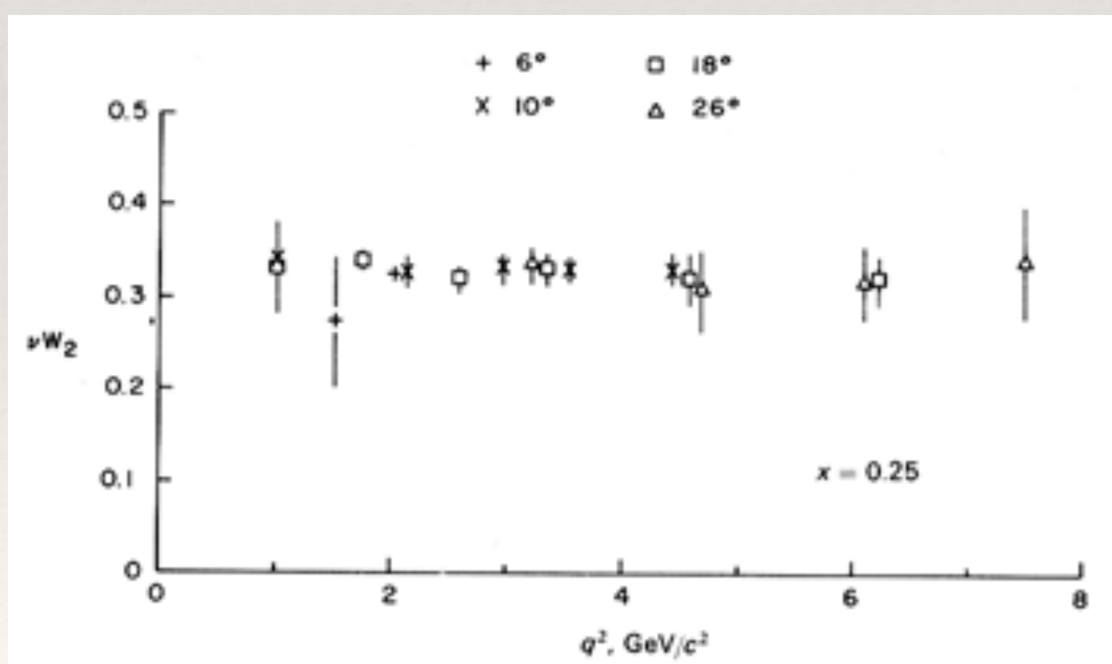


$$\left(\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}y}\right)^{\gamma} = \frac{8\pi\alpha^2 ME}{(Q^2)^2}\left\{\frac{1+(1-y)^2}{2}2xF_1^{\gamma}(x,Q^2)\right.$$
$$\left.+(1-y)[F_2^{\gamma}(x,Q^2) - 2xF_1^{\gamma}(x,Q^2)] - \frac{M}{2E}xyF_2^{\gamma}(x,Q^2)\right\}$$

x =Q²/2M(E-E′) is momentum fraction of struck quark

y=(1-E′/E) is fractional energy loss of electron

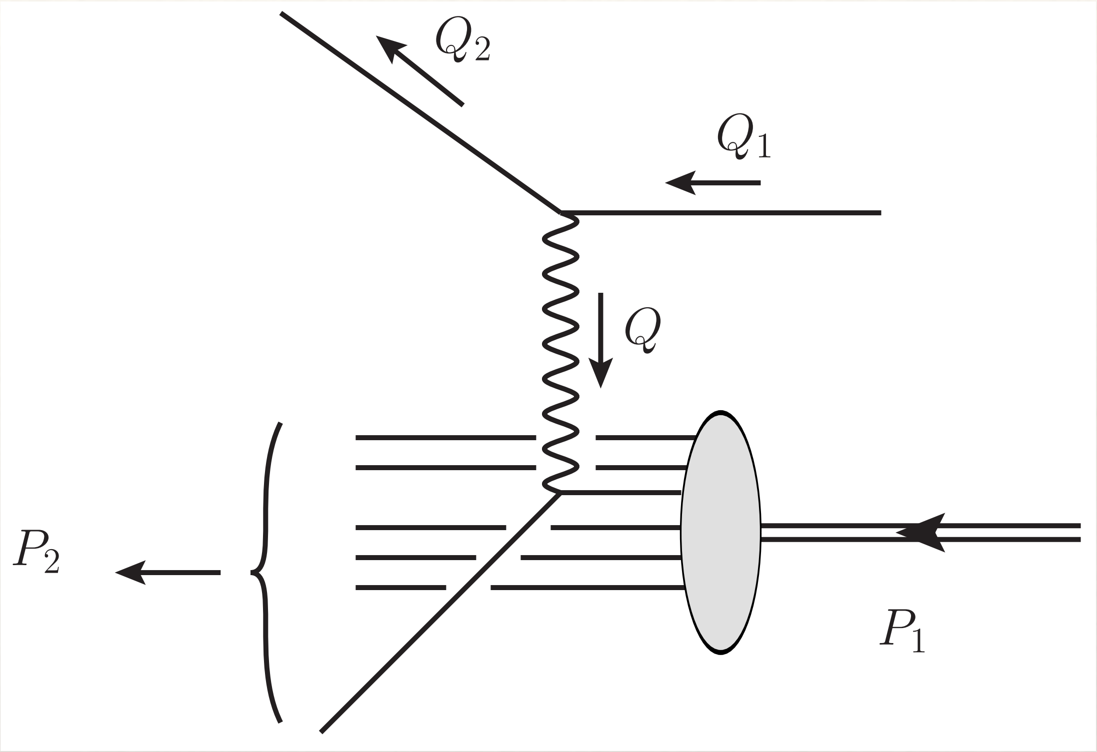
- Outcome: F2 can depend on x and Q2, but seemed to only depend on x
  - “Scaling”



Parton model

Feynman, Paschos

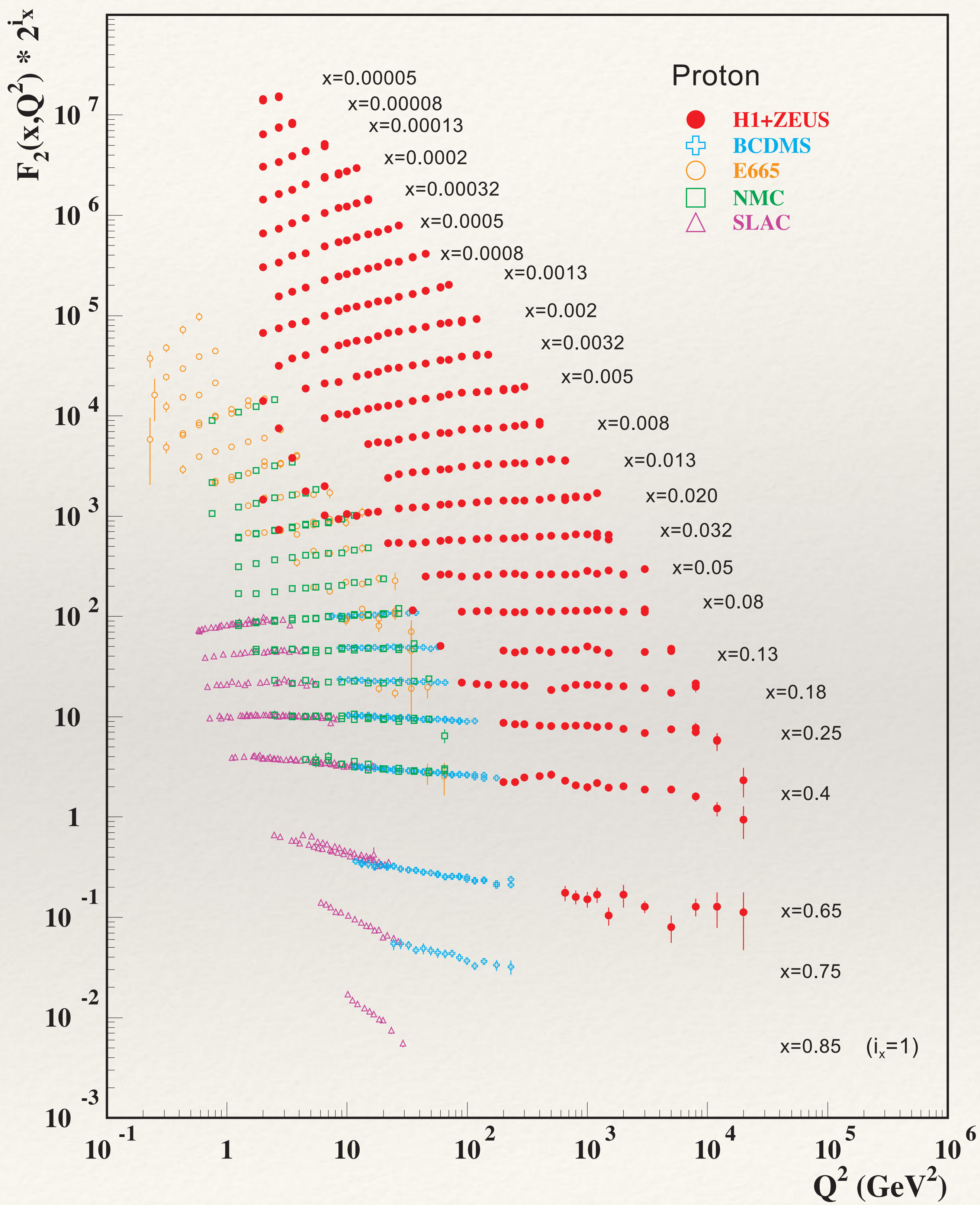
- Solution: the Parton model
- Wonderfully elegant idea, still at the basis of our predictions for the LHC.
- The scene: an electron at high energy hitting a proton (sitting inside a fixed target, or approaching from colliding beam).
- From the electron point of view, two relativistic effects occur
  - The proton is length contracted, looks like a disk
  - The internal proton dynamics is slowed down, due to time dilation
  - Assume interactions beween constituent "partons" are absent (rather wild assumption at the time)
- Introduce now the parton distribution function $\phi_{i/p}(\xi)$, and integrate over all allowed momentum fractions ξ.
- This explains the scaling.



$$\left(\frac{d\sigma}{dx\,dy}\right)^{\gamma} = \frac{8\pi\alpha^2 ME}{(Q^2)^2}\,\frac{(1-y)^2+1}{2}\sum_i q_i^2\, x\phi_{i/p}(x)$$

Deep-inelastic scattering

- But in better measurements: no scaling anymore
    - “violation” mild: logarithmic
- Notice: steeper slopes for smaller x
- Data are large x earlier cannot reach high Q^2
    - QCD can explain all this
- Data take in 90’s and early naughties at fixed target experiments, and at the HERA ep collider in DESY, Hamburg

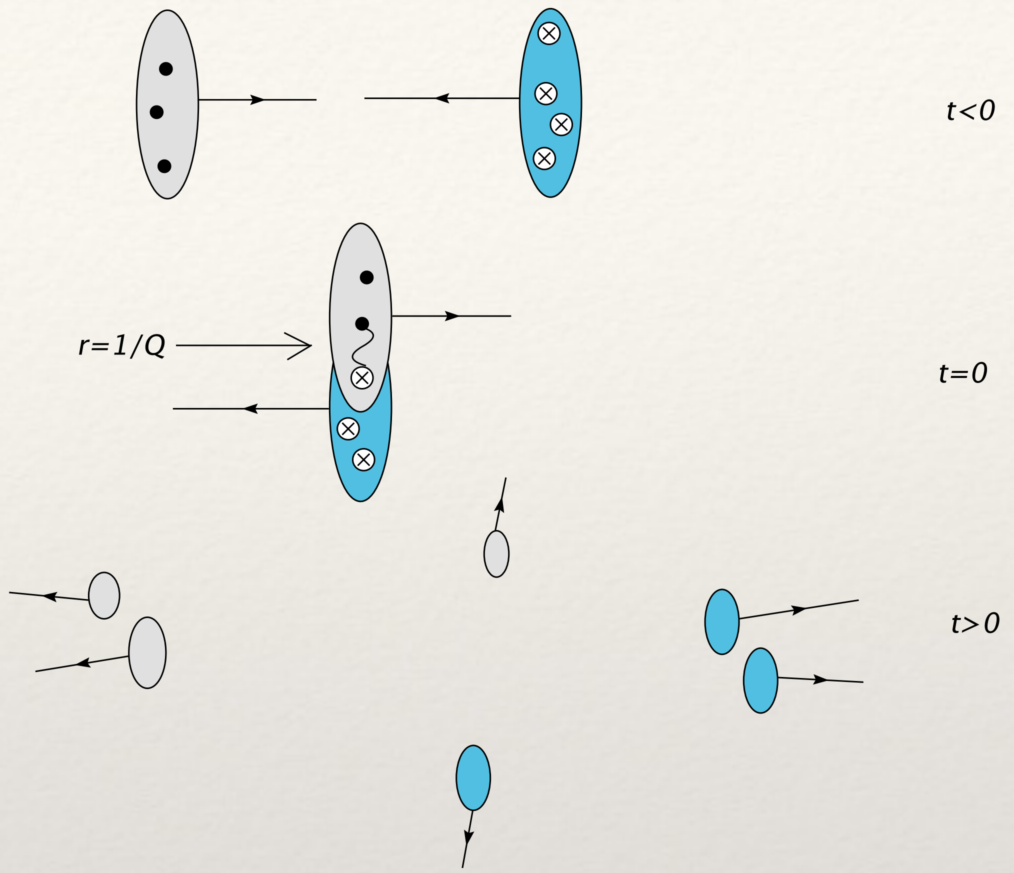


Universality of parton model, and paradox

- Ideas generalizes to hadron-hadron scattering.
  - Brings predictive power, if the PDF's are the same all processes
    - ✓ This is an assumption in the parton model
- In QCD this can be proven. Such proofs, though formal, are important.

Catani, de Florian, Rodrigo; Forshaw, Seymour, Siodmok, ..

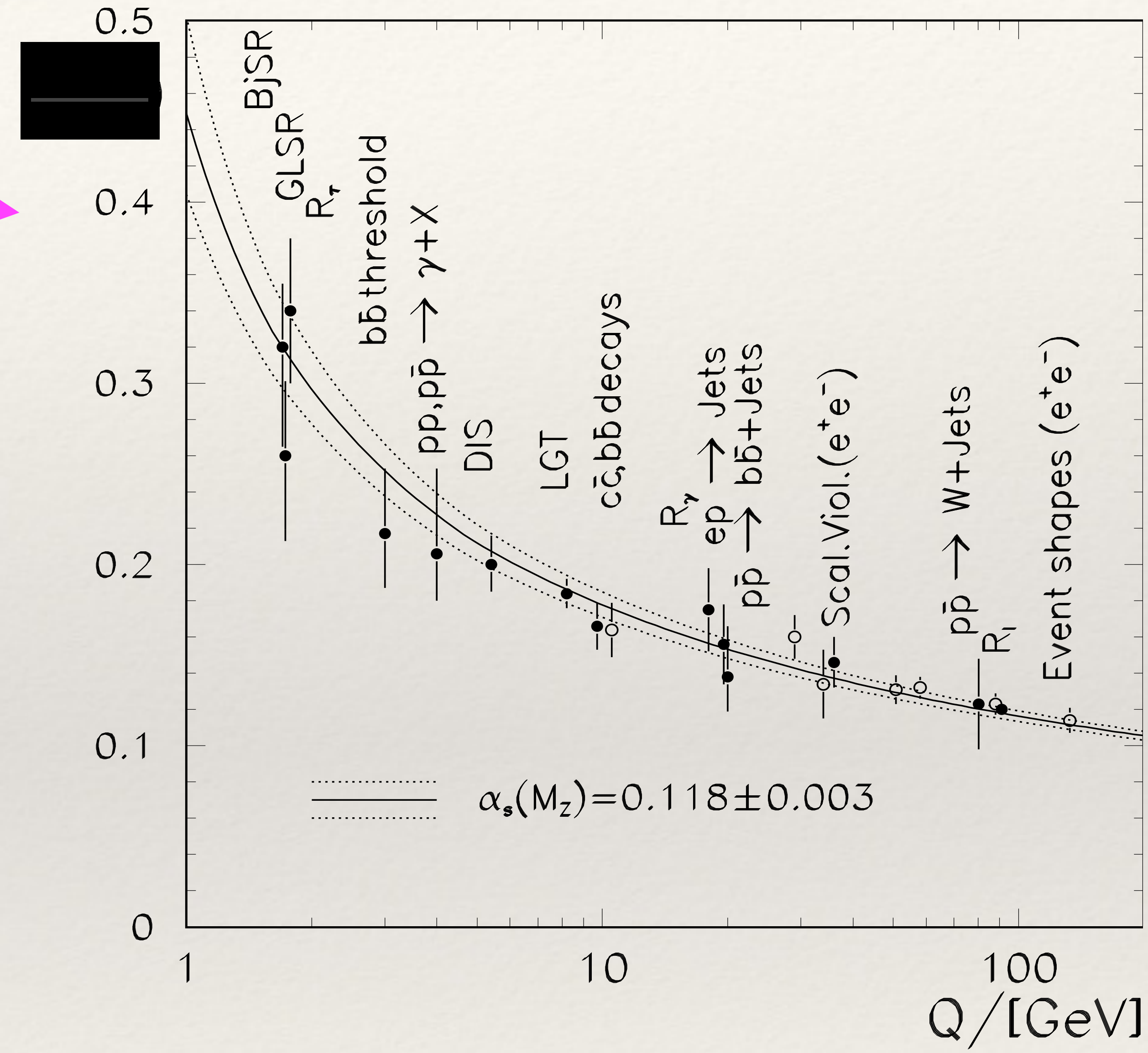
  - They don't work for *every* case
- The succes of the parton model presented a great paradox however



How can quark be both strongly bound into hadrons, and act as free "partons" in deep-inelastic scattering??

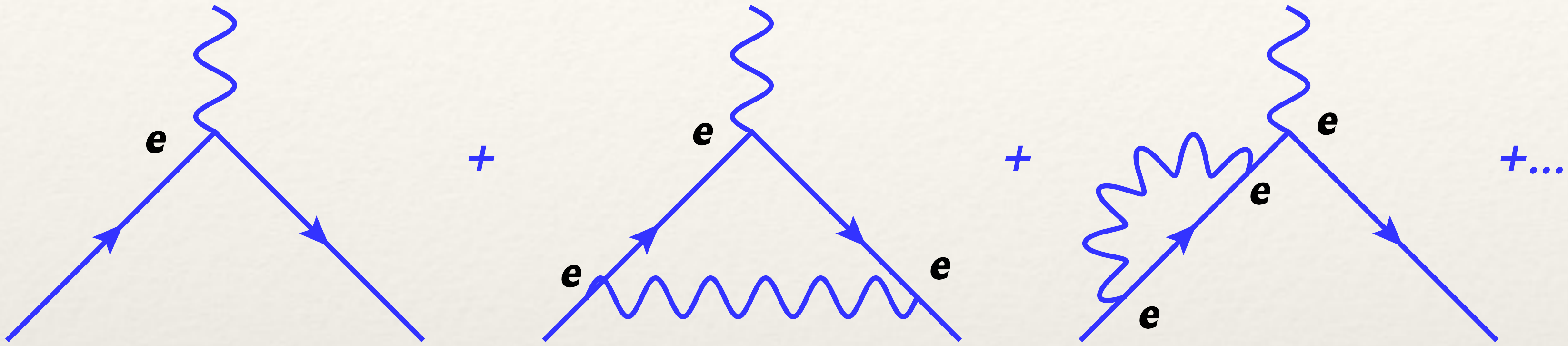
Towards a solution of the paradox

- To solve this paradox, the coupling would have to behave like this
    - At low Q coupling is strong
    - For increasing Q, the coupling decreases
- But: how does a coupling become Q dependent in the first place. In the Lagrangian it is just a number: “g”?
- For this we need to consider the effect of renormalization.



Loops and regularization

- In fact, quantum effects do lead to a scale-dependent coupling, through renormalization.
- Computing any Green function at higher orders in a coupling leads to loops.



- Some loop integrals are divergent, and need to be regularized before being able to “handle” them

$$\int d^4l \, \frac{1}{[l^2 - m^2][(l+p)^2 - m^2]} \sim \int \frac{d^4 l}{(l^2)^2} \sim \mathrm{i} \int \frac{d\Omega \, l^3 dl}{l^4} \sim 2\pi^2 \mathrm{i} \int^{\infty} \frac{dl}{l}$$

- One can put a cut-off on the l integral, but everyone uses dimensional regularization: 4⟶ 4-2ε

$$\int^{\infty} \frac{dl}{l} \to \int^{\infty} \frac{dl}{l^{1+2\varepsilon}} = \frac{-1}{2\varepsilon}$$

- Very elegant. So loop integral results are would-be divergent. How to get rid of this? Renormalize

Renormalization

- We focus on the key point. Write in this case

$$e = Z_e\left(\frac{1}{\varepsilon}, e_R(\mu)\right)\, e_R(\mu)$$

$$Z_e = 1 + e_R^2(\mu)\left(z^{1,1}\frac{1}{\varepsilon} + z^{1,0}\right) + \mathcal{O}(e_R^4)$$

- So beside the loop integrals, there is now a second source of $1/\varepsilon$: the renormalization of the coupling e in the tree-level graph.
    - Choose now the number $z^{1,1}$ such that the $1/\varepsilon$ from the loops is cancelled.
    - You might say I could cancel any $1/\varepsilon$ divergence in that way
    - BUT: the magic of renormalizable theories is that fixing $z^{1,1}$ in this way, will fix this type of $1/\varepsilon$ divergence in *any* other one-loop diagram in this theory.
    - I can renormalize a finite number of quantities: couplings, fields and masses. I can fix the Z-factors in a few calculations, but they must then work also in all other situations.
- Observe that on the right hand side a scale μ appears, in both Z-factor and renormalized coupling e_R. The product does not depend on it. This is the renormalization scale. Sketchwise:

$$\left(1 + e_R^2\ln(\frac{\Lambda}{\mu}) + \mathcal{O}(e_R^4)\right)\times\left(1 + e_R^2\ln(\frac{\mu}{Q}) + \mathcal{O}(e_R^4)\right) = 1 + e_R^2\ln(\frac{\Lambda}{Q}) + \mathcal{O}(e_R^4)$$

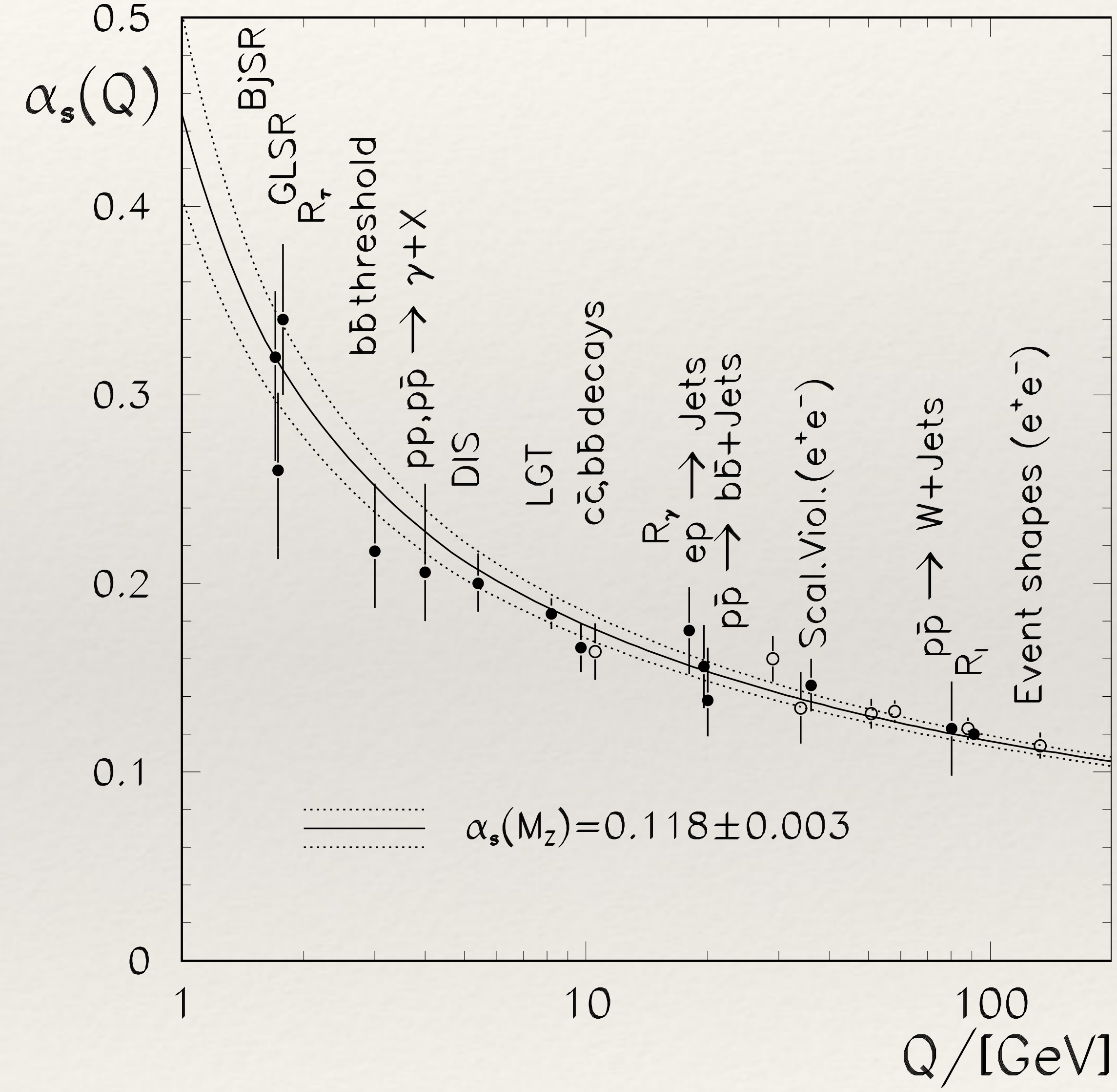
Beta-function

- A very useful reminder when using dimensional regularization:
$$\frac{1}{\varepsilon}\left(\frac{\mu}{Q}\right)^{\varepsilon} = \frac{1}{\varepsilon}\exp\left[\varepsilon \ln\left(\frac{\mu}{Q}\right)\right]$$
$$\simeq \frac{1}{\varepsilon}(1+\varepsilon\ln\left(\frac{\mu}{Q}\right)) = \frac{1}{\varepsilon} + \ln\left(\frac{\mu}{Q}\right)$$
- In analogy to e_R, now for $\alpha_s = g^2/4\pi$
$$\alpha_s = Z_\alpha\left(\frac{1}{\varepsilon}, \alpha_{s,R}(\mu)\right)\alpha_{s,R}(\mu)$$
- We derive from this
$$Z_\alpha = 1 + \frac{\alpha_{s,R}(\mu)}{4\pi}\left(\frac{11C_A - 2n_f}{3}\frac{1}{\varepsilon} + c_\alpha\right) + \mathcal{O}(\alpha_{s,R}^2)$$
$$\mu\frac{d}{d\mu}\ln\alpha_{s,R}(\mu) = -\mu\frac{d}{d\mu}\ln Z_\alpha\left(\frac{1}{\varepsilon}, \alpha_{s,R}(\mu)\right) = -\frac{\beta(\alpha_{s,R}(\mu)}{\alpha_{s,R}(\mu)}$$
- The QCD beta function is known to 4th order, with the 5th order being computed. Keeping only the first term gives the differential equation
$$\mu\frac{d}{d\mu}\alpha_s(\mu) = -\frac{\alpha_s^2(\mu)}{2\pi}\left(\frac{11C_A - 2n_f}{3}\right) = -\frac{\beta_0}{2\pi}\alpha_s^2$$
  - Observe already that an increase in μ leads to decrease in α. But for higher μ the decrease decreases..
  - Solution
$$\alpha_s(\mu) = \frac{4\pi/\beta_0}{\ln\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right)}$$

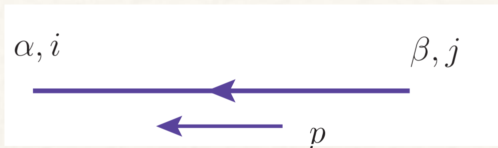
This solves the paradox!

QCD and asymptotic freedom

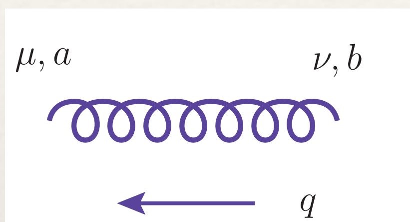
- The QCD couplings is asymptotically free in the UV, and very strong in the IR
- Crucial was the minus sign in front of β_0.
  - higher order terms in β do not spoil this
- In the 70's lots of theories were examined, but only this strange non-abelian gauge theory yielded a negative beta-function
- Nobelprize 2004: Gross, Wilczek, Politzer



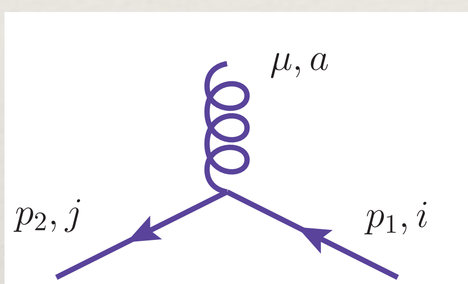
QCD Feynman Rules (not all)



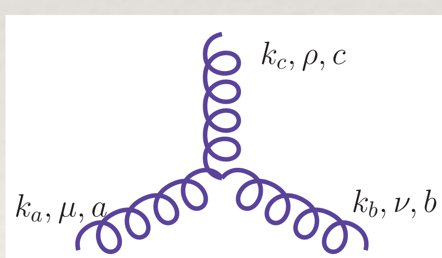
$$\frac{1}{i(2\pi)^4}\frac{\delta_{ij}(-i\not{p}+m)_{\alpha\beta}}{p^2+m^2}$$



$$\frac{1}{i(2\pi)^4}\frac{\delta_{ab}}{q^2}\left(\eta_{\mu\nu}-(1-\lambda^{-2})\frac{q_\mu q_\nu}{q^2}\right)$$



$$i(2\pi)^4\delta^4(p_1-p_2+q)(-g)[\mathbf{T_a}]_{ij}(\gamma_\mu)_{\alpha\beta}$$



$$i(2\pi)^4(-ig)f^{abc}\left[\eta_{\mu\nu}(k^a-k^b)_\rho+\eta_{\nu\rho}(k^b-k^c)_\mu+\eta_{\rho\mu}(k^c-k^a)_\nu\right]$$

- Rules involves Lorentz (vector, spinor) and SU(3) (fundamental, adjoint) parts
  - Not directly linked
  - Omitted 4-gluon vertex, and ghost rules

QCD and UV divergences

- When computing loop integrals, and UV divergences result from them, not all of them can be cancelled by renormalization of just the QCD coupling

$$\alpha_s = Z_\alpha\left(\frac{1}{\varepsilon}, \alpha_{s,R}(\mu)\right)\,\alpha_{s,R}(\mu)$$

$$Z_\alpha = 1 + \frac{\alpha_{s,R}(\mu)}{4\pi}\left(\frac{11C_A - 2n_f}{3}\frac{1}{\varepsilon} + c_\alpha\right) + \mathcal{O}(\alpha_{s,R}^2)$$

- In fact, in general, all the fields, couplings and parameters get their Z-factors.

$$W_\mu{}^a \to \sqrt{Z_W}\, W_\mu{}^a\,, \quad \psi \to \sqrt{Z_\psi}\,\psi\,, \quad c^a \to \sqrt{Z_c}\, c^a\,, \quad b^a \to \sqrt{Z_b}\, b^a\,,$$
$$g \to Z_g\, g\,, \quad m \to Z_m\, m\,, \quad \lambda \to Z_\lambda\,\lambda$$

- As a consequence

$$\mathcal{L} \longrightarrow \mathcal{L} + \Delta\mathcal{L}$$

$$\Delta\mathcal{L} = -\,(Z_\psi - 1)\,\bar\psi \partial\!\!\!/ \psi - m\,(Z_m\, Z_\psi - 1)\,\bar\psi\psi$$
$$+\, g\,(Z_g\, Z_\psi\, Z_W^{1/2} - 1)\, W_\mu^a\, \bar\psi\gamma^\mu t_a \psi + \ldots$$

- From one-loop calculations

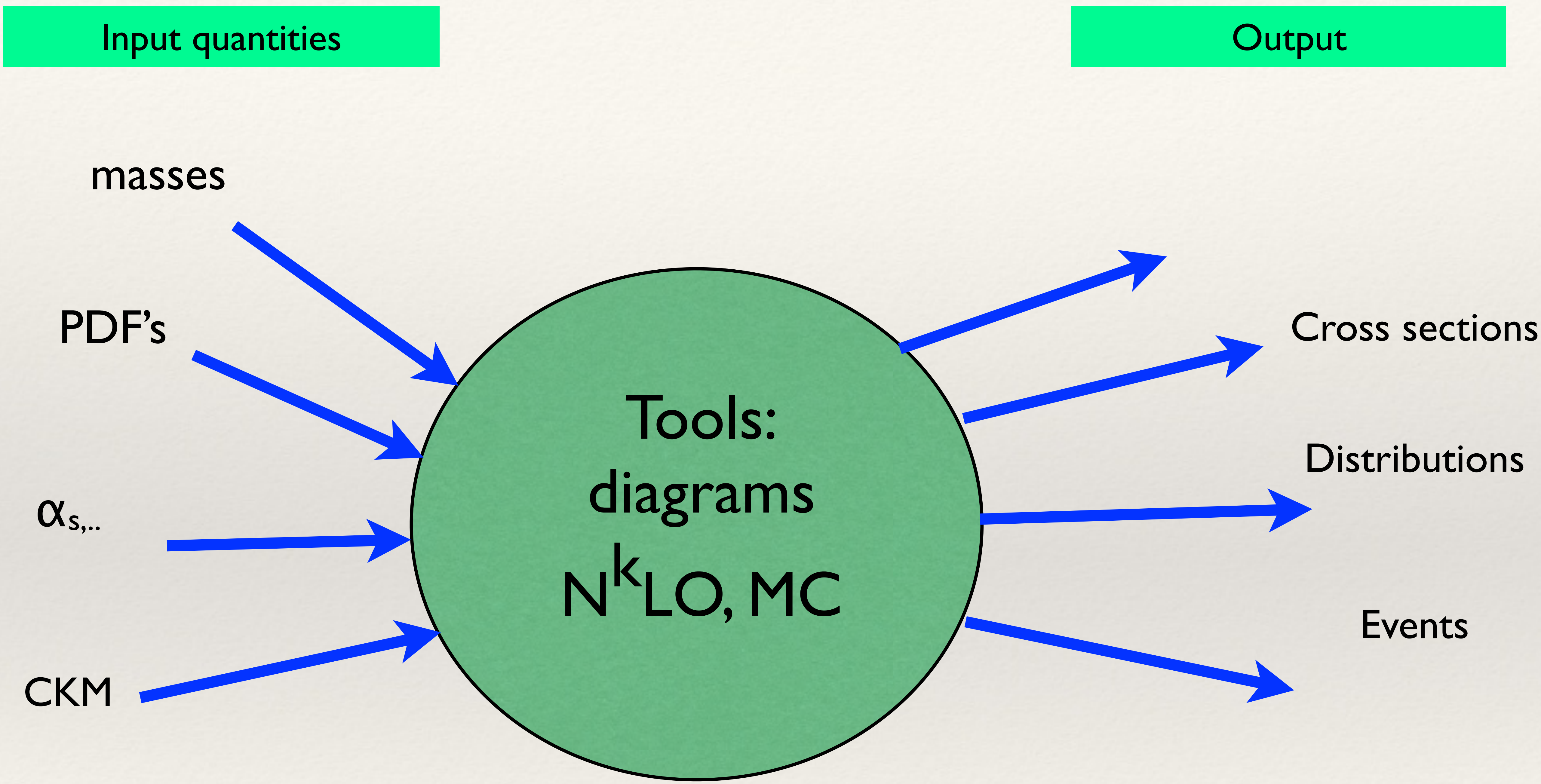
$$\begin{aligned}
Z_W &= 1 + \frac{g^2\mu^\varepsilon}{6\pi^2}\frac{1}{\varepsilon}\left[N_f\, C_2(\mathrm{R})\,\tfrac{\dim \mathrm{R}}{\dim \mathrm{G}} - \tfrac{1}{8} C_2(\mathrm{G})\,(13 - 3\lambda^{-2})\right] + \mathcal{O}(g^4)\,,\\
Z_\lambda &= 1 - \frac{g^2\mu^\varepsilon}{12\pi^2}\frac{1}{\varepsilon}\left[N_f\, C_2(\mathrm{R})\,\tfrac{\dim \mathrm{R}}{\dim \mathrm{G}} - \tfrac{1}{8} C_2(\mathrm{G})\,(13 - 3\lambda^{-2})\right] + \mathcal{O}(g^4)\,,\\
Z_\psi &= 1 + \frac{g^2\mu^\varepsilon}{8\pi^2}\frac{1}{\varepsilon}\, C_2(\mathrm{R})\,\lambda^{-2} + \mathcal{O}(g^4)\,,\\
Z_m &= 1 + \frac{3\,g^2\mu^\varepsilon}{8\pi^2}\frac{1}{\varepsilon}\, C_2(\mathrm{R}) + \mathcal{O}(g^4)\,,\\
Z_g &= 1 - \frac{g^2\mu^\varepsilon}{12\pi^2}\frac{1}{\varepsilon}\left[N_f\, C_2(\mathrm{R})\,\tfrac{\dim \mathrm{R}}{\dim G} - \frac{11}{4} C_2(\mathrm{G})\right] + \mathcal{O}(g^4)\,,\\
\sqrt{Z_b Z_c} &= 1 + \frac{g^2\mu^\varepsilon}{12\pi^2}\frac{1}{\varepsilon}\left[N_f\, C_2(\mathrm{R})\,\tfrac{\dim \mathrm{R}}{\dim G} - \frac{1}{4} C_2(\mathrm{G})\,(11 - 3\lambda^{-2})\right] + \mathcal{O}(g^4)\,.
\end{aligned}$$

Renormalizability of QCD

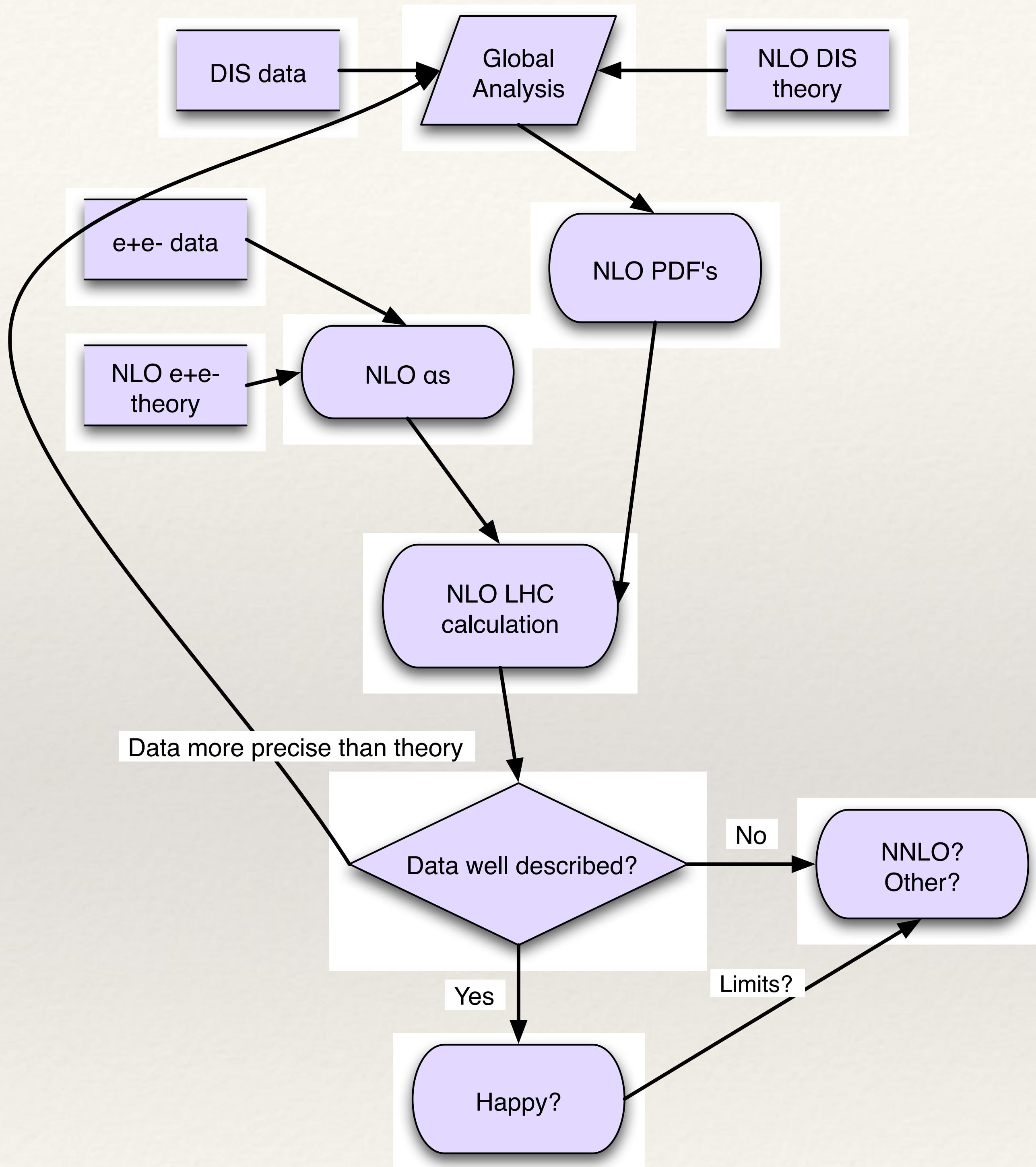
- In fact, with these Z-factors, *every* UV divergence in *any* one-loop QCD amplitude is cancelled.
- But if it goes wrong at higher orders, all is for naught..
- This was a key worry in the early 70's. Renormalizability of QED was known, and of numerous scalar, Yukawa and other field theories. Non-abelian gauge seemed too hard.
- This was the problem that Gerard 't Hooft tackled as a PhD student, together with his advisor Martinus Veltman
- The solution was presented by 't Hooft at a EPS meeting in Amsterdam in 1971, leaving most participants stunned. He and Veltman proved that no new Z-factors are needed to any order. One just needs to determine the same set of Z-factors to higher order.
- They used clever diagrammatic techniques. More modern is the use of BRST symmetry, but the proof was there.
  - Only then was QCD, and in fact the Standard Model, taken more seriously, since it now was a legitimate theory.
- Our problem will be mostly other types of divergences.

Blackboard: RG resummation

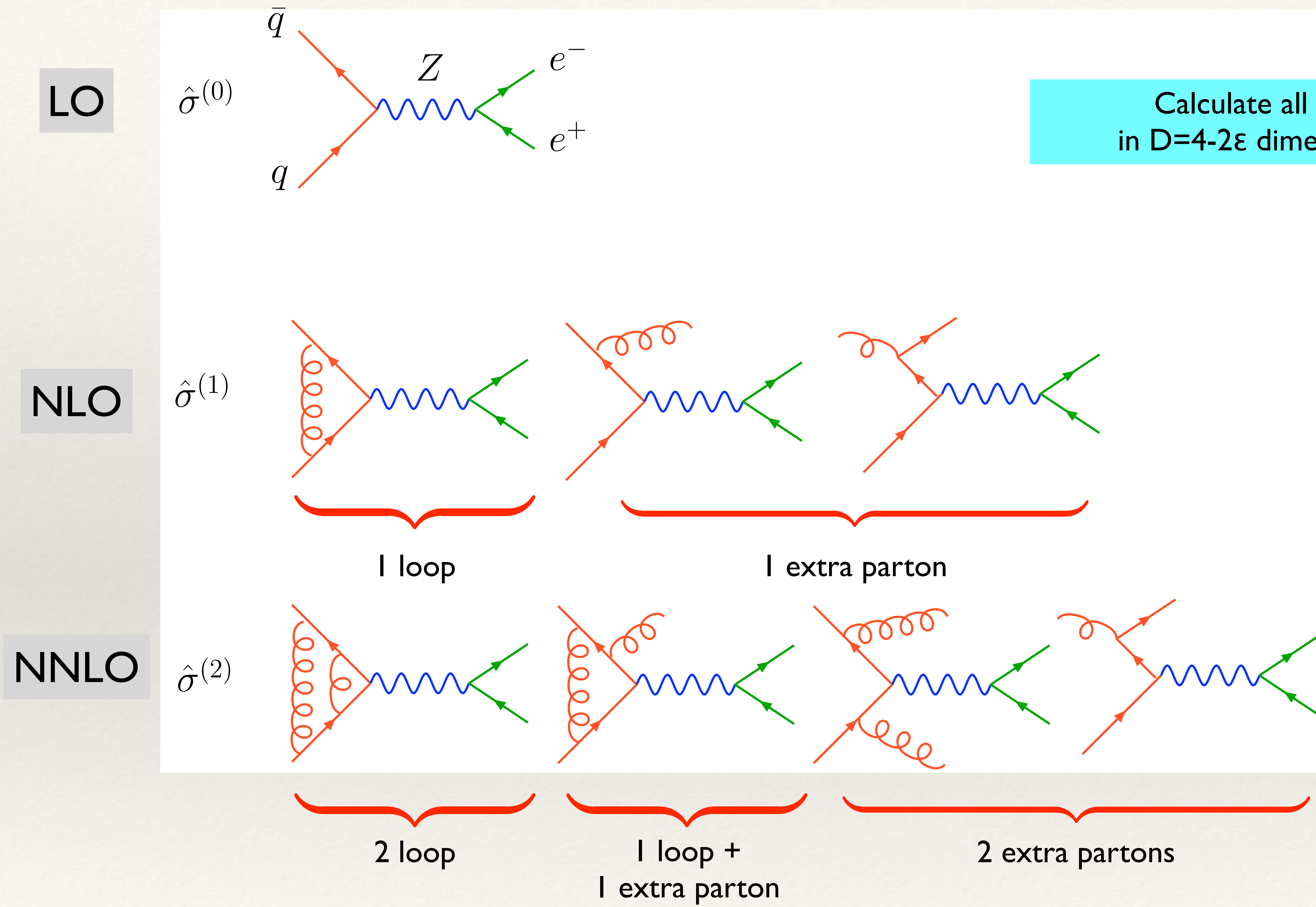
QCD in practice, simplified



How to use QCD in practice, less simplified



LO and higher order amplitudes



Calculate all this
in D=4-2ε dimensions

General structure of LO, NLO,.. cross sections

Multi-differential hadronic NLO cross section

NLO PDF's

$$\frac{d\sigma^{pp\to X}}{d^3p_1\ldots d^3p_n} = \sum_{a,b}\int dx_1 dx_2 \phi_{a/p}(x_1,\mu_F)\phi_{b/p}(x_2,\mu_F)$$

$$\times\hat{\sigma}_{ab}(p_a+p_b\to p_X,\alpha_s(\mu_R),\mu_R,\mu_F)+\mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

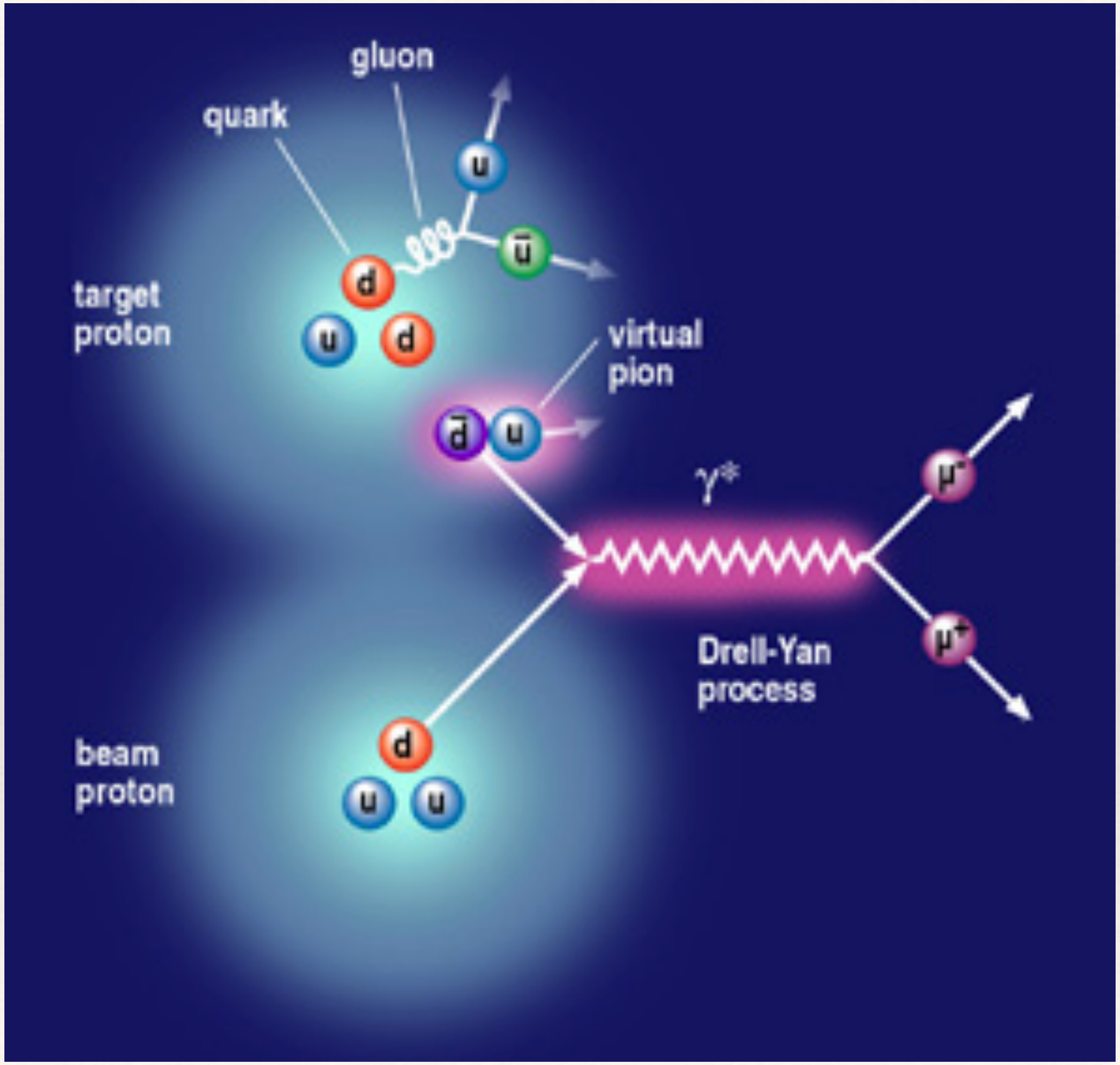
Multi-differential partonlevel NLO cross section

Power corrections. Hard!

Renormalization and Factorization scale

For NNLO, add "N" in front of every occurrence of "NLO"..

KLN, Drell-Yan and its lessons



Parton distribution functions

- Before concentrating on the computing the partonic cross sections, let us discuss the PDF's. In the parton model they only depend on the momentum fraction. But we had seen that structure function depend logarithmically on Q, so we expect that PDF's might also. Indeed that is the case, as we'll see. How does one determine them?

- Crucial at hadron colliders, must be known very accurately. But they cannot be computed from first principles.

- Answer: use their universality, as follows.
  - We need to determine 11 PDF (5 quarks + antiquarks + gluon), *and their uncertainties*
  - Choose with care a set of measurements/observables [e.g. DIS structure functions, or hadron collider cross sections] Each is described as a PDF ⊗ partonic cross sections. We then have the set of equations

$$(O_n \pm \Delta O_n)^{\rm exp} = \sum_{j=1}^{n_f} \phi_{j/p} \otimes [\hat{\sigma}_{n,j} \pm \delta\sigma_{n,j}]^{\rm th}$$

  - From the comparison one fits the $\varphi_{j/P}(x,\mu)$.
    - ✓ Various groups, employing slightly different approaches
      - MSTW, CTEQ, NNPDF, GJR, HERAPDF, ABKM…
  - If the partonic calculation is LO, NLO, NNLO etc, then the PDF thus fitted are also labelled LO, NLO etc.
    - ✓ NLO PDF's must be used with NLO calculations. NNLO also ok, LO not

Aside: PDF's as operator matrix elements

- Although they cannot yet be fully computed from first principles, one can give a precise definition of PDF's, in terms of operators. Essentially, these are counting operators (cf $a^\dagger a$ in QM)

$$\phi_{q/p}(\xi) = \frac{1}{4\pi}\int_{-\infty}^{\infty} dy^- e^{-i\xi p^+ y^-} \langle p|\bar{q}(0,y^-,0_T)\gamma^+ q(0,0,0_T)|p\rangle$$

proton state

Quark field

$$p^\pm = \frac{p^0 \pm p^3}{\sqrt{2}}$$

$$p \cdot q = -p^+q^- - p^-q^+ + p_1q_1 + p_2q_2$$

  - in a certain gauge. The non-perturbative part sits in the hadronic state in which this counting operator is inserted.
  - Benefit: once you have an operator, one can compute its renormalization, and derive an RG equation for it (just like for the coupling constant). This is in fact the DGLAP equation
    - ✓ There are other ways of deriving it. We will see another method later.
  - To do so, just replace the proton states with quark states (and keep the operator). At lowest order this is just

$$\delta(1-\xi)$$

  - At next order it has the form

$$\frac{\alpha_s}{2\pi} C_F \frac{1}{\varepsilon}\left(\frac{1+\xi^2}{1-\xi}\right)_+ + \ldots$$

quark-to-quark splitting function!

    - Plus distribution:

$$\int_0^1 dz \left[\frac{a(z)}{1-z}\right]_+ g(z) = \int_0^1 (g(z)-g(1))\left[\frac{a(z)}{1-z}\right]$$

Parton distribution functions

- The logic is thus very similar to running coupling, we now have "running functions":

$$\mu\frac{d}{d\mu}\phi_{i/H}(x,\mu) = \int_x^1 \frac{dz}{z} P_{ij}(z,\alpha_s(\mu))\,\phi_{j/H}\left(\frac{x}{z},\mu\right) \qquad \left[\equiv P_{ij}\otimes\phi_{j/H}\right](x,\mu)$$

  - DGLAP equations. P_{ij} are the splitting functions, aka parton evolution kernels. They are now known to NNLO (3rd order)
  - Logic: determine the PDF's at some scale Q, then compute them at all other scales by solving the DGLAP equations.

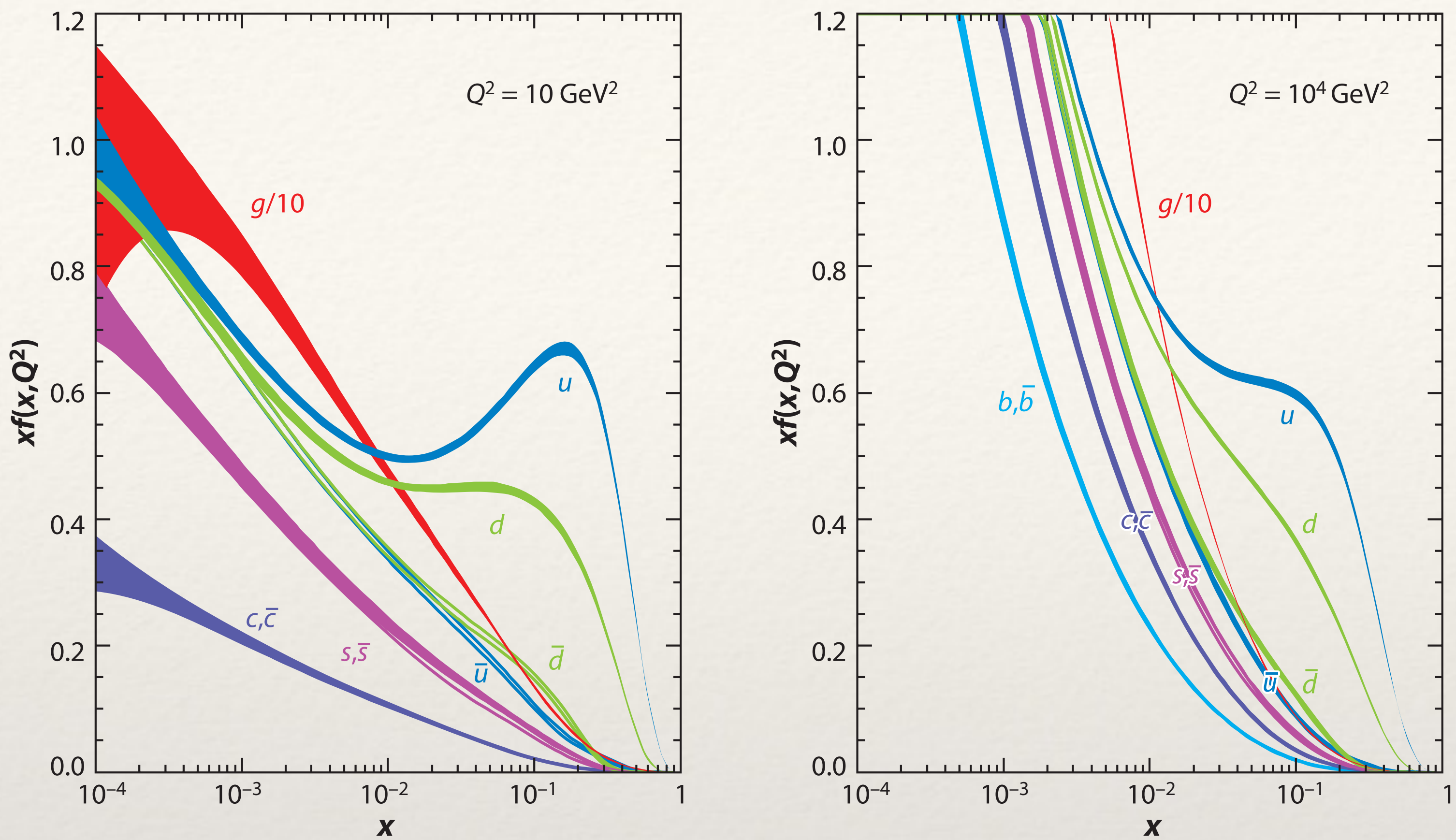
- Note:
  - for LO PDF's, use one-loop splitting and beta-function
  - for NLO PDF's use two-loop splitting and beta-function, etc.
  - in 2004 the three-loop splitting functions [Moch, Vermaseren, Vogt] were computed, so also NNLO sets are now available (NNLO partonic cross sections for DIS, Drell-Yan etc were already available).

- To determine the PDF's from the equation

$$(O_n \pm \Delta O_n)^{\rm exp} = \sum_{j=1}^{n_f} \phi_{j/p}\otimes\left[\hat\sigma_{n,j}\pm\delta\sigma_{n,j}\right]^{\rm th}$$

- one must choose the data on the lhs well.

Form of PDF's



MSTW08 at two values of Q^2

- Notice how evolving the sets to high scale narrows the uncertainty.
  - and how all PDF's grow towards small x: driven by the gluon density in the evolution
- Only u and d still show some bumps: a memory of them being partly valence quarks
- For hadronic collisions one often makes out of the two PDF's the parton luminosity [for "simple enough" cross sections]

$$\sigma_H(s,M^2) = \sum_{a,b}\int_\tau^1 \frac{dx}{x}\mathcal{L}_{ab}(x,M^2)\hat{\sigma}_{ab}(\frac{\tau}{x},M^2,\alpha_s(M^2)) \qquad \tau = M^2/s$$

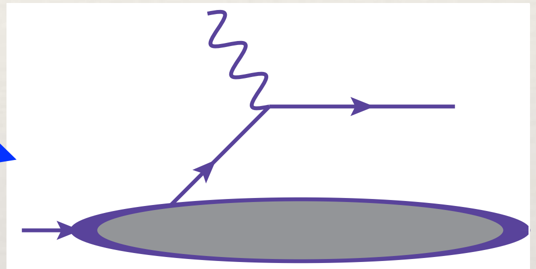
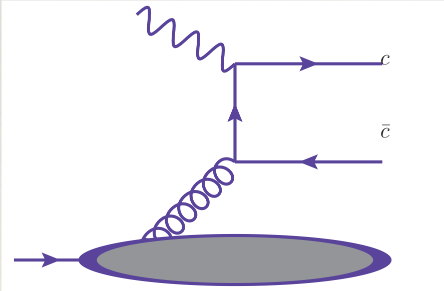
$$\mathcal{L}_{ab}(x,M^2) = \int_x^1 \frac{dz}{z}\phi_{a/p}(z,M^2)\phi_{b/p}(\frac{x}{z},M^2)$$

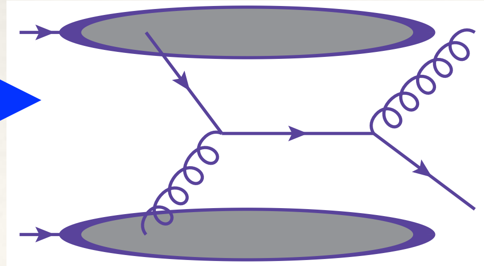
PDF input data

See e.g. Forte, Watt '13

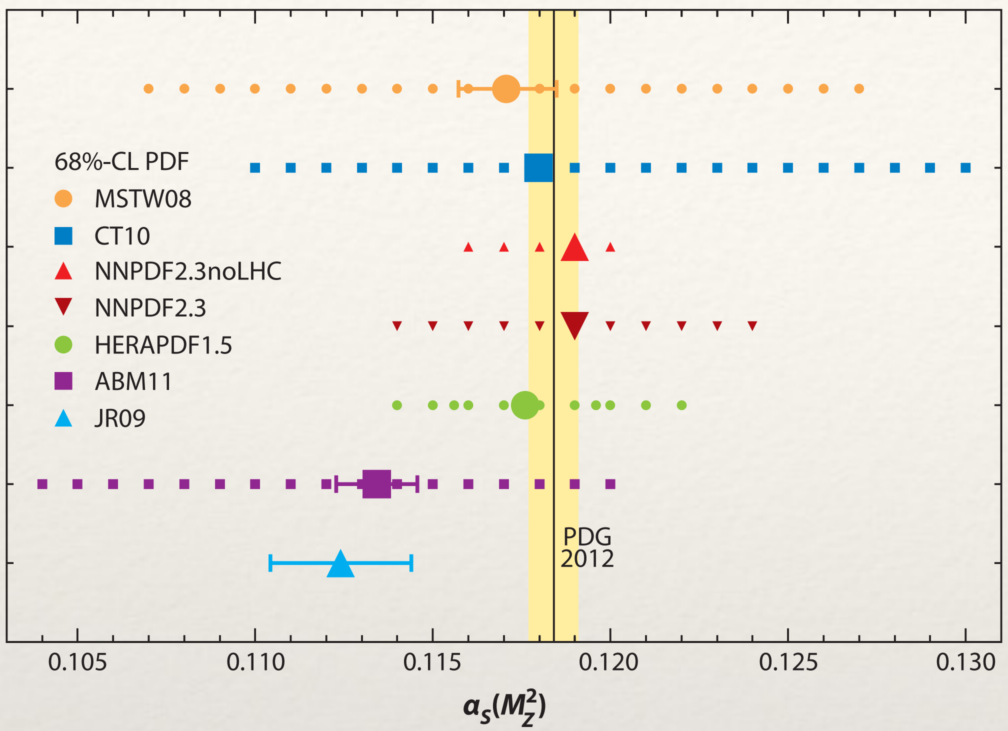
- What data to choose as inputs to fit to?
  - Those that single out particular parton distributions
    - ✓ DIS structure functions most sensitive to valence (u-ū etc) quarks. Prompt photon production sensitive to gluon density etc.
  - Those that provide extra information in certain x ranges (e.g. jet production gives large-x gluon information)

Process	Subprocess	Partons	x range
$\ell^{\pm}\{p,n\} \to \ell^{\pm} X$	$\gamma^* q \to q$	$q, \bar{q}, g$	$x \gtrsim 0.01$
$\ell^{\pm} n/p \to \ell^{\pm} X$	$\gamma^* d/u \to d/u$	d/u	$x \gtrsim 0.01$
$pp \to \mu^+\mu^- X$	$u\bar{u}, d\bar{d} \to \gamma^*$	$\bar{q}$	$0.015 \lesssim x \lesssim 0.35$
$pn/pp \to \mu^+\mu^- X$	$(u\bar{d})/(u\bar{u}) \to \gamma^*$	$\bar{d}/\bar{u}$	$0.015 \lesssim x \lesssim 0.35$
$\nu(\bar{\nu})\, N \to \mu^-(\mu^+)\, X$	$W^* q \to q'$	$q, \bar{q}$	$0.01 \lesssim x \lesssim 0.5$
$\nu\, N \to \mu^-\mu^+ X$	$W^* s \to c$	s	$0.01 \lesssim x \lesssim 0.2$
$\bar{\nu}\, N \to \mu^+\mu^- X$	$W^* \bar{s} \to \bar{c}$	$\bar{s}$	$0.01 \lesssim x \lesssim 0.2$
$e^{\pm} p \to e^{\pm} X$	$\gamma^* q \to q$	$g, q, \bar{q}$	$0.0001 \lesssim x \lesssim 0.1$
$e^+ p \to \bar{\nu} X$	$W^+\{d,s\} \to \{u,c\}$	d, s	$x \gtrsim 0.01$
$e^{\pm} p \to e^{\pm} c\bar{c}\, X$	$\gamma^* c \to c,\, \gamma^* g \to c\bar{c}$	c, g	$0.0001 \lesssim x \lesssim 0.01$
$e^{\pm} p \to \text{jet} + X$	$\gamma^* g \to q\bar{q}$	g	$0.01 \lesssim x \lesssim 0.1$
$p\bar{p} \to \text{jet} + X$	$gg, qg, qq \to 2j$	g, q	$0.01 \lesssim x \lesssim 0.5$
$p\bar{p} \to (W^{\pm} \to \ell^{\pm}\nu)\, X$	$ud \to W,\, \bar{u}\bar{d} \to W$	$u, d, \bar{u}, \bar{d}$	$x \gtrsim 0.05$
$p\bar{p} \to (Z \to \ell^+\ell^-)\, X$	$uu,\, dd \to Z$	d	$x \gtrsim 0.05$

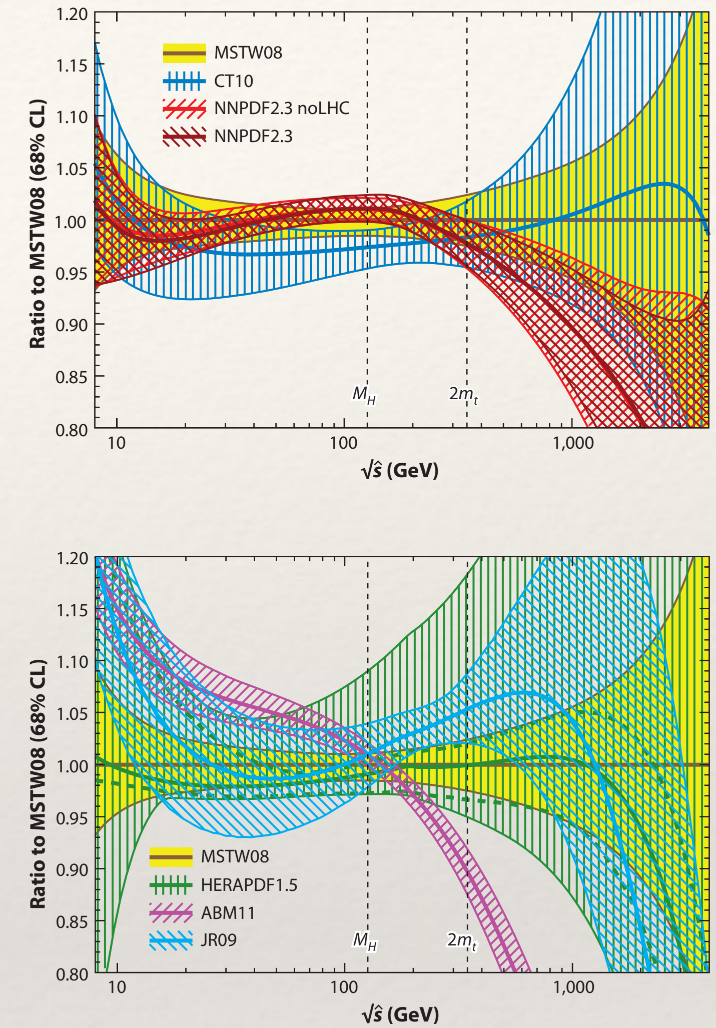





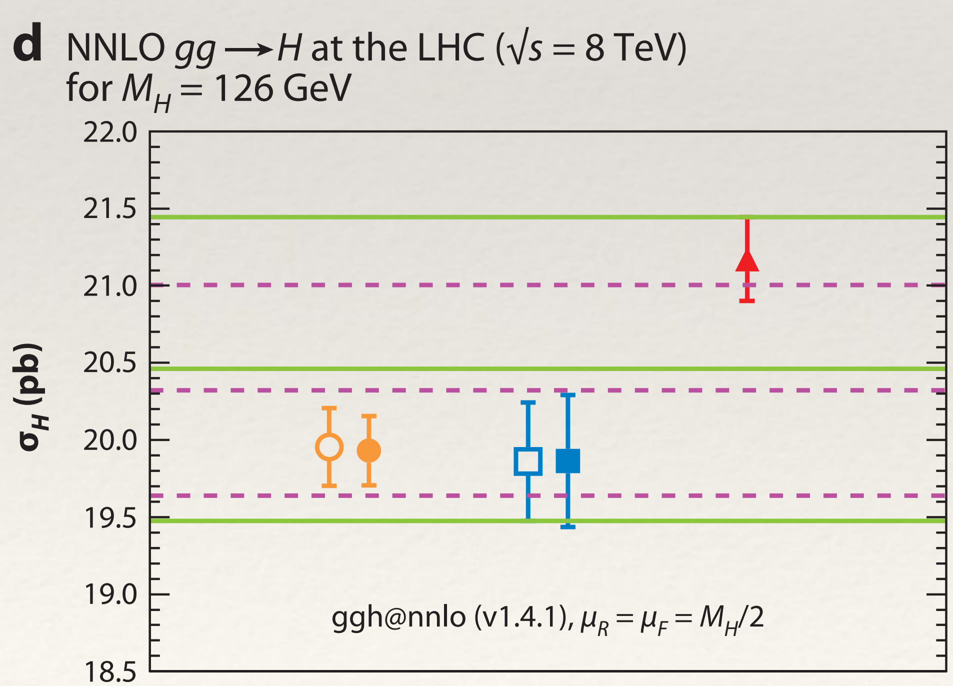
Comparing NNLO PDF sets



α_s values in NNLO sets



gg luminosities at 8 TeV, relative to MSTW08

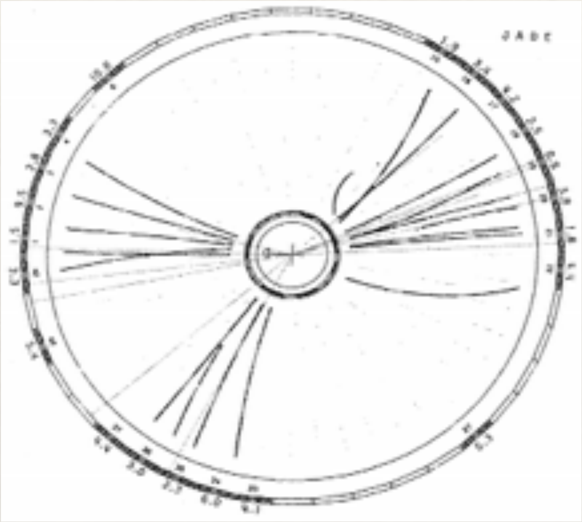


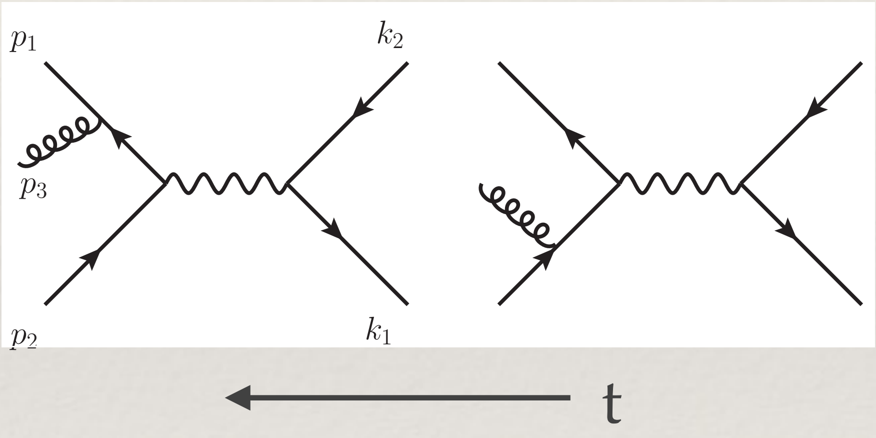
MSTW08 CT10 NNPDF2.3

Impact from LHC

QCD and e^+e^- collisions

- But before turning to hadronic collisions in more details, let us review what QCD does in a simpler setting.
- The cleanest place to study and test QCD is at a e^+e^- collider, where QCD is only active in the final state. We saw already the importance of the R ratio in establishing the number of colors.
- But the R ratio just involves a total cross section: nothing is asked of the final state. It often has an interesting structure, possibly reflecting certain diagrams.





- Two classes of observables do take structure into account
  - Jet cross sections (more on these later)
  - Event shapes

Event shapes - Thrust

- There are *many*. A famous one is Thrust (maximum directed momentum)

$$T = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \vec{\hat{n}}|}{\sum_i |\vec{p}_i|}$$

  - Exercise: show that T=1/2 for spherical final states, and T=1 for two very narrow jets.

$$s_{ij} = -(p_i + p_j)^2$$

$$x_i = E_i/E$$

$$x_1 + x_2 + x_3 = 2$$

- Reaction

$$e^+(k_1) + e^-(k_2) \to \gamma(q) \to q(p_1) + \bar{q}(p_2) + g(p_3)$$

- Phase space measure

$$\frac{1}{(2\pi)^5} \int \frac{d^3p_1}{2E_1} \int \frac{d^3p_2}{2E_2} \int \frac{d^3p_3}{2E_3} = \frac{1}{(2\pi)^5} \int \frac{1}{32q^2} ds_{13} ds_{23} d\phi d\sin\theta d\chi$$

- Squaring the two diagrams and integrating over ϕ and χ

$$\frac{d^3\sigma}{ds_{13} ds_{23} d\sin\theta} = \frac{\alpha_e^2}{8} \frac{\alpha_s}{q^2} \left({x_1}^2 + {x_2}^2\right)\left(2 + \cos^2\theta\right) \frac{1}{s_{13} s_{23}}$$

- Integrating over θ

$$\sigma_T^{-1} \frac{d^2\sigma}{dx_1 dx_2} = \frac{2}{3\pi} \alpha_s \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}, \qquad \sigma_T = \frac{4\pi\alpha^2}{3s}$$

  - ✓ Notice divergences near x_1 or x_2 near 1.

Divergences

- The formula for the 3-parton ($qg\bar{q}$) final state

$$\frac{d^2\sigma}{dx_1 dx_2} = \sigma_T \frac{2}{3\pi}\alpha_s \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

- If we wish to compute the NLO QCD correction to the total cross section, we must integrate this over x_1 and x_2 ($=E_1/E$, E_2/E).
  - but there is an obvious problem if these x's are near 1.
  - $x_1=1$ means that the quark takes half the cm energy, leaving only half the anti-quark plus gluon. It would work out well if the gluon wasn't there. The gluon can imitate "not being there" by having either zero energy and momentum (infrared), or by being perfectly collinear with the massless antiquark

$$p_2^2 = 0,\ p_3^2 = 0, \qquad (p_2 + p_3)^2 = 2p_2 \cdot p_3 = 0 \ \text{ iff } \ p_3^\mu = z p_2^\mu$$

  - Clearly these are divergent situations
    - ✓ Infrared divergence ($p_3^\mu \longrightarrow 0$) and collinear divergence ($p_3^\mu \longrightarrow z p_2^\mu$)
  - Let us see how the occur in practice. We regularized UV divergence using dimensional regularization
  - DimReg can also be used for IR and COL divergences

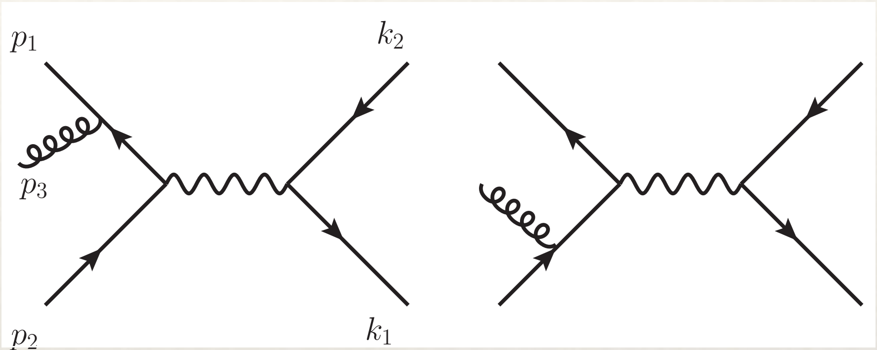
Final state IR and COL divergences

- To use DimReg, we should really have written the final state phase space measure also in n=4-2ε dimensions

$$\int \frac{d^3p_1}{2E_1}\frac{d^3p_2}{2E_2}\frac{d^3p_3}{2E_3} \to \int \frac{d^{n-1}p_1}{2E_1}\frac{d^{n-1}p_2}{2E_2}\frac{d^{n-1}p_3}{2E_3}$$

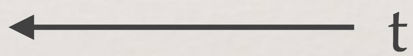
- Then we find

$$\sigma_{qg\bar{q}}(\varepsilon) = \sigma_T 3\frac{\alpha_s}{2\pi}\sum_f Q_f^2 H(\varepsilon)\int dx_1 dx_2 \frac{x_1^2+x_2^2-\varepsilon(2-x_1-x_2)}{(1-x_1)^{1+\varepsilon}(1-x_2)^{1+\varepsilon}}$$



- which yields

$$\sigma_{qg\bar{q}}(\varepsilon) = \sigma_T 3\frac{\alpha_s}{2\pi}\sum_f Q_f^2 H(\varepsilon)\left[\frac{2}{\varepsilon^2}+\frac{3}{\varepsilon}+\frac{19}{2}+\mathcal{O}(\varepsilon)\right]$$



- Double and single poles in ε, from IR and COL regios of phase space. How do they cancel?

Virtual contribution

- But this is not the only contribution to NLO, we also need the virtual contribution. The result of the doing the loop integral in n-dimensions is

$$\sigma_{q\bar{q},V}(\varepsilon) = \sigma_T 3 \frac{\alpha_s C_F}{2\pi} \sum_f Q_f^2 H(\varepsilon) \left[-\frac{2}{\varepsilon^2} - \frac{3}{\varepsilon} - 8 + \mathcal{O}(\varepsilon) \right]$$

- We just found

$$\sigma_{qg\bar{q},R}(\varepsilon) = \sigma_T 3 \frac{\alpha_s}{2\pi} \sum_f Q_f^2 H(\varepsilon) \left[\frac{2}{\varepsilon^2} + \frac{3}{\varepsilon} + \frac{19}{2} + \mathcal{O}(\varepsilon) \right]$$

- Add up and add the LO contribution

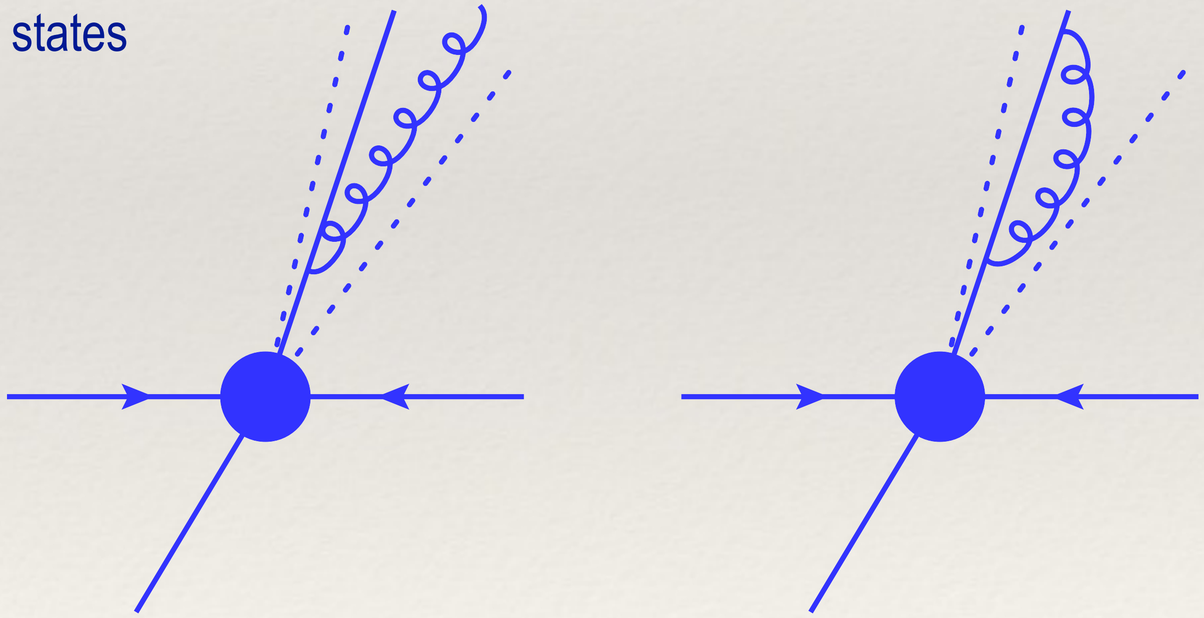
$$\sigma_{NLO} = \sigma_T 3 \sum_f Q_f^2 \left[1 + \frac{\alpha_s}{\pi} \right]$$

- The IR and COL divergence cancel nicely. All we had to do was add the real and virtual contributions.

- This is in fact a very general phenomenon, and it known as the KLN theorem.

Kinoshita-Lee-Nauenberg theorem

- Theorem not only for QCD, but very generally for quantum mechanical transition probabilities
- In essence it says that if one computes the transition probability not just to one very specific state, but to a collection of degenerate states [E-ΔE, E+ΔE] one gets a finite answer.
  - Clearly, a state of just 2 quarks and a state with 2 quarks plus a soft or collinear gluon are degenerate.
  - This is why inclusive, or semi-inclusive cross sections are finite
  - But is also why we look at jets.
    - ✓ A quark with a correction and a quark with a soft of collinear gluon are part of the same jet
      - so a jet defines a collection of degenerate states
      - also event shapes are ***infrared-safe***

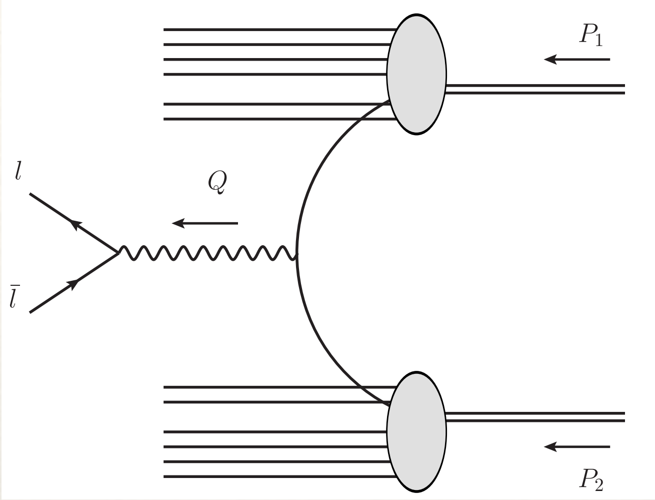


- Now we turn to hadronic collisions.

Drell-Yan

- Production of lepton pair in hadronic collision, either through photon, W or Z

$$p + \bar{p}/p \to l + \bar{l} + X$$



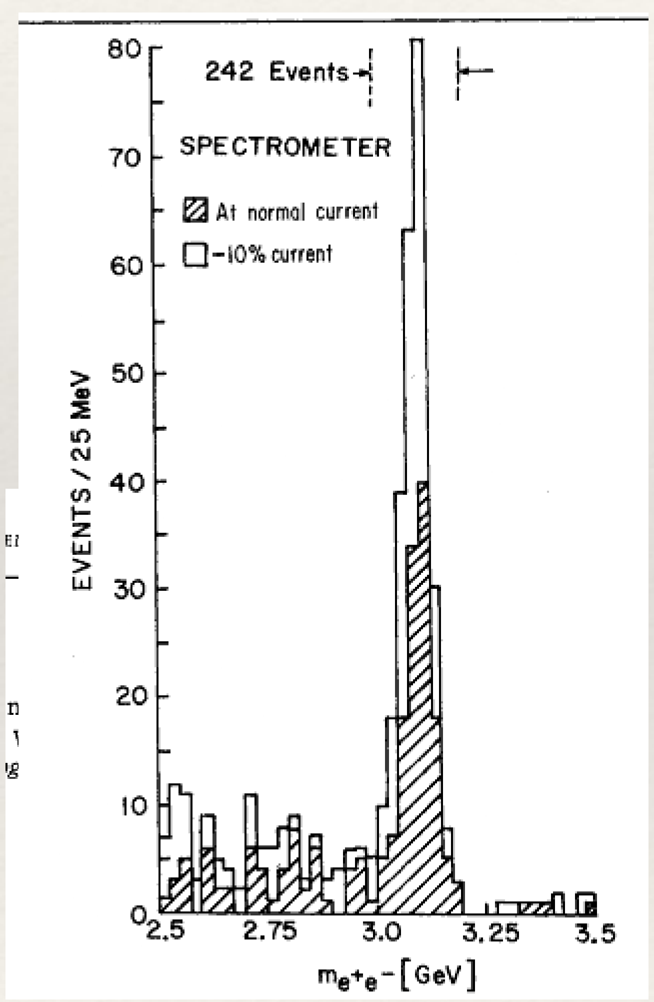
- Storied physics background (next slide)

- These days: often a "theory" laboratory. All the key complications without many external legs. Higgs production is just "Drell-Yan with initial state gluons".

- To illustrate typical issues in QCD higher-order calculations, we shall compute Drell-Yan to NLO.
  - Infrared and collinear divergences, KLN theorem, factorization

Drell-Yan history

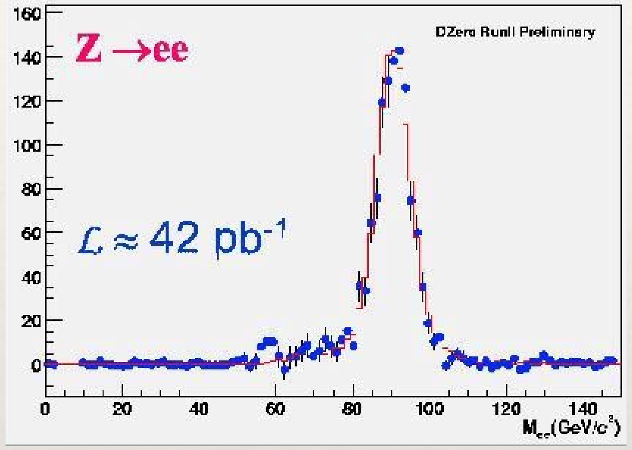
To predict DY cross section
could use the PDF's from DIS.
This worked well.



J/Psi discovery
at BNL AGS and SLAC in '74

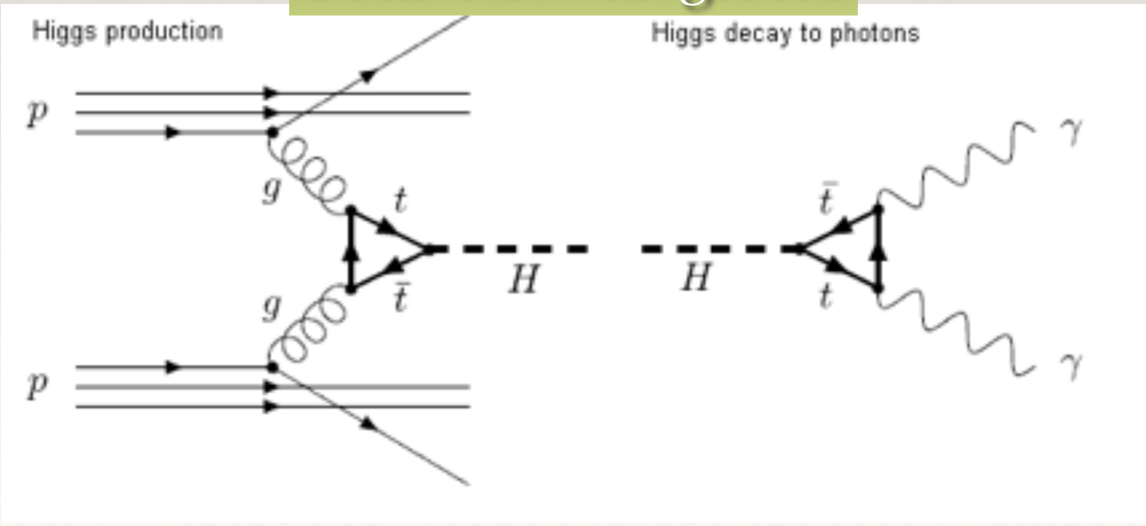
$p+N \rightarrow \Upsilon\,(b\bar{b})+X$
bottom discovery '77
Fermilab E288 exp.

$p+\bar{p} \rightarrow W/Z+X$
W/Z discovery '83
at CERN UA1/UA2



Not discovery but a nice peak!

Last but not least:
Drell-Yan with gluons

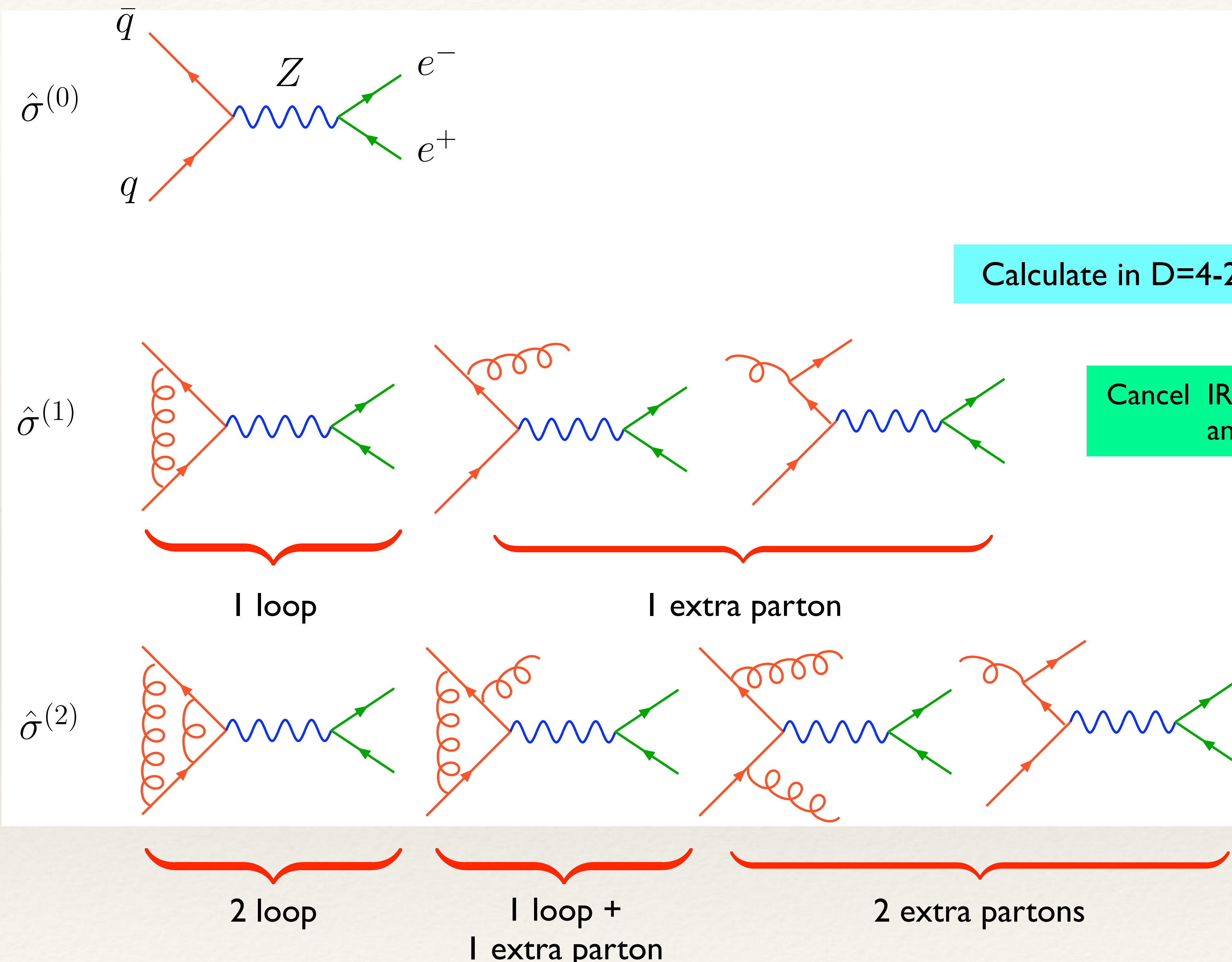


Recall: LO and higher order amplitudes

LO

NLO

NNLO

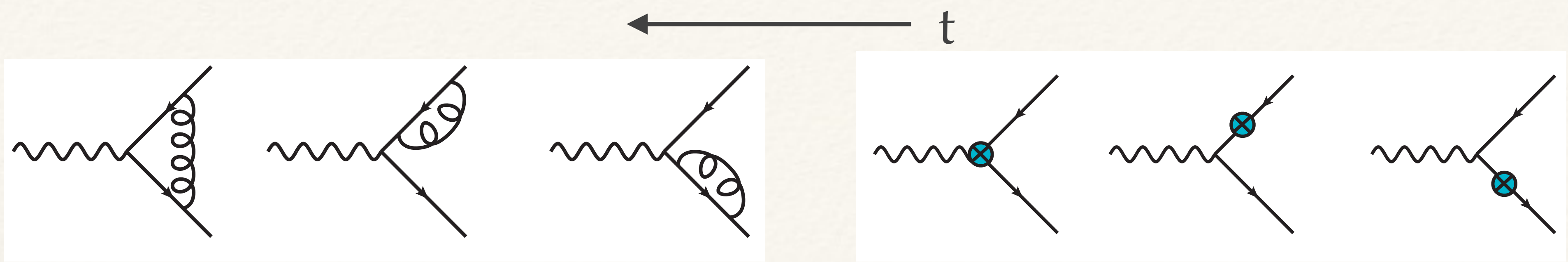


Calculate in D=4-2ε dimensions

Cancel IR poles $1/\varepsilon^2$ before anything else

Cancel IR poles $1/\varepsilon^4$ etc before anything else; hard!

NLO Drell-Yan: virtual diagrams



- Time here from right to left (apologies). 6 diagrams, but we are in luck
  - Sum of three “counterterm contributions” = 0
    - because QCD corrections should not affect the electric charge of the quark
  - Self-energy diagrams = 0, leaves only triangle graph (leftmost one). We suspect (from the e+e- case) that the loop integral will produce IR and COL divergences/
    - ✓ Indeed we find

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2}\Big|_{\text{virtual}} = \sigma^{(0)}_{\gamma} Q_f^2 \frac{1}{2\pi} C_2(R) \left(\frac{4\pi\mu^2}{\hat{s}}\right)^{-\varepsilon/2} \frac{\Gamma(1+\varepsilon/2)}{\Gamma(1+\varepsilon)}$$
$$\times \left[-\frac{8}{\varepsilon^2} + \frac{6}{\varepsilon} - 8 + \frac{2\pi^2}{3} + O(\varepsilon)\right] \delta(1-x)\,, \qquad x = \frac{Q^2}{s}$$

    - ✓ Observe again double and single pole

NLO Drell-Yan: real diagrams

- Now there are two diagrams, with a gluon radiated of either incoming quark. Result

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2}|_{\text{real}} = \sigma^{(0)}_{\gamma} Q_f^2 \frac{1}{2\pi} C_2(R) \left(\frac{4\pi\mu^2}{\hat{s}}\right)^{-\varepsilon/2} \frac{\Gamma(1+\varepsilon/2)}{\Gamma(1+\varepsilon)} \frac{4}{\varepsilon}$$
$$\times \left[2x^{1-\varepsilon/2}(1-x)^{-1+\varepsilon} + x^{-\varepsilon/2}(1-x)^{1+\varepsilon}\right]$$

- We see a single pole, but no double pole! Trouble with KLN?

- No. To see this, express the functions of x in terms of “plus-distributions”

$$\frac{1}{(1-x)^{1-\varepsilon}} = \frac{1}{\varepsilon}\delta(1-x) + \left[\frac{1}{1-x}\right]_+ + \varepsilon\left[\frac{\ln(1-x)}{1-x}\right]_+ + O(\varepsilon^2)$$

  - Now do get double pole

Proof: use a test function F

$$\int_0^1 dx\, \frac{F(x)}{(1-x)^{1-\varepsilon}} = \frac{1}{\varepsilon}\int_0^1 dx\, F(x)\delta(1-x) + \int_0^1 dx\, \frac{F(x)-F(1)}{1-x}$$
$$+\varepsilon\int_0^1 dx [F(x)-F(1)]\frac{\ln(1-x)}{1-x} + O(\varepsilon^2)$$

- Use, and add to virtual. Result

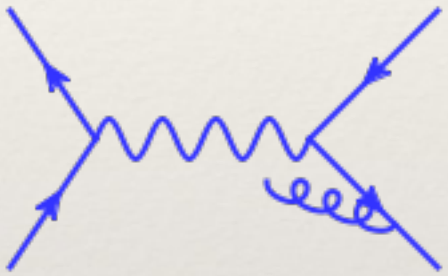
$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2} = \sigma^{(0)}_{\gamma} Q_f^2 \frac{1}{2\pi} C_2(R) \left(\frac{4\pi\mu^2}{\hat{s}}\right)^{-\varepsilon/2} \frac{\Gamma(1+\varepsilon/2)}{\Gamma(1+\varepsilon)}$$
$$\times \left\{\frac{4}{\varepsilon}\left((1+x^2)\left[\frac{1}{1-x}\right]_+ + \frac{3}{2}\delta(1-x)\right) + 4(1+x^2)\left[\frac{\ln(1-x)}{1-x}\right]_+\right.$$
$$\left. -2(1+x^2)\frac{\ln x}{1-x} + (4\zeta(2)-8)\delta(1-x) + O(\varepsilon)\right\}$$

NLO Drell-Yan: sum of real and virtual

- Again, now expressed in terms of the splitting function $P_{qq}(x)$.

$$\frac{d\sigma^{(1)}_{q\bar{q}}}{dQ^2} = \sigma^{(0)}_{\gamma} \frac{Q_f^2}{2\pi} C_\varepsilon \times \left\{ \frac{4}{\varepsilon} P_{qq}(x) + 4(1+x^2)\left[\frac{\ln(1-x)}{1-x}\right]_+ - 2(1+x^2)\frac{\ln x}{1-x} + (4\zeta(2)-8)\delta(1-x) \right\}$$

- Even with KLN helping, there is a remaining divergence!
  - ✓ Initial state collinear divergence



- How to get rid of it?
- Answer: very analogous to use of Z-factor for renormalization of coupling. Renormalize the PDF's as

$$\phi_{q/A}(\xi) = \int_0^1 dz \int_0^1 dy\, \phi_{q/A}(y,\mu_F) T^{-1}_{qq}(z,\mu_F)\delta(\xi - zy)$$

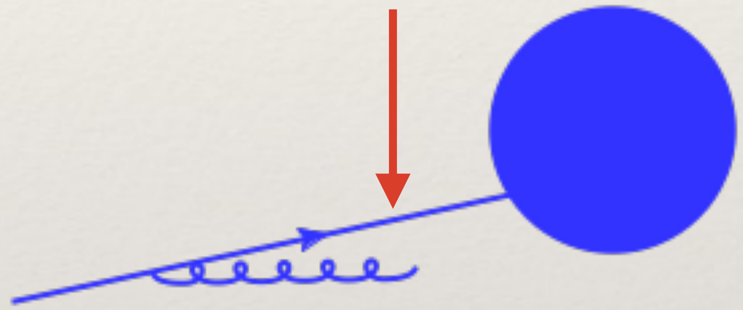
- To first order

$$\phi_{q/A}(\xi) = \phi_{q/A}(\xi,\mu_F) - \int_\xi^1 \frac{dz}{z}\phi_{q/A}\left(\frac{\xi}{z},\mu_F\right) \times \left\{\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}P_{qq}(z)\right\}$$

  - ✓ This new divergence cancels the above one.
- Notice: this new contribution shows no information about this being the Drell-Yan process

QCD Factorization

- What you just witnessed is called “factorization”. It turns out:
  - For any process this removes the remaining initial state collinear divergence!
    - ✓ Works to all orders [Collins, Soper Sterman]
    - ✓ KLN theorem helps cancel all IR and all final state collinear divergences
- As a result, the “renormalized” PDF depends on $\mu_{F,}$ through the DGLAP equation.
- Why does KLN not solve this?
  - The initial state is precisely defined, there is no set of degenerate initial states.



- Physical picture:

$$\phi_{q/A}(\xi) = \phi_{q/A}(\xi,\mu_F) - \int_{\xi}^{1} \frac{dz}{z}\phi_{q/A}\left(\frac{\xi}{z},\mu_F\right) \times \left\{\frac{\alpha_s}{2\pi}\frac{1}{\varepsilon}P_{qq}(z)\right\}$$

- Consider the indicated propagator. If the gluon is very collinear, the virtuality of that line is very small.
  - Therefore, that state could be very long-lived: the gluon could have been radiated off long before the hard scattering. The very collinear gluon thus should be grouped with the proton.

A brief aside on IR analysis, Landau equations…

All orders in QCD: resummation

Predictive power in QFT

- Observable, computed in perturbation theory

$$\hat{O} = \sum_n c_n \, \alpha^n + R_n$$

- Finite order: only take lowest few "n". Please complete then this checklist

☑ α is small enough?

☑ Is R_n small enough ?

☑ c_n does not grow too fast with n?

Resummation

- If it does, sum up perturbative series to all orders
  - why would one do that?
  - what can one sum?
  - when should one do that?
  - to what accuracy?
- Answer: a black box

Perturbative series in QFT

- Typical perturbative behavior of observable

$$\hat{O}_2 = 1 + \alpha(L^2 + L + 1) + \alpha^2(L^4 + L^3 + L^2 + L + 1) + \ldots$$

  - α is the coupling of the theory (QCD, QED, ..)
  - L is some numerically large logarithm
  - “1” = π^2, ln2, anything no
  - Notice: *effective* expansion parameter is αL^2. Problem occurs if is this >1!!
  - Possible fix: reorganize/resum terms such that

$$\begin{aligned}\hat{O} &= 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \ldots \\ &= \exp\left(\underbrace{\underbrace{Lg_1(\alpha_s L)}_{LL} + g_2(\alpha_s L)}_{NLL} + \alpha_s g_3(\alpha_s L) + \ldots\right)\underbrace{C(\alpha_s)}_{\text{constants}} \\ &\quad + \text{ suppressed terms}\end{aligned}$$

- Notice the definition of LL, NLL, etc

LL, NLL,.. and matching to fixed order

- This is nomenclature you see very often: leading-log, next-to-leading log, etc
  - Here is the schematic overview of accuracy in resummation

$$O = \alpha_s^p \left(\underbrace{\underbrace{C_0}_{\text{LL,NLL}} + C_1 \alpha_s}_{\text{NNLL}} + \dots \right) \exp\left[\underbrace{\underbrace{\underbrace{(\sum_{n=1} \alpha_s^n L^{n+1} c_n)}_{\text{LL}} + (\sum_{n=1} \alpha_s^n L^n d_n)}_{\text{NLL}} + (\sum_{n=1} \alpha_s^n L^{n-1} e_n)}_{\text{NNLL}} + \dots \right]$$

  - This is a systematic expansion in α_s in the exponent
    - ✓ If we can find the coefficients c_n, d_n, e_n, C_0, C_1 etc
  - It is directly clear how to combine this with an exact NLO or NNLO calculation
    - ✓ Expand the resummed version to the next order in α_s . Add the NLO and resummed, but subtract the order α_s - expanded resummed result, to avoid double counting.

$$O_{\text{NLO matched}} = O_{\text{NLO}} + O_{\text{resummed}} - (O_{\text{resummed}})\Big|_{\text{expanded to } \mathcal{O}(\alpha_s)}$$

- generalization to NNLO is obvious

- But what can L be the logarithm of?

Benefits of resummation

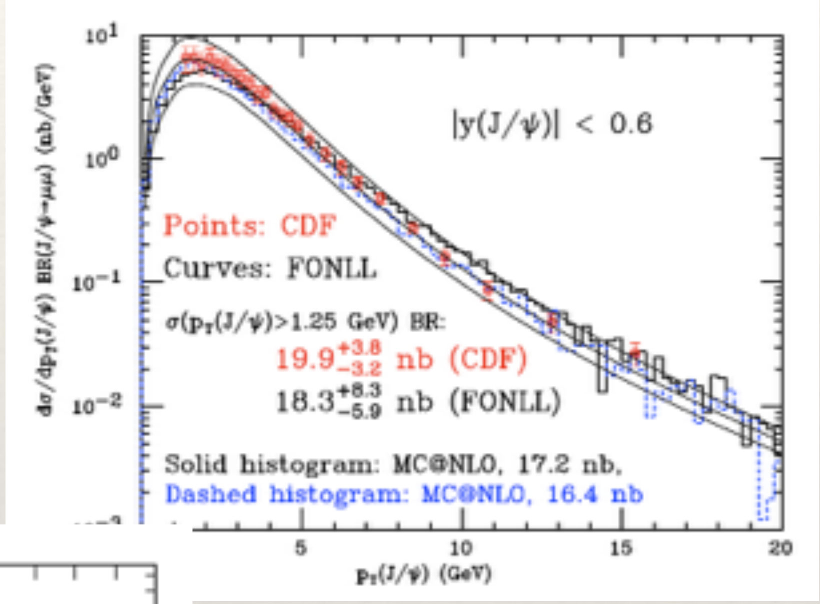
- It can rescue predictive power
  - when perturbative series converges poorly
  - and can predict terms in next order when they are not known exactly yet (“approximate NNLO”)
    - ✓ by expanding the resummed cross section to that order
- Better physics description (small p_T e.g., more later)
- Lessens the renormalization/factorization scale uncertainty,
  - the inclusive top quark cross section
  - the Higgs cross section

Resummation of what logarithm?

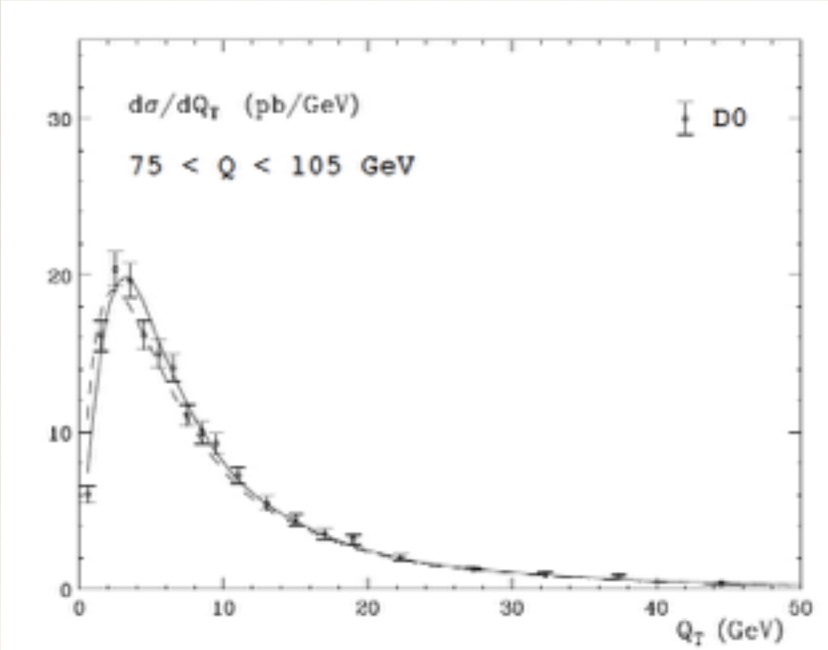
- So many variables, so many logs,…

$\ln(1-T)$ $\ln(p_T/m_Z)$ $\ln(k_T)$ $\ln(1-x)$ $\ln(1/x)$ $\ln(N)$ $\ln(\mu)$ $\ln(b)$

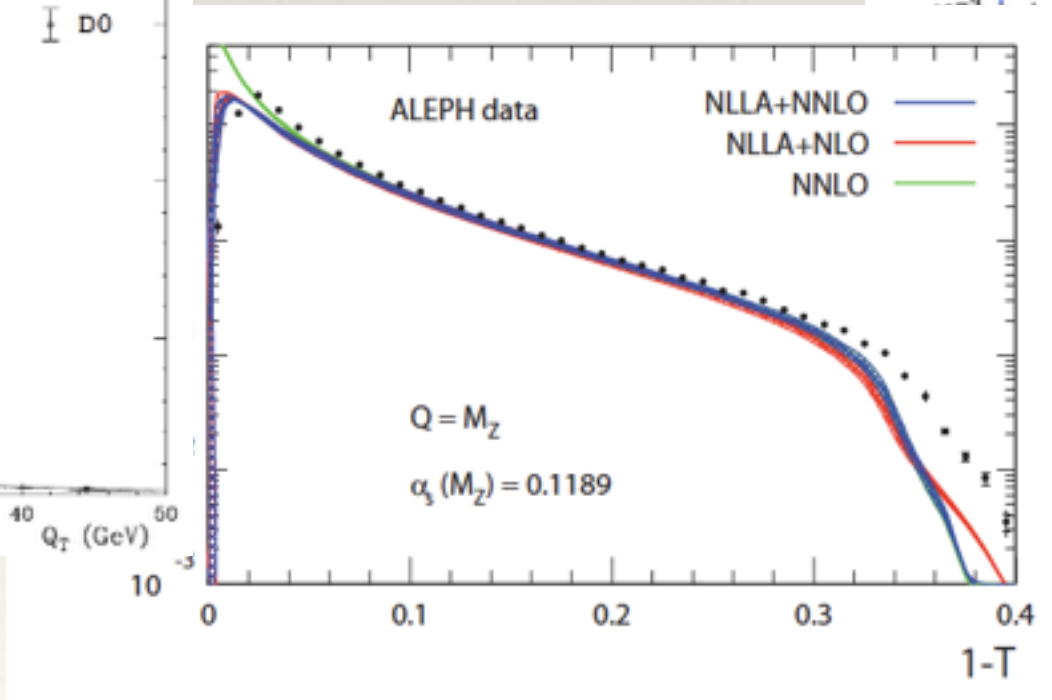
pT of b @ Tevatron



pT of Z @ Tevatron



Thrust T @ LEP



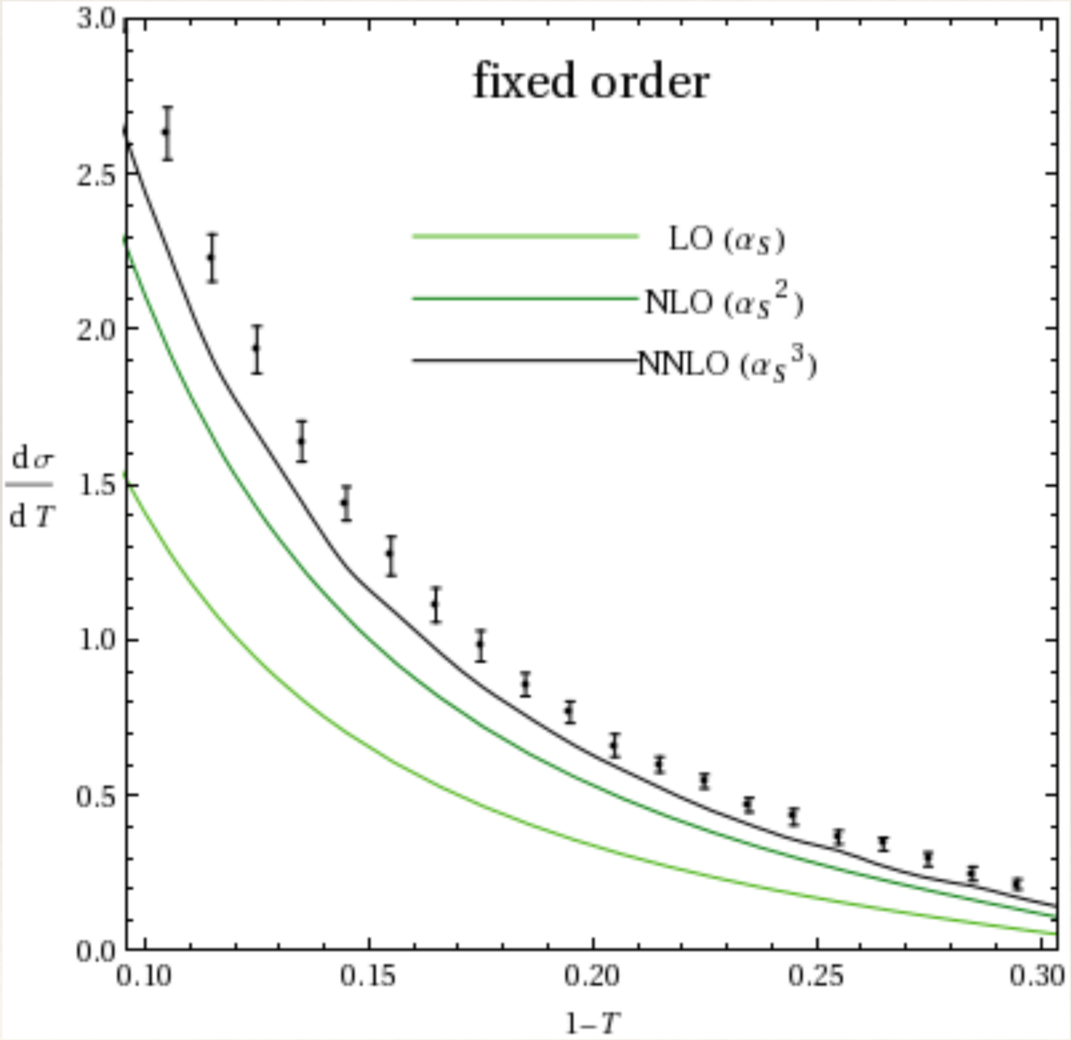
1st example of double logs: thrust

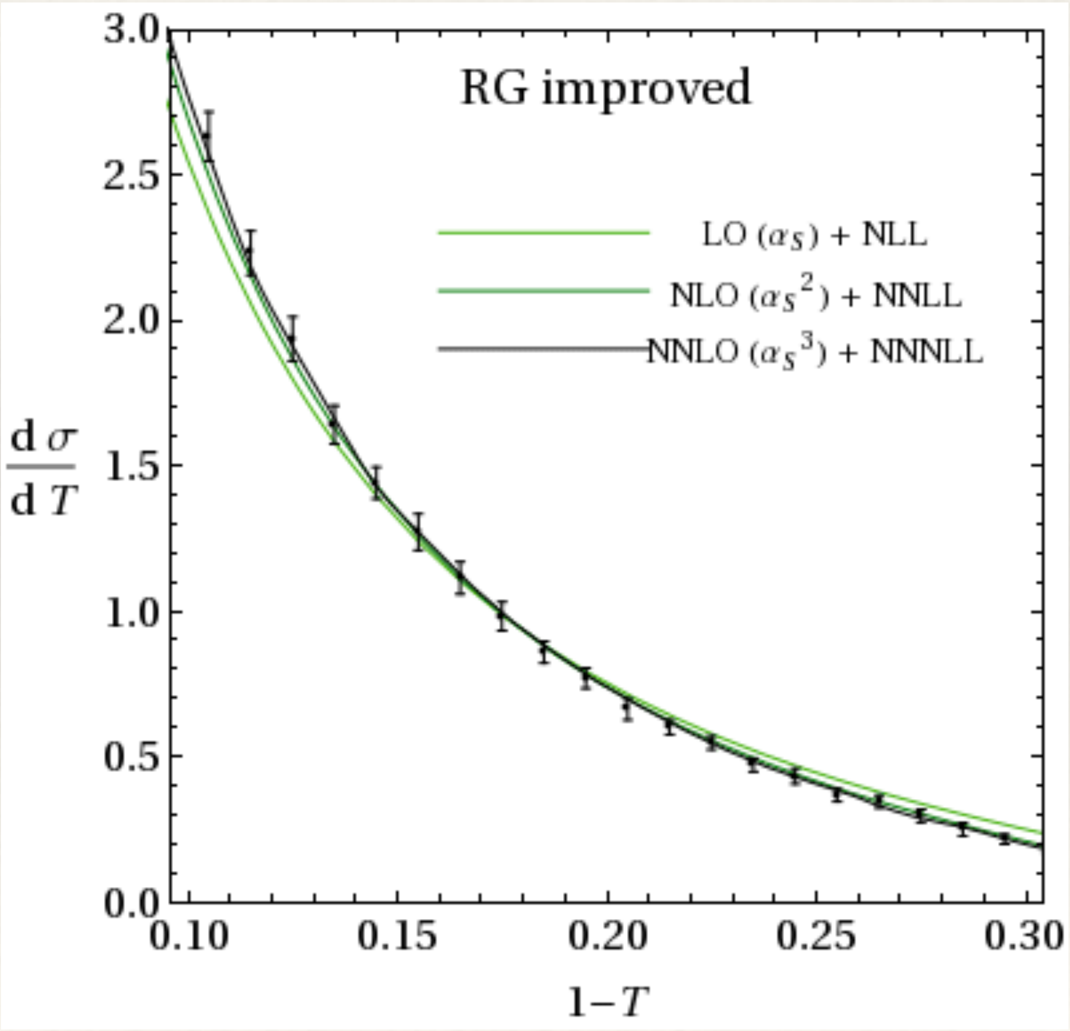
- Near T=1 the final state looks like two very narrow jets
  - emission must then be either very soft, and/or very collinear. Large logs:

$$\ln^2(1-T)$$

  - Data (ALEPH) vs fixed order and vs resummation

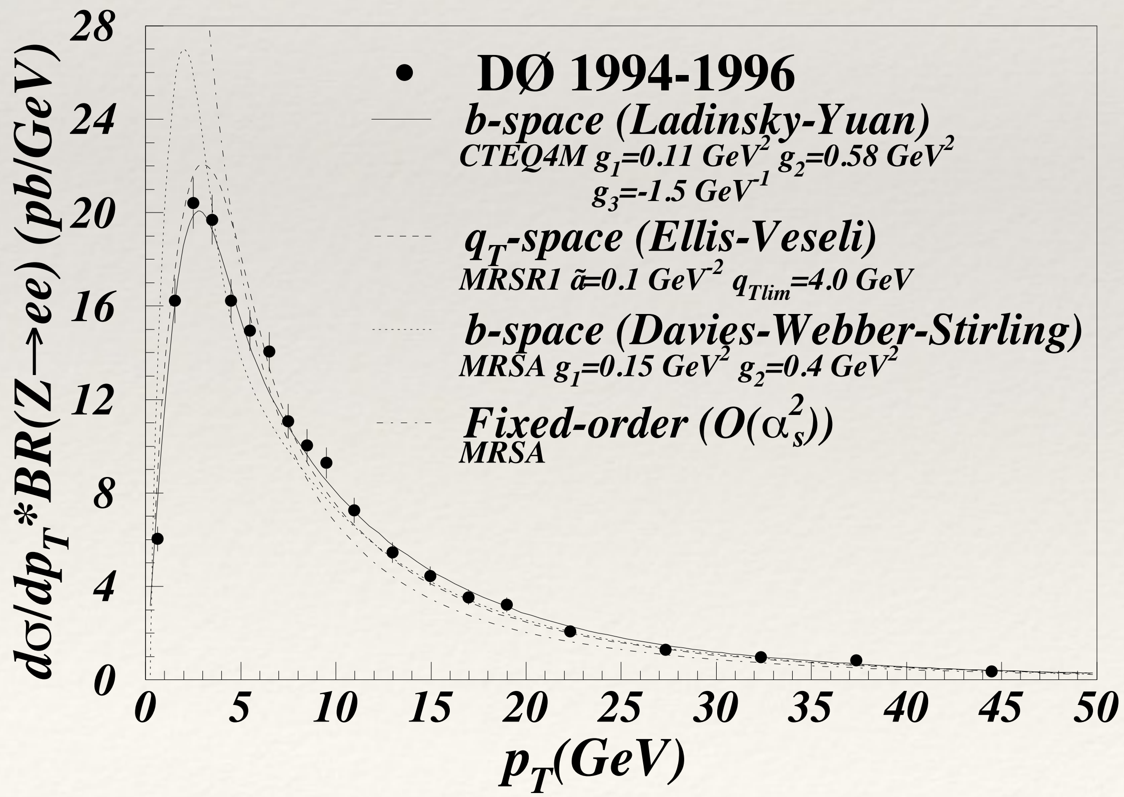
Becher, Schwartz

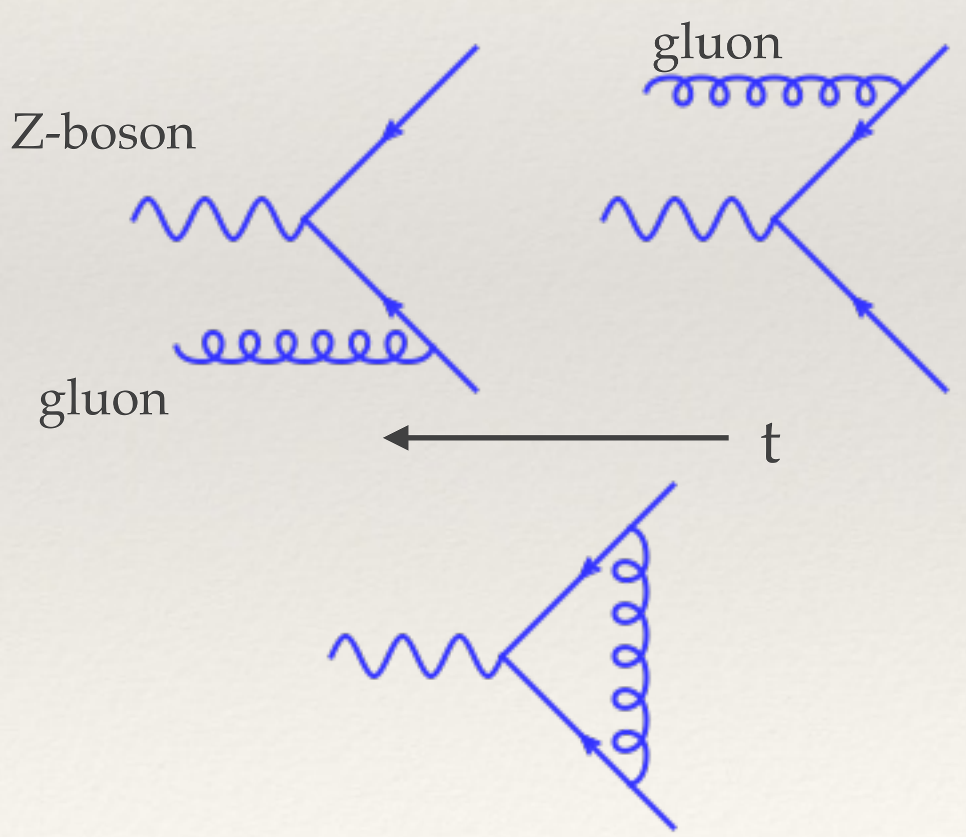




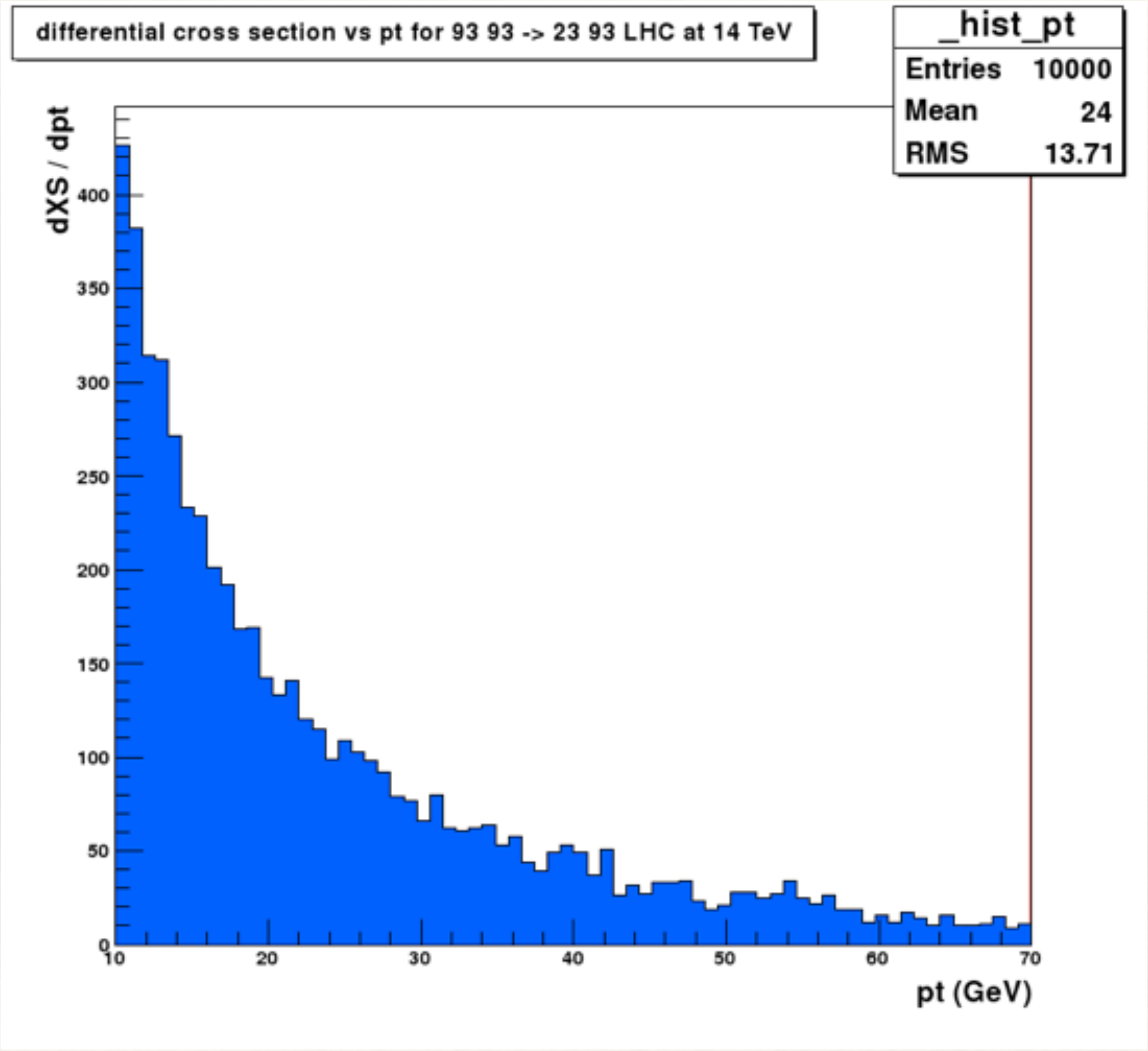
2nd example of double log: recoil logs

- Eg. pT of Z-bosons produced at Tevatron
  - Z-boson gets p_T from recoil agains (soft) gluons
  - Visible logs (argument made of measured quantities)
    - ✓ 1 emission: with gluon very soft: divergent
      - virtual: large negative bin at pT=0
  - The turn-over at pT around 5 GeV is only explained by resummation, not by finite order calculations





Divergence near p_T=0



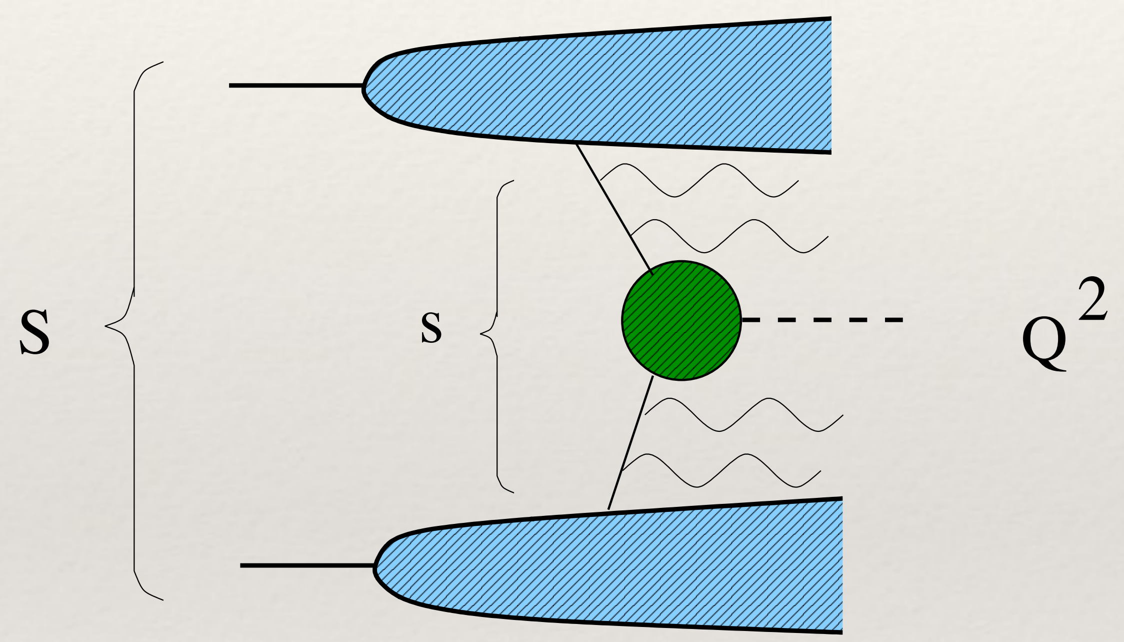
⇔

Physics near small p_T

- At finite order
$$\frac{d\sigma}{dp_T} = c_0\delta(p_T) + \alpha_s\left(c_2^1\frac{\ln p_T}{p_T} + c_1^1\frac{1}{p_T} + c_0^1\delta(p_T)\right) + \ldots$$
  - hence the real divergence toward p_T near zero
- Resummed
$$\frac{d\sigma}{dp_T} = c_0 \exp\left[-c_2^1\alpha_s\ln^2(p_T) + \ldots\right]$$
    - ✓ this is also the effective behaviour of the parton shower there
- Notice:
  - finite order oscillates wildly near small p_T, and may be negative
  - resummed is positive, and it tracks the data well
- Physics of resummed answer:
  - probability of the process **not** to emit at small p_T is vanishingly small
    - ✓ There is violent acceleration of color charges after all..

3rd example of double log: threshold logs

- Logarithm of "energy above threshold Q²" $\ln^2(1 - Q^2/s)$
  - "Invisible" logs": argument made up of integration variables
  - Typical effect: enhancement of cross section



$$S \geq s \geq Q^2$$

Threshold log rule of thumb

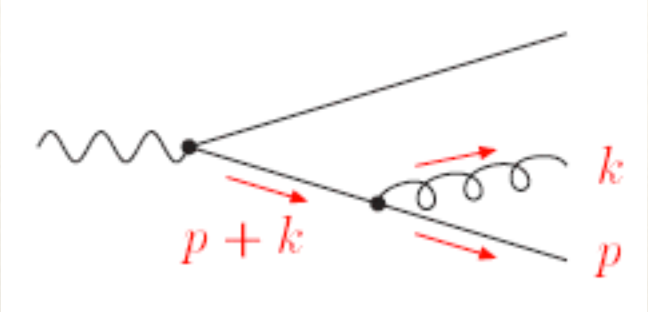
- Why do they increase the cross section? (N large = near threshold)

$$\sigma_{partonic,resum}(N) = \frac{\sigma_{hadronic}(N)}{\phi^2(N)} = \frac{\exp(-\ln^2 N)}{\left(\exp(-\ln^2 N)\right)^2} = \exp(+\ln^2 N)$$

- In words:
  - The hadronic cross section is a product/convolution of PDF’s and the partonic cross section
  - In both factors emissions may, and should occur.
    - ✓ The contribution from the PDF’s is too stingy
    - ✓ The partonic cross section has to overcompensate in order to get the right amount for the hadronic cross section

Reminder of origin of double (“Sudakov”) logs

- Double logarithms in cross sections are related to IR divergences

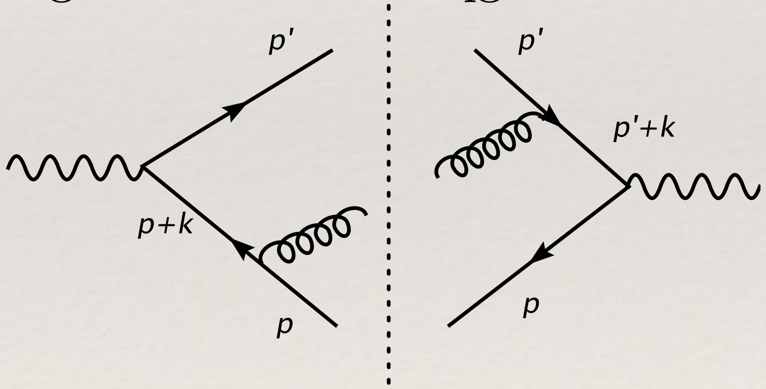


$$\frac{1}{(p+k)^2} = \frac{1}{2p\cdot k} = \frac{1}{2E_{\rm g}E_{\rm q}(1-\cos\theta_{\rm qg})}$$

Phase space integration

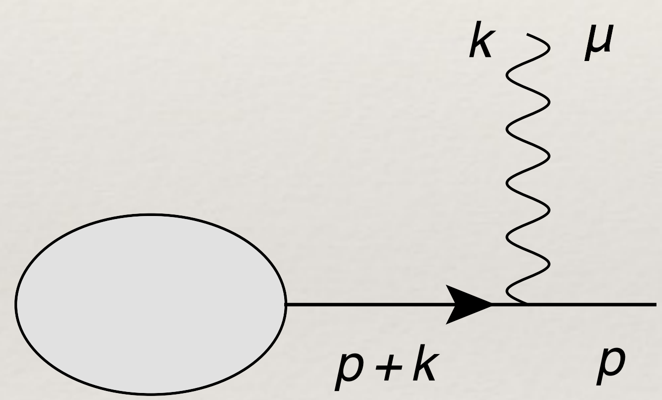
$$\alpha_s \int \frac{d^{4-2\epsilon}k}{(2\pi)^4}\,\frac{p\cdot p'}{p\cdot k\, p'\cdot k} \sim \alpha_s \int^K \frac{dE_{\rm g} E_{\rm g}^{-\epsilon}}{E_{\rm g}} \int \frac{d\theta_{\rm qg}\,\sin^{-\epsilon}\theta_{\rm qg}}{\theta_{\rm qg}}$$

$$\sim \alpha_s\,(\frac{1}{\epsilon^2} + \ln^2(K))\,.$$



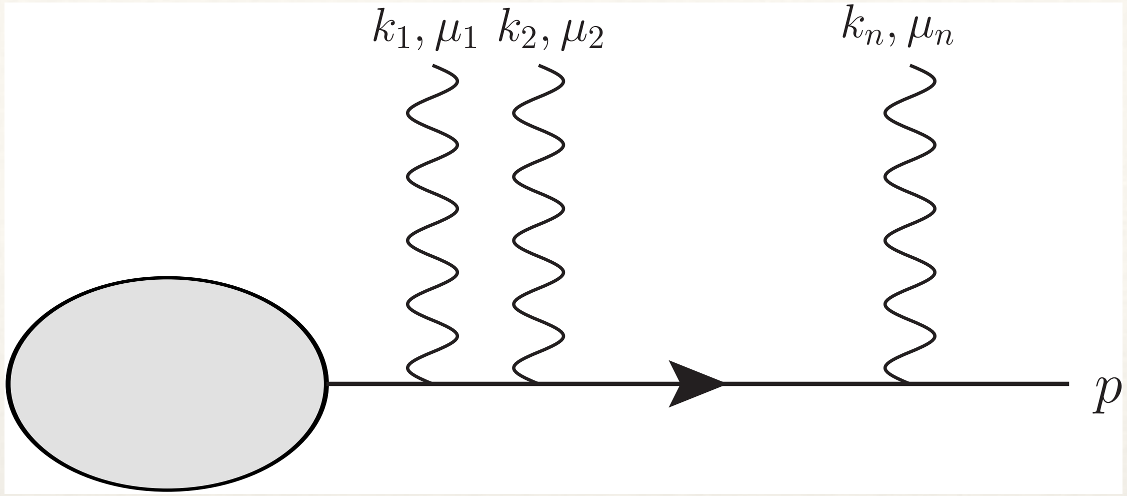
Basics of eikonal approximation: QED

- Charged particle emits softly
  - Propagator: expand numerator & denominator in soft momentum, keep lowest order
  - Vertex: expand in soft momentum, keep lowest order



$$\frac{(p+k)^\mu + p^\mu}{2p\cdot k + k^2} \longrightarrow \frac{2p^\mu}{2p\cdot k}$$

Basics of eikonal approximation in QED



Exact: $$\frac{1}{(p+K_1)^2}(2p+K_2+K_1)^{\mu_1}\dots\frac{1}{(p+K_n)^2}(2p+K_n)^{\mu_n}, \quad K_i=\sum_{m=i}^{n} k_m.$$

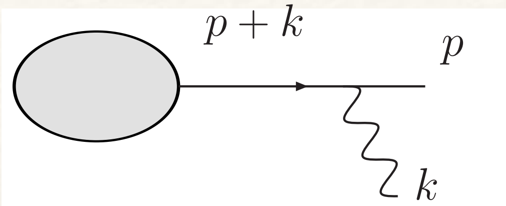
Approx: $$\frac{1}{2pK_1}2p^{\mu_1}\dots\frac{1}{2pK_n}2p^{\mu_n}$$

Eikonal identity: $$\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_2}+\frac{1}{p\cdot(k_1+k_2)\,p\cdot k_1}=\frac{1}{p\cdot k_1\,p\cdot k_2}$$

Sum over all perm's: $$\prod_i \frac{p^{\mu_i}}{p\cdot k_i}.$$

Independent, uncorrelated emissions, Poisson process

Eikonal approximation: no dependence on emitter spin

- Emitter spin becomes irrelevant in eikonal approximation
  - Fermion  $M(\frac{i(\not p+\not k)}{(p+k)^2}(-ig_s\gamma^\mu)\,u(p)$

  - Approximate, and use Dirac equation $\not p u(p) = 0$
  - Result:

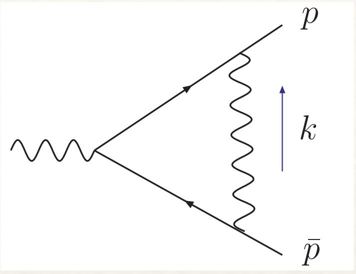
$$g\,(M\,u(p)) \times \frac{p^\mu}{p\cdot k}$$

  - Two things have happened
    - ✓ No sign of emitter spin anymore
    - ✓ Coupling of photon proportional to p^μ !
- Decoupling again of emission and emitter

Eikonal exponentiation

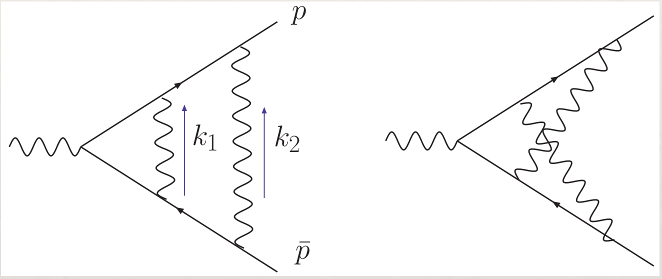
- In the eikonal approximation, suddenly we see very interesting patterns.

One loop vertex correction, in eikonal approximation



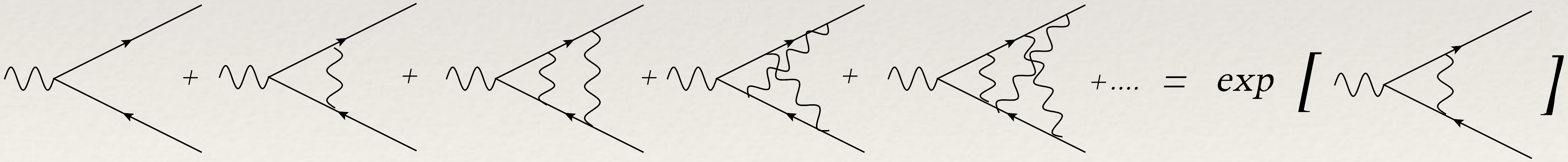
$$\mathcal{A}_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$

Two loop vertex correction, in eikonal approximation



$$\mathcal{A}_0 \, \frac{1}{2} \left(\int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

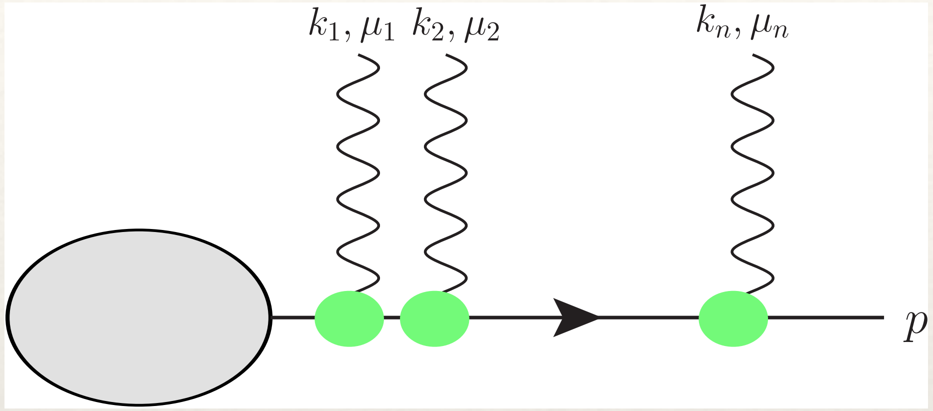
Exponential series! A really beautiful result



Yennie, Frautschi, Suura

Non-abelian eikonal approximation

- Same methods as for QED, but organization harder: SU(3) generator at every vertex



  - now no obvious decorrelation

Order the T_a according to λ

- Key “object”: Wilson line

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[ig \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A^a(\lambda n)\, T_a\right]$$

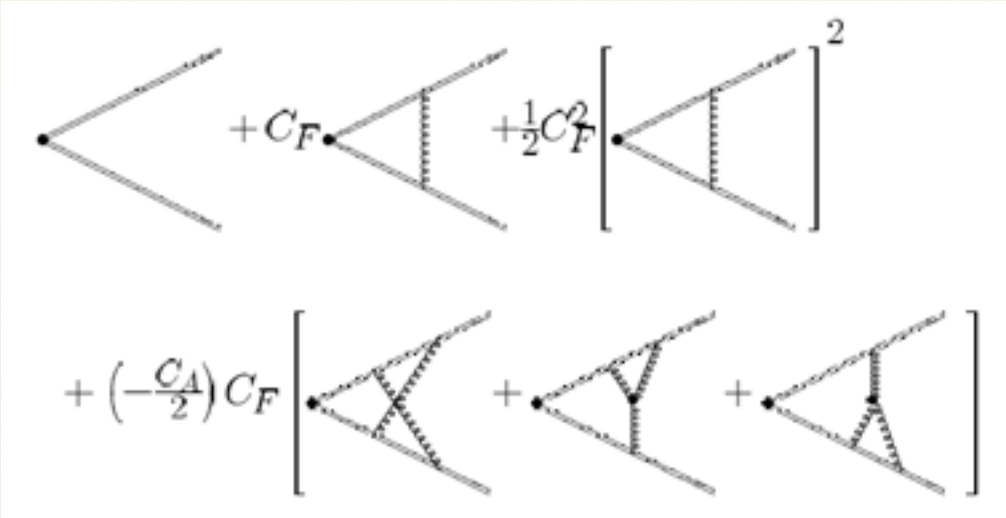
  - Order by order in “g”, it generates QCD eikonal Feynman rules, including the SU(3) generators

Non-abelian exponentiation: webs

Gatheral; Frenkel, Taylor; Sterman

- Take quark - antiquark line, connect with soft gluons in all possible ways, and use eikonal approximation
- Exponentiation still occurs! Sum of all eikonal diagrams D with color factor C and momentum space part F

$$\sum C(D)\mathcal{F}(D) = \exp\left[\bar{C}(D)W(D)\right]$$



- A selection of diagrams in exponent, but with modified color weights: “webs”
  - ✓ Easy to select webs: they must be two-eikonal line irreducible
  - ✓ More difficult to compute the modified color factors, but can be done also

Resummation using path integrals

EL, Stavenga, White

Use textbook result

$$\text{Sum of all diagrams} = \exp\Big(\text{Connected diagrams}\Big)$$

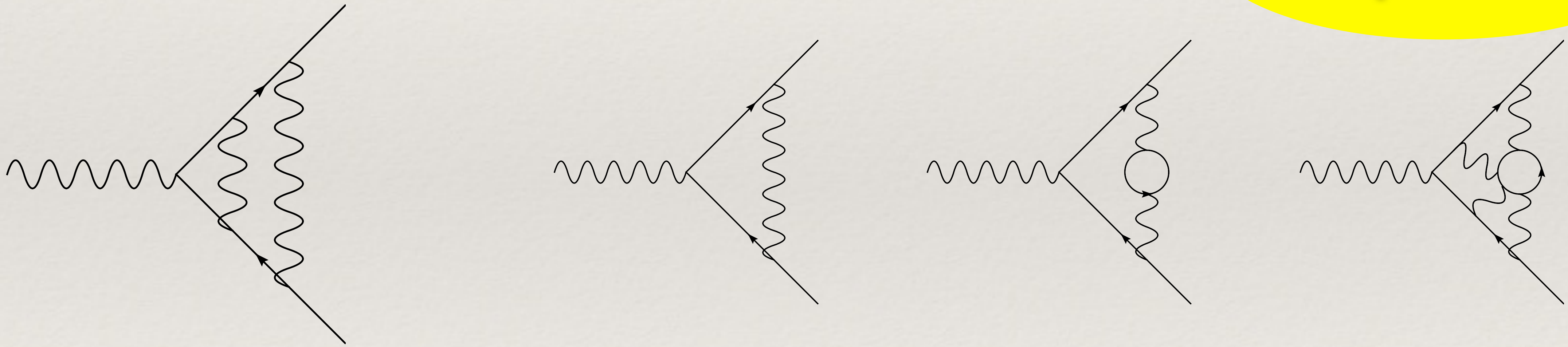
$$f = e^{i\int dt(\frac{1}{2}\dot{x}^2 + p\cdot A + ..)}$$

Write scattering amplitude as first-quantized path integral

$$M(p_1, p_2, \{k\}) = \int \mathcal{D}A_s\, \mathcal{D}x(t)\, H[x]\, f_1[A_s, x(t)]\, f_2[A_s, x(t)]\, e^{iS[A_s]}$$

x(t): path of charged particle

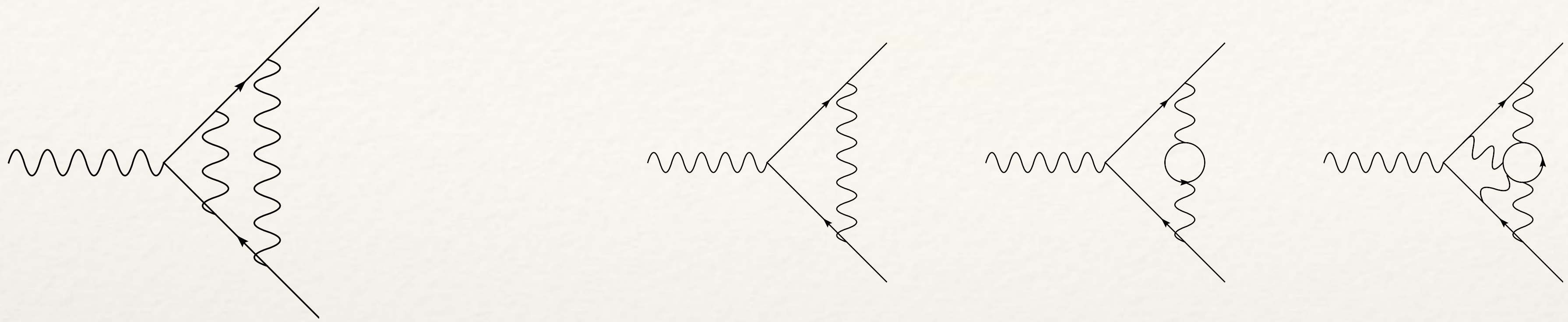
Eikonal vertices are sources for gauge bosons along line



Disconnected

Connected

Path integral method, non-abelian



- Not immediately obvious how this could work (the path integral must be a real exponential), since
  - Source terms have non-abelian charges, so don't commute
  - External line factors are path-ordered exponentials
  - Nevertheless

Gatheral; Frenkel, Taylor; Sterman

$$\sum_D \mathcal{F}_D C_D = \exp\left[\sum_i \bar{C}_i w_i\right]$$

- To prove, use replica trick (from statistical physics)

Replica trick

EL, Stavenga, White

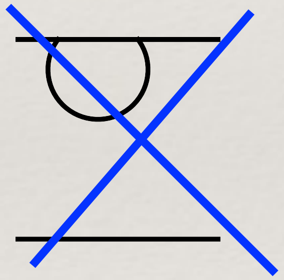
- Relates exponentiation of soft gauge fields to that of connected diagrams in QFT.
- Consider a N copies of a scalar theory

  - If Z is exponential, find out what contributes to log Z

$$Z[J]^N = \int \mathcal{D}\phi_1 \ldots \mathcal{D}\phi_N e^{iS[\phi_1]+..+iS[\phi_N]+J\phi_1+..J\phi_N}$$

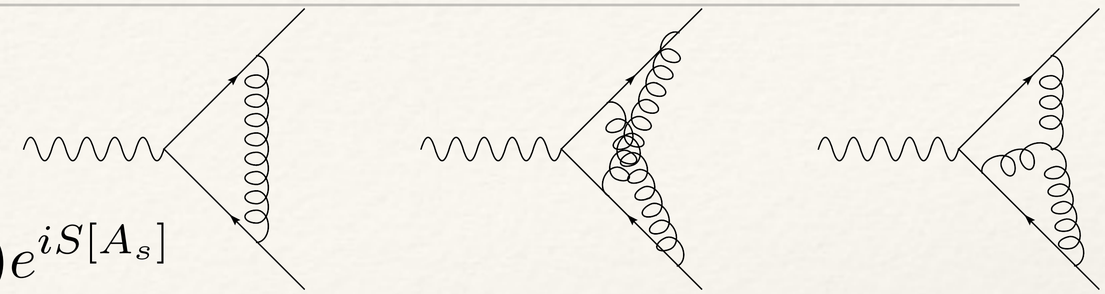
  - Amounts to diagrams that allow only one replica $\longrightarrow$ connected!

$$Z^N = 1 + N\log Z + \mathcal{O}(N^2)$$



Application to QCD

Amplitude for two colored lines

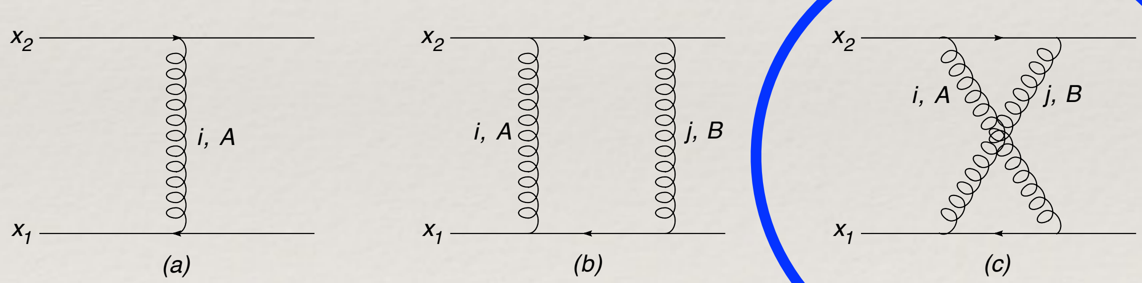


$$S(p_1,p_2) = H(p_1,p_2)\int \mathcal{D}A_s f(\infty) e^{iS[A_s]}$$

Replicate, and introduce replica ordering operator R

$$f(\infty) = \mathcal{P}\exp\left[\int dx\cdot A(x)\right] \qquad \prod_{i=1}^{N}\mathcal{P}\exp\left[\int dx\cdot A_i(x)\right] = \mathcal{R}\mathcal{P}\exp\left[\sum_{i=1}^{N}\int dx\cdot A_i(x)\right]$$

Look for diagrams of replica multiplicity N. These will go into exponent



Web
Modified color factor

(a) is order N

(b) for equal replica number (i=j): C_F^2. For i≠j also C_F^2. Sum: $NC_F^2 + N(N-1)C_F^2 = N^2C_F^2$

(c) for equal replica number (i=j): C_F^2-C_F C_A /2.
For i≠j C_F^2. Term linear in N: $N\left(C_F^2 - \frac{C_F C_A}{2}\right) + (-N)C_F^2 = N\left(-\frac{C_F C_A}{2}\right)$

Multiple colored lines

- Structure

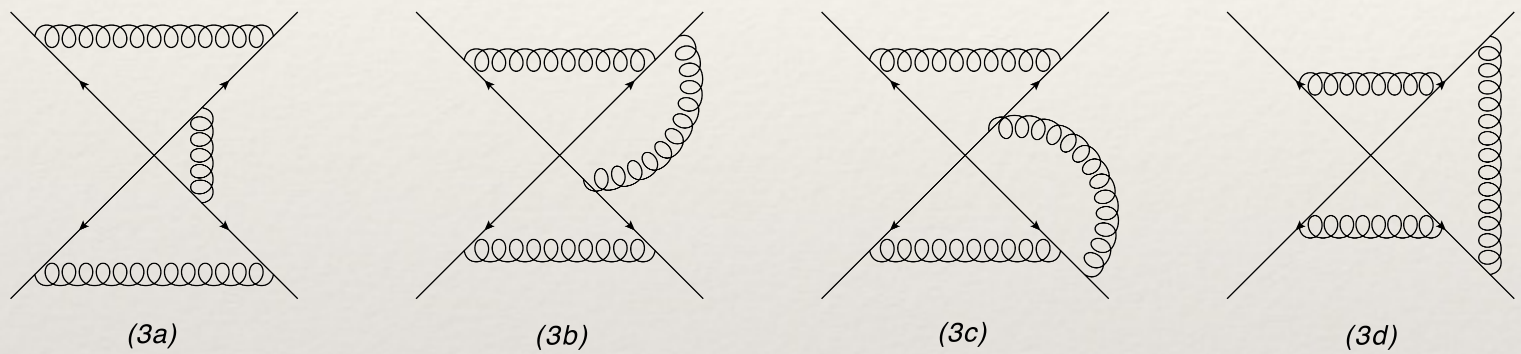
Projector matrix

$$\sum \mathcal{F}(D)C(D) = \exp[\sum_{d,d'} \mathcal{F}(d)R_{dd'}C(d')]$$

$$\sum_{d'} R_{dd'} = 0$$

Eigenvalues 0 or 1

  - multi-parton webs are "closed sets" of diagrams, with modified color factors



= Multiparton Web

- Closed form solution for modified color factor

$$\frac{1}{6}\Big[C(3a) - C(3b) - C(3c) + C(3d)\Big] \times \Big[M(3a) - 2M(3b) - 2M(3c) + M(3d)\Big]$$

  - Interesting properties of projector matrix (reduces degree of divergence)

Projector matrix

$$\sum \mathcal{F}(D)C(D) = \exp[\sum_{d,d'} \mathcal{F}(d) R_{dd'} C(d')]$$

Gardi, White

- Projects out contributions that come from exponentiation of lower order diagrams
  - ✓ Interesting combinatorial aspects (Stirling numbers)
  - ✓ Proof of idempotency and zero sum row property
- Combinatorics involves quite interesting for mathematicians

How to resum?

- There are many ways, depending on
  - the observable
  - the logarithm
  - the resummer
- Here we take as key notions
  - factorization
  - approximations for kinematic limit (eikonal approximation e.g.)

Resummation 101

- Cross section for n extra gluons

Phase space measure **Squared matrix element**

$$\sigma(n) = \frac{1}{2s}\int d\Phi_{n+1}(P, k_1, \ldots, k_n) \times |\mathcal{M}(P, k_1, \ldots, k_n)|^2$$

- When emissions are soft, can factorize phase space measure and matrix element [eikonal approximation]

$$d\Phi_{n+1}(P, k_1, \ldots, k_n) \longrightarrow d\Phi(P) \times \Big(d\Phi_1(k)\Big)^n \frac{1}{n!}$$

- Sum over all orders

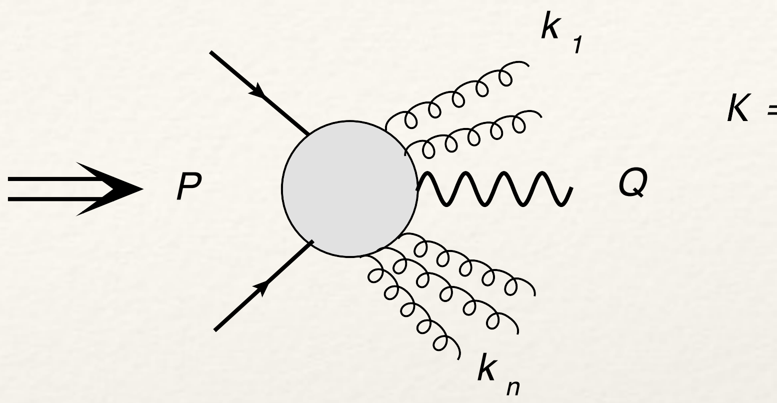
$$|\mathcal{M}(P, k_1, \ldots, k_n)|^2 \longrightarrow |\mathcal{M}(P)|^2 \times \Big(|\mathcal{M}_{1\ \text{emission}}(k)|^2\Big)^n$$

$$\sum_n \sigma(n) = \sigma(0) \times \exp\left[\int d\Phi_1(k) |\mathcal{M}_{1\ \text{emission}}(k)|^2\right]$$

- Incorporate Theta or Delta functions in space space
  - but these must factorize similarly, or they cannot go into exponent

Phase space in resummation

- Kinematic condition expresses "z" in terms of gluon energies



$$s = Q^2 - 2P\cdot K - K^2 \qquad \delta\Big(1 - \frac{Q^2}{s} - \sum_i \frac{2k_i^0}{\sqrt{s}}\Big)$$

  - or conservation of transverse momentum

$$\delta^2(Q_T - \sum_i p_T^i)$$

- Transform (e.g. Laplace/Mellin or Fourier) factorizes the phase space

$$\int_0^\infty dw\, e^{-w\,N}\delta\Big(w - \sum_i w_i\Big) = \prod_i \exp(-w_i N) \qquad \int d^2Q_T\, e^{ib\cdot Q_T}\,\delta^2(Q_T - \sum_i p_T^i) = \prod_i e^{ib\cdot p_T^i}$$

- So can go into exponent

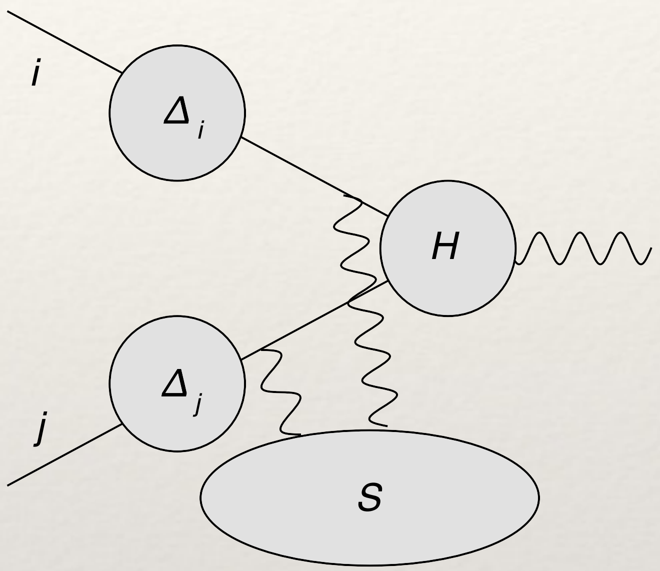
$$\sum_n \sigma(n) = \sigma(0) \times \exp\Big[\int d\Phi_1(k)|\mathcal{M}_{1\ \mathrm{emission}}(k)|^2(\exp(-w\,N) - 1)\Big]$$

  - Large logs: ln(N) or ln(bQ)

Factorization and resummation for Drell-Yan

$$\sigma(N) = \Delta(N,\mu,\xi_1)\Delta(N,\mu,\xi_2)S(N,\mu,\xi_1,\xi_2)H(\mu)$$

- Near threshold, cross section is equivalent to product of 4 well-defined functions
- Demand independence of
  - renormalization scale μ
  - gauge dependence parameter ξ
    - ✓ find exponent of double logarithm



$$0 = \mu\frac{d}{d\mu}\sigma(N) = \xi_1\frac{d}{d\xi_1}\sigma(N) = \xi_2\frac{d}{d\xi_2}\sigma(N)$$

Contopanagos, EL, Sterman

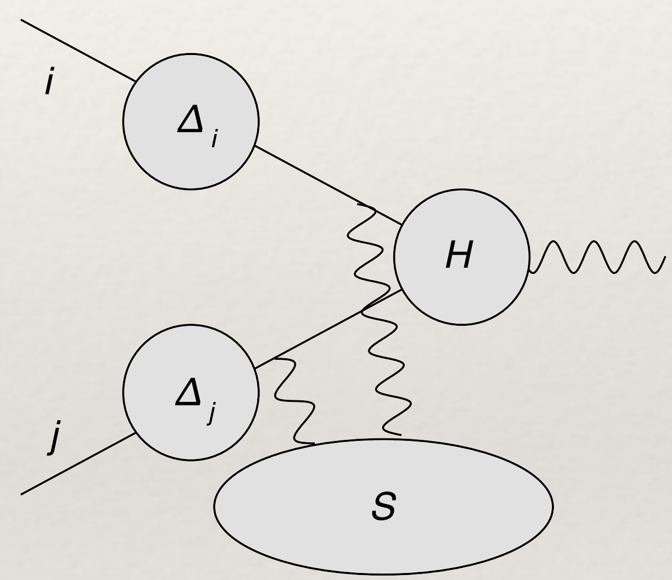
$$\Delta = \exp[\int\frac{d\mu}{\mu}\int\frac{d\xi}{\xi}..]$$

Factorization for threshold resummation

- $\Delta_i(N)$: initial state soft+collinear radiation effects
  - real+virtual
  - $\alpha_s^n \ln^{2n} N$

$$\sigma(N) = \sum_{ij} \phi_i(N)\phi_j(N) \times \underbrace{\left[\Delta_i(N)\Delta_j(N)S_{ij}(N)\, H_{ij}\right]}_{\hat{\sigma}_{ij}(N)}$$

- $S_{ij}(N)$: soft, non-collinear radiation effects
  - $\alpha_s^n \ln^n N$
- H: hard function, no soft and collinear effects



$$\Delta_i(N) = \exp\left[\ln N \frac{C_F}{2\pi b_0 \lambda}\{2\lambda + (1-2\lambda)\ln(1-2\lambda)\} + ..\right]$$
$$= \exp\left[\frac{2\alpha_s C_F}{\pi}\ln^2 N + ..\right]$$

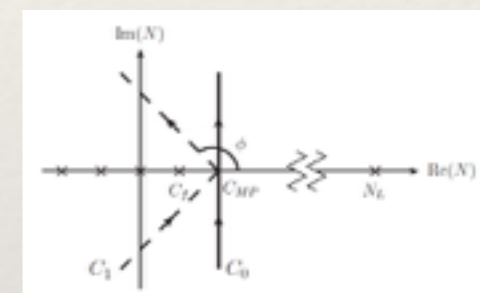
From N space back to momentum-space

- Parton cross section derived in N space
- PDF's in N space
  - Use initial conditions in N-space, then QCD-PEGASUS evolution (A. Vogt)

$$\sigma^{(\mathrm{res})}_{h_1h_2\to kl}(\rho^2,\{m^2\},\mu_R^2,\mu_F^2) = \frac{1}{\pi}\int_0^\infty dy\,\mathrm{Im}\,[e^{i\phi}\,\rho^{-C_{\mathrm{MP}}-ye^{i\phi}} \times \sigma^{(\mathrm{res})}_{h_1h_2\to kl}(N=C_{\mathrm{MP}}+ye^{i\phi},\{m^2\},\mu_R^2,\mu_F^2)\,]$$

- Use inverse Mellin transform
  - Avoid Landau pole singularity with Minimal Prescription (go left..)
    - ✓ gives Good numerical stability

Catani, Mangano
Nason, Trentadue



- Exercise:
  - function $f(x) = x^p$
  - Melling transform $f(N) = \int_0^1 dx\, x^{N-1} x^p = \frac{1}{N+p}$
  - Inverse Mellin transform $f(x) = \frac{1}{2\pi i}\int dN x^{-N}\frac{1}{N+p} = x^p$
    - ✓ Correct!

Resummed Drell-Yan/Higgs cross section

Sterman; Catani, Trentadue

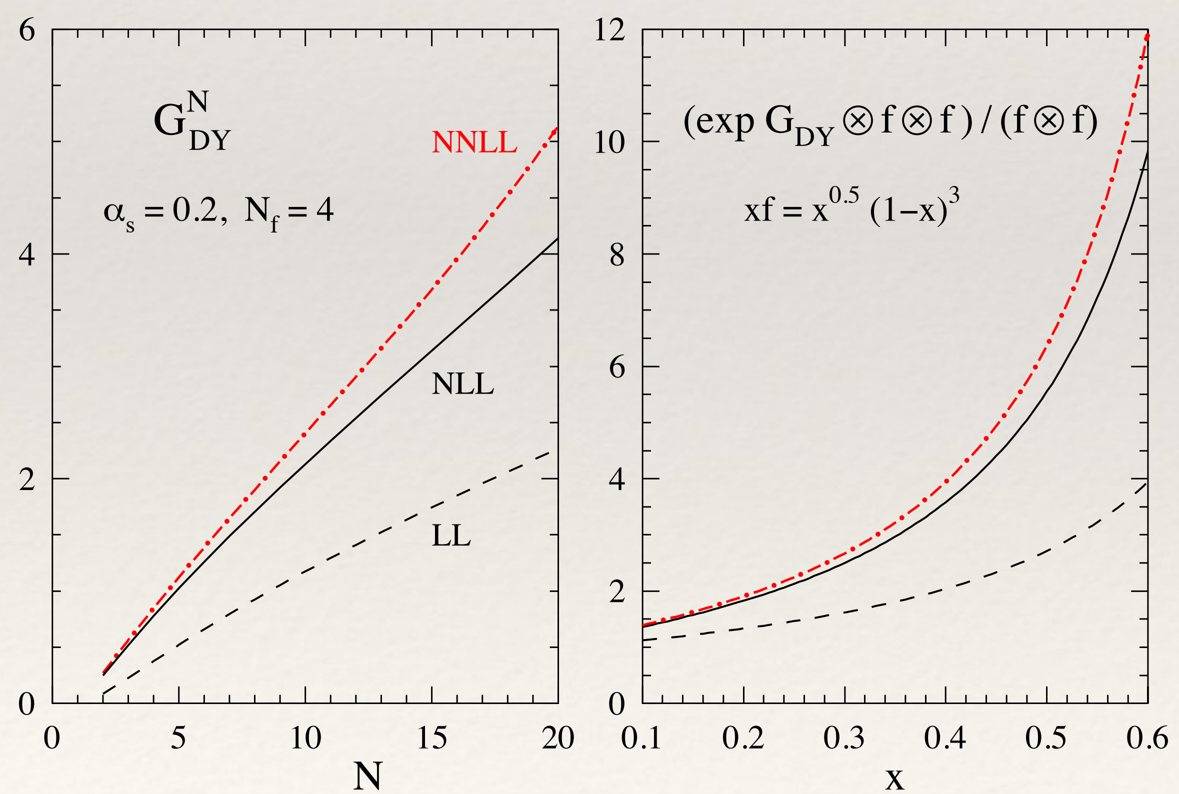
Threshold-resummed Drell-Yan cross section

Functions in exponent depend only on coupling

$$
\begin{aligned}
\frac{d\sigma^{\text{resum}}}{dQ^2}(z) &= \int_C \frac{dN}{2\pi\, i}\, z^{-N}\, \hat{\sigma}(N) \\
\sigma(N) &= \exp\left[-\int_0^1 dx\, \frac{x^{N-1}-1}{1-x}\left\{\int_{Q^2}^{Q^2(1-x)^2}\frac{d\mu}{\mu}A(\alpha_s(\mu))\right.\right. \\
&\quad \left.\left. +D(\alpha_s((1-x)Q))\right\}\right] \times \left(1+\alpha_s(Q^2)\frac{C_F}{\pi}+\ldots\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{\sigma}_{DY}(N,Q^2) &= g_0(Q^2)\,\exp\left[G^N_{DY}(Q^2)\right] \\
G^N_{DY} &= \ln N g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \ldots, \qquad \lambda = \beta_0\alpha_s \ln N
\end{aligned}
$$

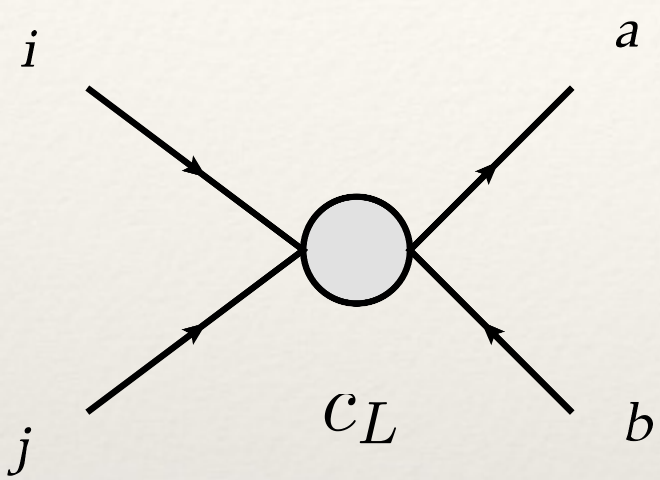
A. Vogt



Good convergence in exponent

More color: 2 → 2 parton scattering

- Four external partons can connect in multiple ways



For example: $q(i)\bar{q}(j) \to q'(a)\bar{q}'(b)$ L=1,2

$\delta_{ij}\delta_{ab}$ $T^c_{ij}T^c_{ab}$

- For gg -> gg, (at least) 6 ways.
  - (Different basis choices possible in this space of color tensors)

Colorful 2 → 2 scattering

- Factorization by "usual" methods into Δ, S, H functions

Kidonakis, Oderda, Sterman;

  - Δ's color diagonal (collinear quarks and gluons)
  - Soft emissions mix the color tensors, and the effective vertices H

- Represent scattering amplitude as vector in color tensor space

$$M_{\{\alpha_i\}}(\frac{p_i}{\mu}, \alpha_s(\mu), \epsilon) = M_L(..)(c_L)_{\{\alpha_i\}}$$

$$M_L(..) = S_{LK} H_K \times \Delta\Delta$$

Soft anomalous dimensions

Kidonakis, Oderda, Sterman

- Define soft amplitude as VEV of Wilson lines with velocities β_i
  - represent external particles

$$\mathbf{S} = \langle 0| \prod_i \Phi_{\beta_i}(\infty, 0)_{\alpha_i \eta_i} |0\rangle c_{K,\eta_i}$$

- Wilson line composite operator has anomalous dimension

$$\mu \frac{d}{d\mu} \mathbf{S} = \mathbf{\Gamma_S}\, \mathbf{S}$$

- Soft function is square of amplitude, at fixed energy, depends on ratio (Q/Nμ), so can control N dependence through μ dependence
  - To do resummation beyond LL, need to understand soft anomalous dimensions

Soft anomalous dimensions

- For two lines (Drell-Yan, DIS, Higgs), aka cusp anomalous dimension, known to 3 loops

$$\mathbf{s} = \langle 0| \prod_i \Phi_{\beta_i}(\infty, 0)_{\alpha_i \eta_i} |0\rangle (c_K)_{\eta_i}$$

Moch, Vermaseren, Vogt; Berger

$$\left(\frac{\alpha_s}{\pi}\right) C_F + \left(\frac{\alpha_s}{\pi}\right)^2 C_F \underbrace{\left(C_A[\frac{67}{18} - \zeta_2] - C_F n_F \frac{5}{9}\right)}_{K} + \ldots$$

Aybat, Dixon, Sterman, Mitov, Czakon

- For 2 →2 one loop Γ is a matrix (known to 2 loops)
  - depends only on velocities and color states of external lines
  - for squark and gluino production some new possibilities with respect to Standard Model

$$T(m) = \log\left(\frac{m^2 - t}{\sqrt{s m^2}}\right) - \frac{1 - i\pi}{2} \quad \text{and} \quad U(m) = \log\left(\frac{m^2 - u}{\sqrt{s m^2}}\right)$$

$$\bar{\Gamma}_{qg\to\tilde{q}\tilde{g}} = \frac{\alpha_s}{2\pi} \begin{pmatrix} \bar{\Gamma}_{11,\,qg} & \frac{4N_C^2(N_C-2)}{(N_C^2-1)(N_C-1)}\,\Omega & \frac{4N_C^2(N_C+2)}{(N_C^2-1)(N_C+1)}\,\Omega \\ \frac{1}{2}\,\Omega & \bar{\Gamma}_{22,\,qg} & \frac{N_C(N_C+2)}{N_C+1}\,\Omega \\ \frac{1}{2}\,\Omega & \frac{N_C(N_C-2)}{N_C-1}\,\Omega & \bar{\Gamma}_{33,\,qg} \end{pmatrix},$$

$$\Lambda \equiv \frac{1}{2}\left[T(m_3) + T(m_4) + U(m_3) + U(m_4)\right],$$

$$\Omega \equiv \frac{1}{2}\left[T(m_3) + T(m_4) - U(m_3) - U(m_4)\right],$$

Soft anomalous dimension

- Matrices become diagonal in β⟶0 limit

$$\lim_{\beta\to 0} \bar{S}_{IJ}\big(Q/(N\mu),\mu^2\big) \;=\; \delta_{IJ}\, S^{(0)}_{IJ}\, \Delta^{(\mathrm{s})}_I\big(Q/(N\mu),\mu^2\big)$$

$$\Delta^{(\mathrm{s})}_I\big(Q/(N\mu),\mu^2\big) \;=\; \exp\Big[\int_{\mu}^{Q/N} \frac{dq}{q}\,\frac{\alpha_{\mathrm{s}}(q)}{\pi}\, D_I\Big],$$

  - even true for pT distributions
  - but not true for double-differential distributions
  - for squarks and gluinos e.g.

Kulesza, Motyka

$$\{D_{qq\to\tilde{q}\tilde{q},I}\} \;=\; \{-4/3, -10/3\}$$

$$\{D_{qg\to\tilde{q}\tilde{g},I}\} \;=\; \{-4/3, -10/3, -16/3\}$$

Soft Collinear Effective Theory

Bauer, Fleming, Pirjol, Stewart,...

- Previous "(d)QCD" analysis was essentially diagram based
- Effective field theory approach: SCET
  - Distinguish separate **fields** for soft, collinear, hard partons, and ultrasoft gluons

$$\mathcal{L}_{SCET,qq} = \bar{\xi}_n (in\cdot D + i\not{D}_{c,\perp}\frac{1}{i\bar{n}\cdot D_c} i\not{D}_{c,\perp})\frac{\not{\bar{n}}}{2}\xi_n - \frac{1}{4}\mathrm{Tr}\{G^c_{\mu\nu}G^{c,\mu\nu}\}$$

    - ✓ Powerful power counting. Using +,-,T notation

$$p_h \sim Q(1,1,1) \quad p_c \sim Q(\lambda, 1, \sqrt{\lambda}) \quad p_s \sim Q(\lambda,\lambda,\lambda)$$

    - ✓ Fields scale similarly:

$$\xi_n \sim \lambda \quad \xi_{\bar{n}} \sim \lambda^2 \quad A_s \sim \lambda \quad \bar{n}\cdot A_c \sim \lambda^0$$

  - 2 gauge transformations, collinear and ultrasoft
    - ✓ and two types of Wilson lines: $W_c(x)$ $\quad S_n(x)$

Soft Collinear Effective Theory

Bauer, Fleming, Pirjol, Stewart,...

- Decouple soft gluons from collinear via field redefinition $\xi_n(x) \to S_n(x)\xi_n^{(0)}(x)$

$$\bar{\xi}\frac{\bar{\not{n}}}{2}n\cdot D\,\xi \to \bar{\xi}^{(0)}\frac{\bar{\not{n}}}{2}n\cdot D_c\,\xi^{(0)}$$

  - Soft gluons do not of course fully disappear from every observable
  - Can form soft functions (matrix elements of soft Wilson lines)
- Resummation: match and run
  - Write observable (e.g. σ_{DY}) as

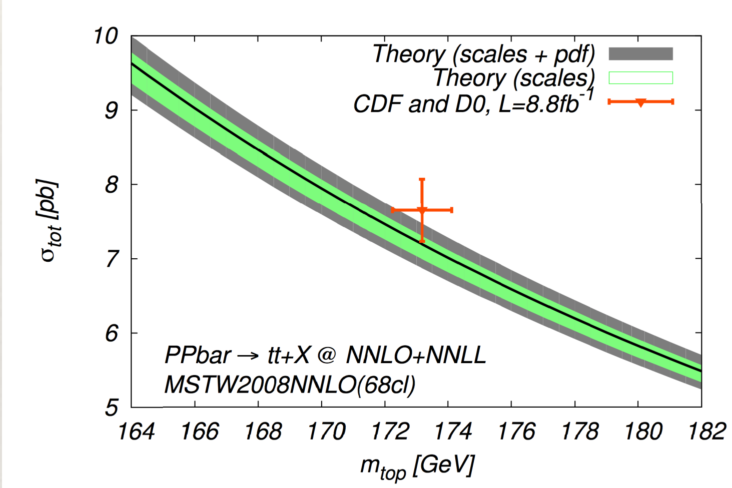
$$\langle O_{QCD}\rangle \qquad \text{and as} \qquad \prod_i \langle O^i_{SCET}\rangle \times C^i_{\mathrm{match}}$$

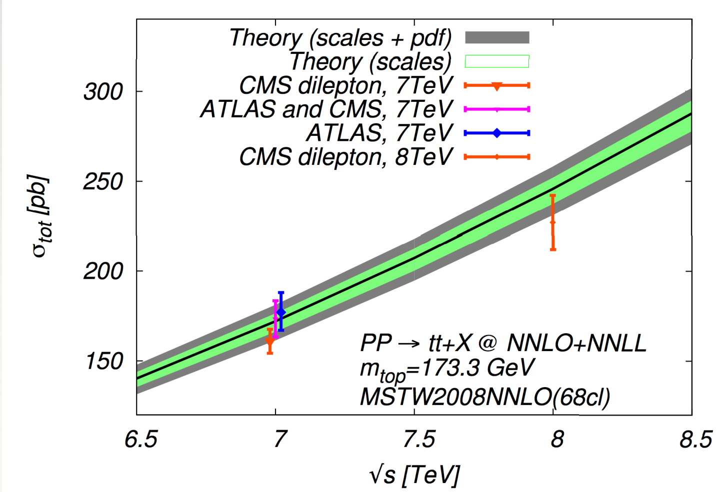
  - Solve RG equations for O^i_{SCET}
  - Find C by 1-loop (or 2-loop) calculations on both sides
- Powerful method

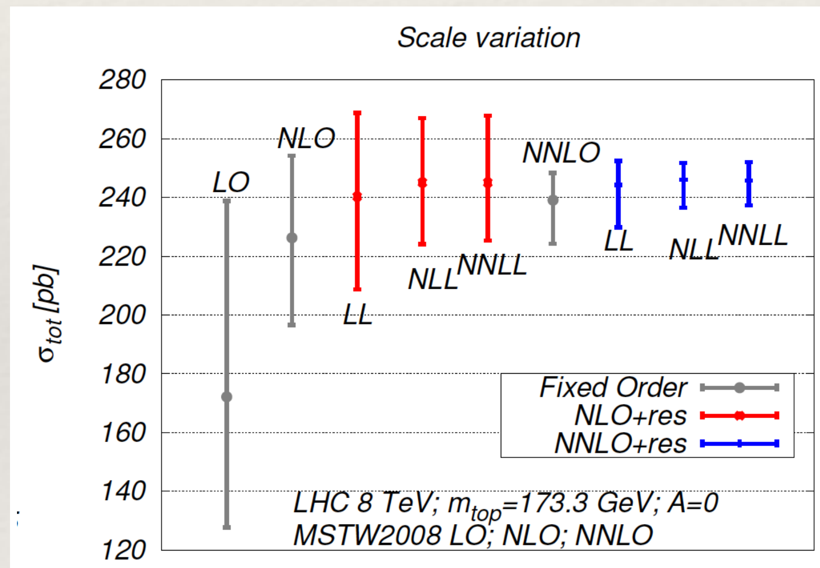
NNLO-NNLL inclusive cross section

Baernreuther, Fiedler, Mitov, Czakon

- A milestone in QCD, with clear benefits
  - precision top physics is here
  - new calculational methods developed
  - use for gluon density at large x, and α_s Czakon, Mitov, Mangano, Rojo







Concurrent uncertainties:

Scales	~ 3%
pdf (at 68%cl)	~ 2-3%
α_S (parametric)	~ 1.5%
m_{top} (parametric)	~ 3%

Soft gluon resummation makes a difference

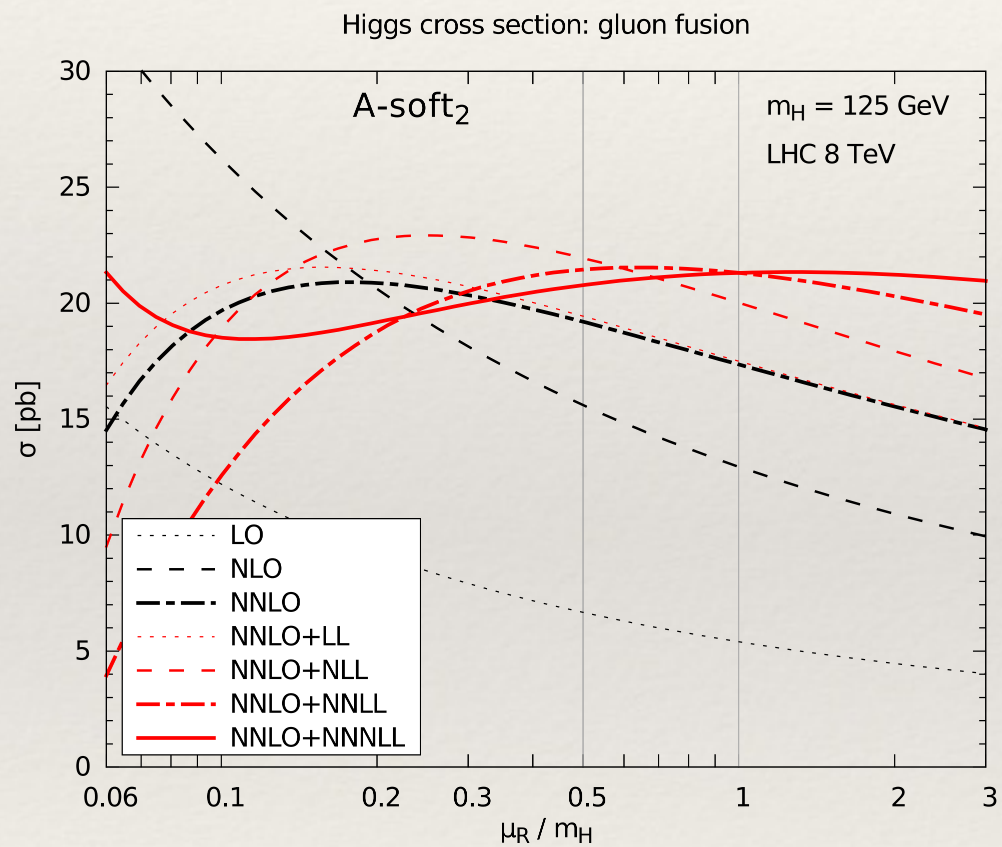
5% -> 3%

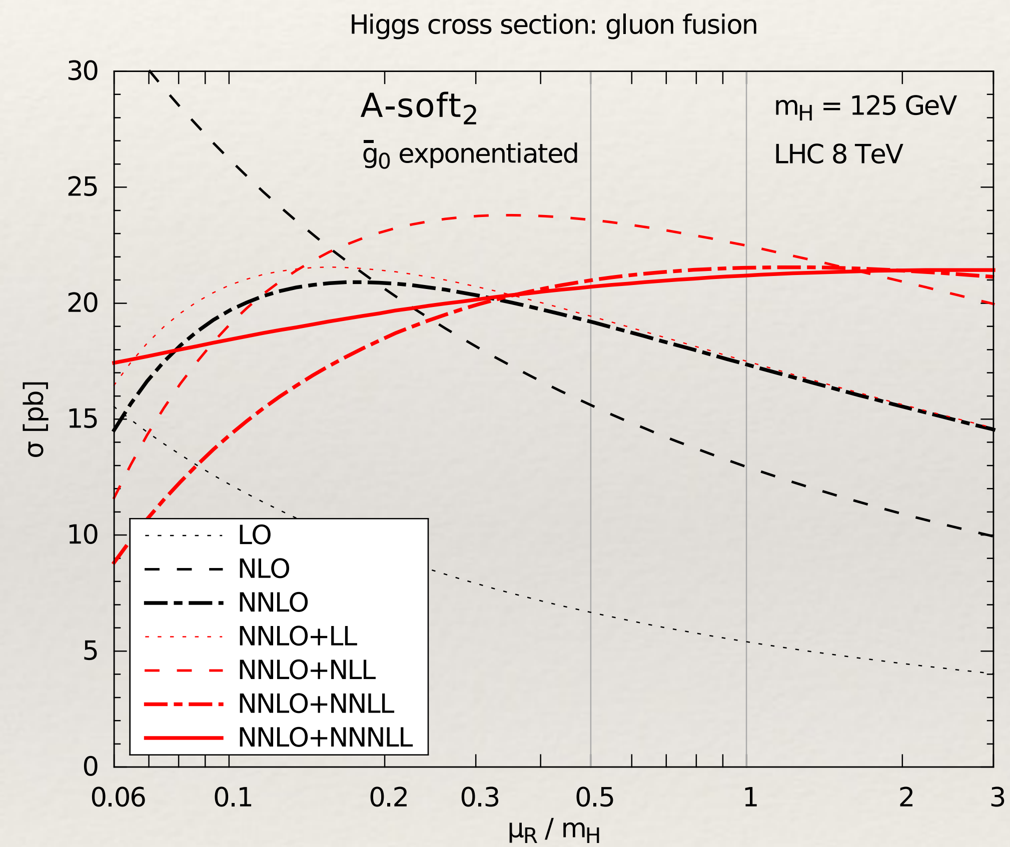
N^3LL resummation for Higgs production

- Logarithm is again threshold logarithm

Bonvini, Marzani

  - For inverse Mellin transform, employ both Minimal Prescription and Borel prescription
  - Nice progression, especially with exponentiated constants





  - Code: ResHiggs and ggHigs

Resummation vs parton shower

- Both account for emission to all orders in perturbative QCD. It’s accuracy vs flexibility
  - Resummation: a formula
    - ✓ accuracy to LL, NLL, NNLL depending on what the theorists did. For specific observables
  - Parton shower: generate events
    - ✓ very flexible, can use for any observables
    - ✓ but, on the downside, in essence only LL accuracte (it never has all the NLL information in it, because that is to some extent observable dependent).
      - Progress is being made here however

Final summary

- Many concepts in perturbative QCD were discussed, in both their essence and some technical aspects
  - Qenormalization, asymptotic freedom
  - Finite orders, IR and COL divergence-handling
  - All-orders: resummation, why and how
    - ✓ here there is quite a bit of physics insight possible
- My hope: that when you see such concepts in workshops or talks, you now have a sense about what this is about.
    - ✓ and ask about it!