Quantum Entanglement
and
Local Excitations

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NORDITA

HMSCS, GGI, 12/03/2015

Based on :

- "Entanglement of local operators in large-N conformal field theories"
  with Masahiro Nozaki, Tadashi Takayanagi
  PTEP 2014 (2014) 9, 093B06

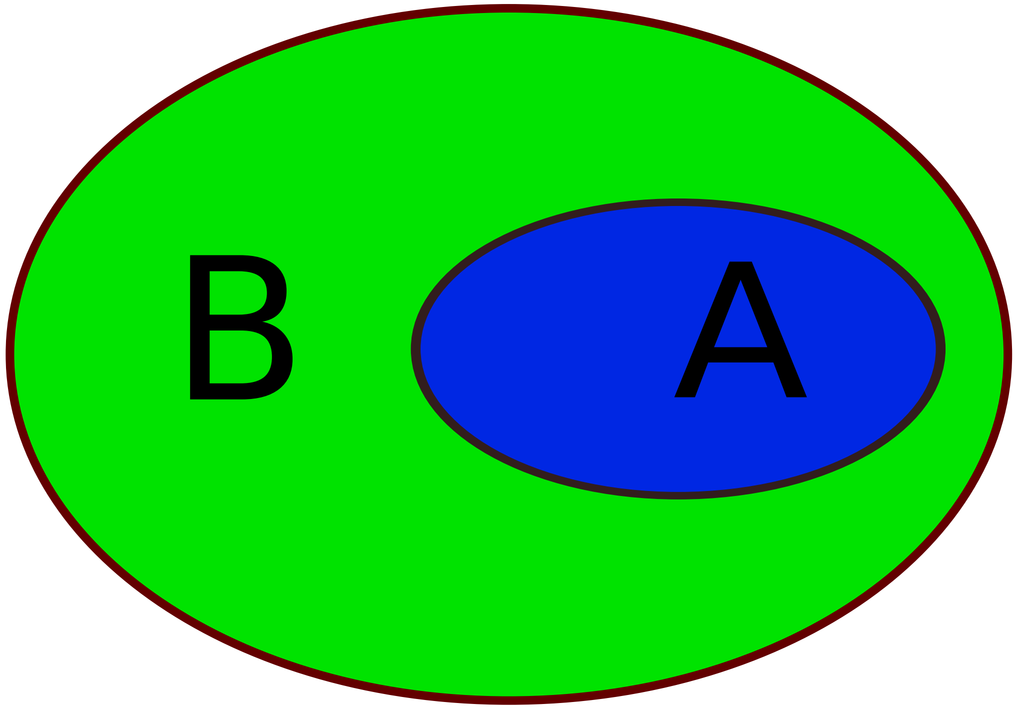
- "Quantum Entanglement of Localised Excited States at Finite Temperature"
  with Joan Simon, Andrius Stikonas, Tadashi Takayanagi
  JHEP 1501 (2015) 102

- "To appear…"
  with Joan Simon, Andrius Stikonas, Tadashi Takayanagi, Kento Watanabe

Entanglement Renyi Entropies

$$\rho = |\psi\rangle \langle\psi|$$

$$\rho_A = Tr_B \rho$$



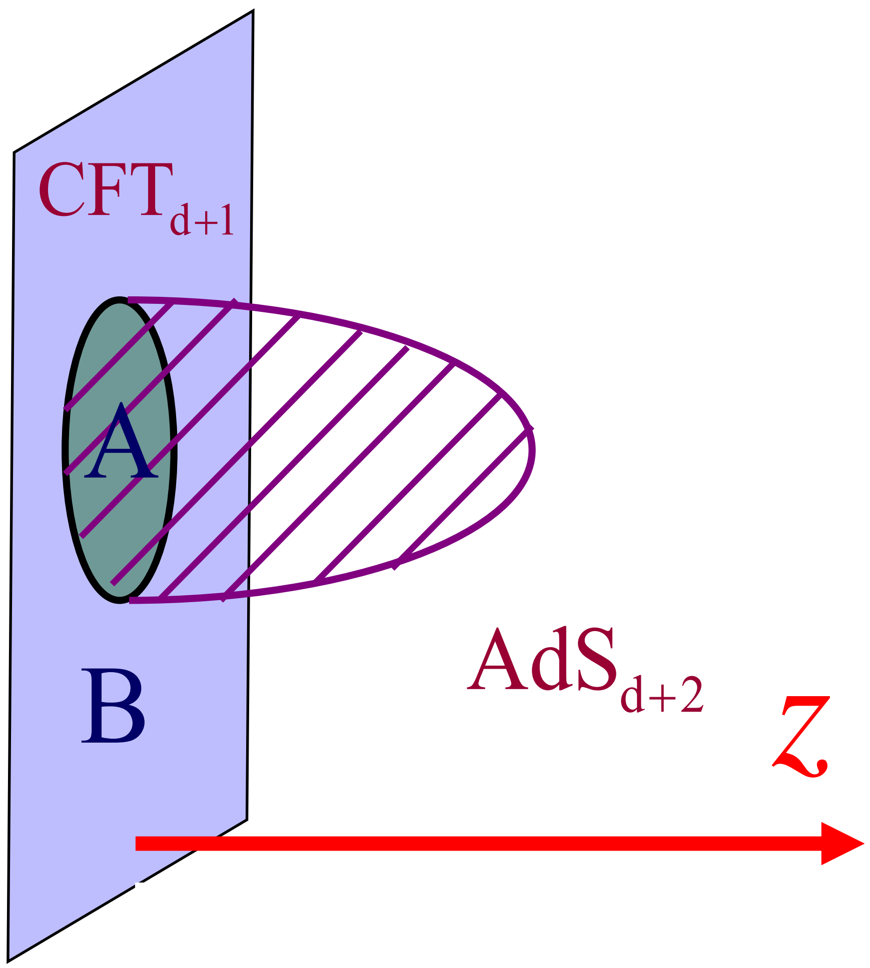
Renyi Entropies

$$S_A^{(n)} = \frac{1}{1-n} \ln Tr(\rho_A^n)$$

von-Neumann

$$S_A^{(1)} = -Tr(\rho_A \ln \rho_A)$$

Entanglement Entropy in AdS/CFT



[Ryu,Takayanagi'06]

$$S_A = \frac{\mathrm{Area}(\gamma_A^d)}{4G_N^{d+2}}$$

[Hubeny,Rangamani,Takayanagi'07]

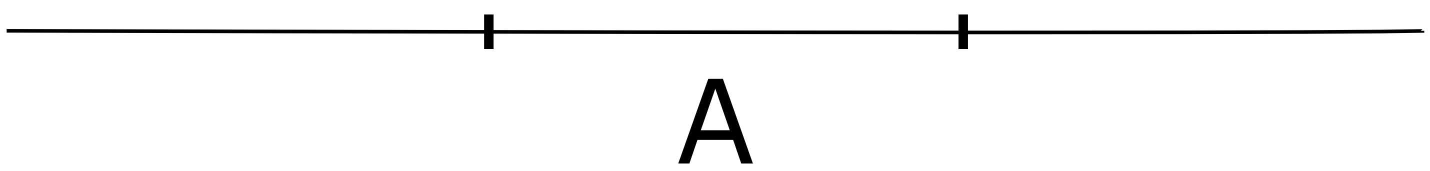
Covariant

Disconnected regions (Mutual Information)

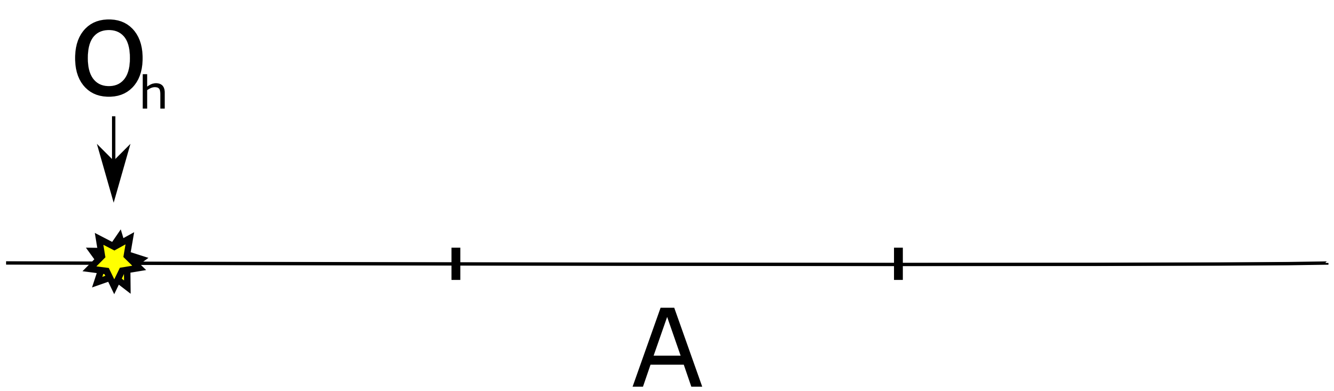
$$I_{A:B} = S_A + S_B - S_{A\cup B}$$

Question: CFT in 1+1d

[see Cardy,Calabrese…]



$$S_A \sim \frac{c}{3} \log \frac{|A|}{\varepsilon}$$



$$\rho(t) = e^{-iHt} O(x) \left|0\right\rangle \left\langle 0\right| O^\dagger(x) e^{iHt}$$

$$S_A(t)\,?$$

Motivation (CFT):

Characterise operators from the perspective of quantum entanglement

Motivation (AdS/CFT):

This Talk: Modest step towards this…

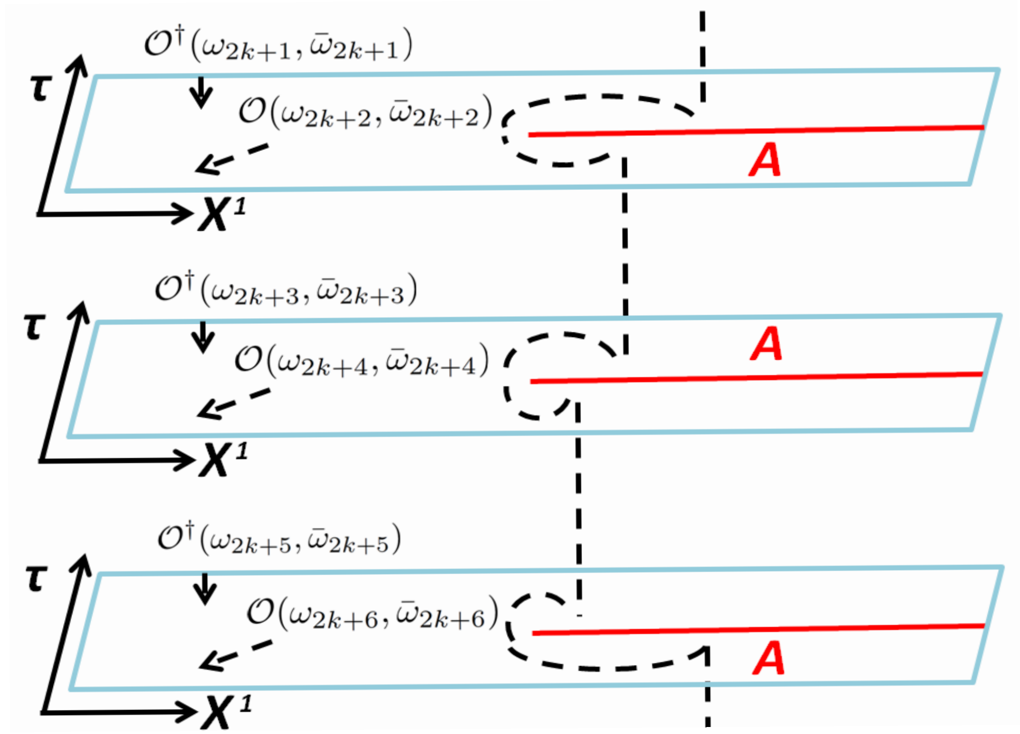
Plan

- Entanglement and locally exited states
- Large c limit and AdS/CFT
- Finite temperature
- Mutual information
- Scrambling time

Entanglement and locally exited states

$$\rho(t,x) = \mathcal{N} e^{-iHt} e^{-\epsilon H} O(x) \left|0\right\rangle \left\langle 0\right| O(x) e^{-\epsilon H} e^{iHt}$$

$$Tr(\rho_A^n)$$



$$\boxed{\Delta S_A^{(n)} = \frac{1}{1-n}\log\left(\frac{\mathrm{Tr}(\rho_A^n)}{\mathrm{Tr}(\rho_A^{(0)})^n}\right) = \frac{1}{1-n}\log\left[\frac{\langle O(w_1,\bar{w}_1)O^\dagger(w_2,\bar{w}_2)\ldots O(w_{2n},\bar{w}_{2n})\rangle_{\Sigma_n}}{(\langle O(w_1,\bar{w}_1)O^\dagger(w_2,\bar{w}_2)\rangle_{\Sigma_1})^n}\right]}$$

[Sierra et al.,'12] [Nozaki,Numasawa,Takayanagi,'13]

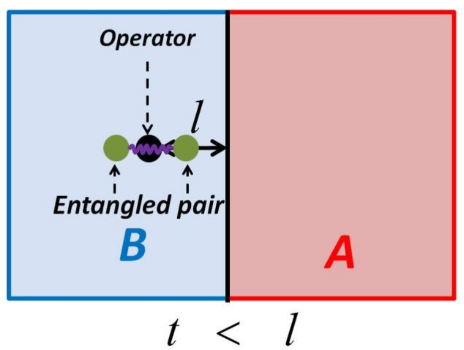
Rational CFT

[He,Numasawa,Takayanagi,Watanabe'14]

(n=2)

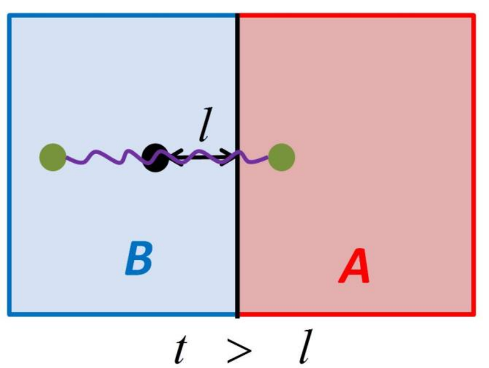
$$\frac{\langle O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)O(w_3,\bar{w}_3)O(w_4,\bar{w}_4)\rangle_{\Sigma_2}}{(\langle O(w_1,\bar{w}_1)O(w_2,\bar{w}_2)\rangle_{\Sigma_1})^2} = |z|^{2\Delta_O}|1-z|^{2\Delta_0}G_O(z,\bar{z})$$

At late time $(z,\bar{z}) \to (1,0)$

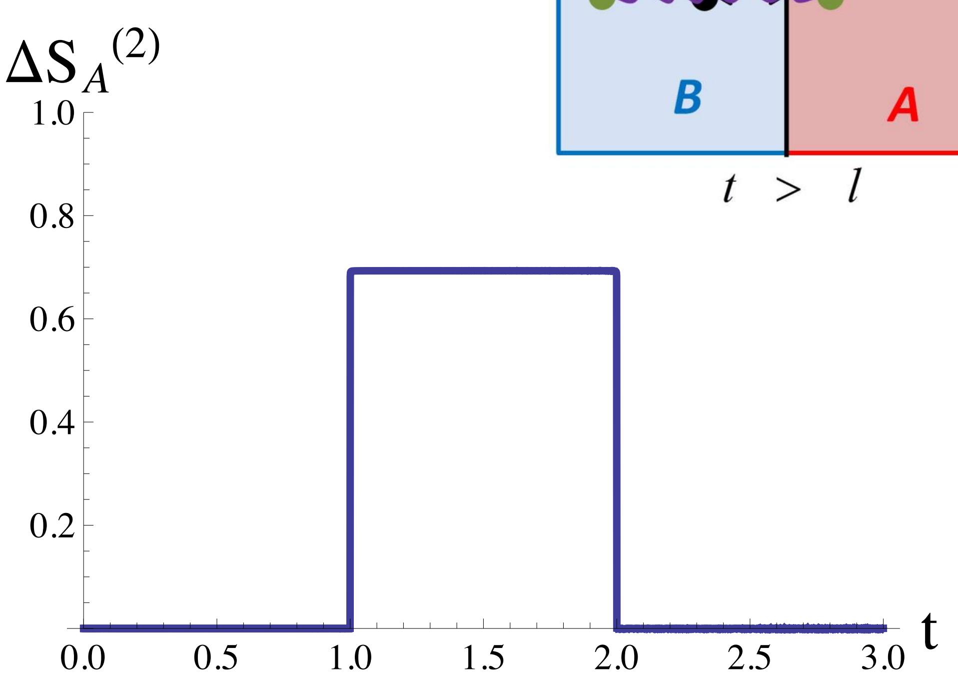


In rational CFTs

$$G(z,\bar{z}) \simeq F_{00}[O]\cdot(1-z)^{-2\Delta_O}\bar{z}^{-2\Delta_O}$$



$$\Delta S_A^{(2)} = -\log F_{00}[O] = \log d_O$$



$$d_\Delta = \frac{S_{0\Delta}}{S_{00}}$$ quantum dimension

"EPR pair propagating through the system"

Large c

[PC,M.Nozaki,T.Takayanagi14]

Conformal block expansion

$$G(z,\bar{z}) = \sum_b (C^b_{OO^\dagger})^2 F_O(b|z)\bar{F}_O(b|\bar{z})$$

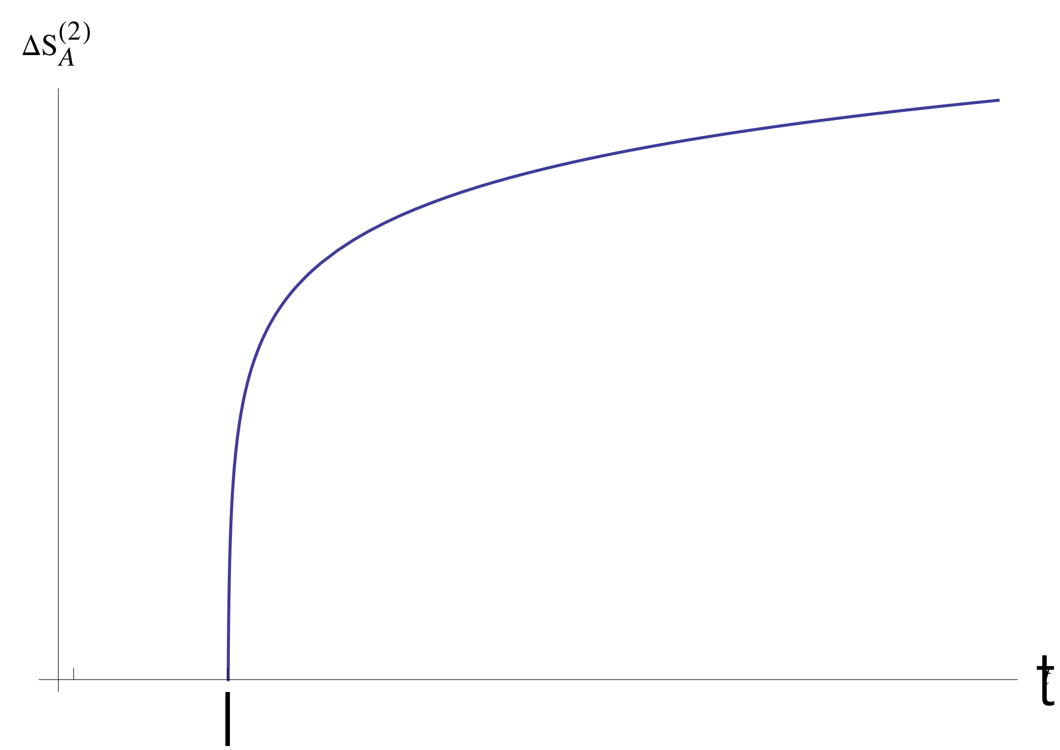
at large central charge c

[Fateev,Ribault'11]

$$F_O(b|z) \simeq z^{\Delta_b - 2\Delta_O} \cdot {}_2F_1(\Delta_b, \Delta_b, 2\Delta_b, z)$$

at late time

$$\Delta S_A^{(2)} \simeq 4\Delta_O \cdot \log \frac{2t}{\epsilon}$$

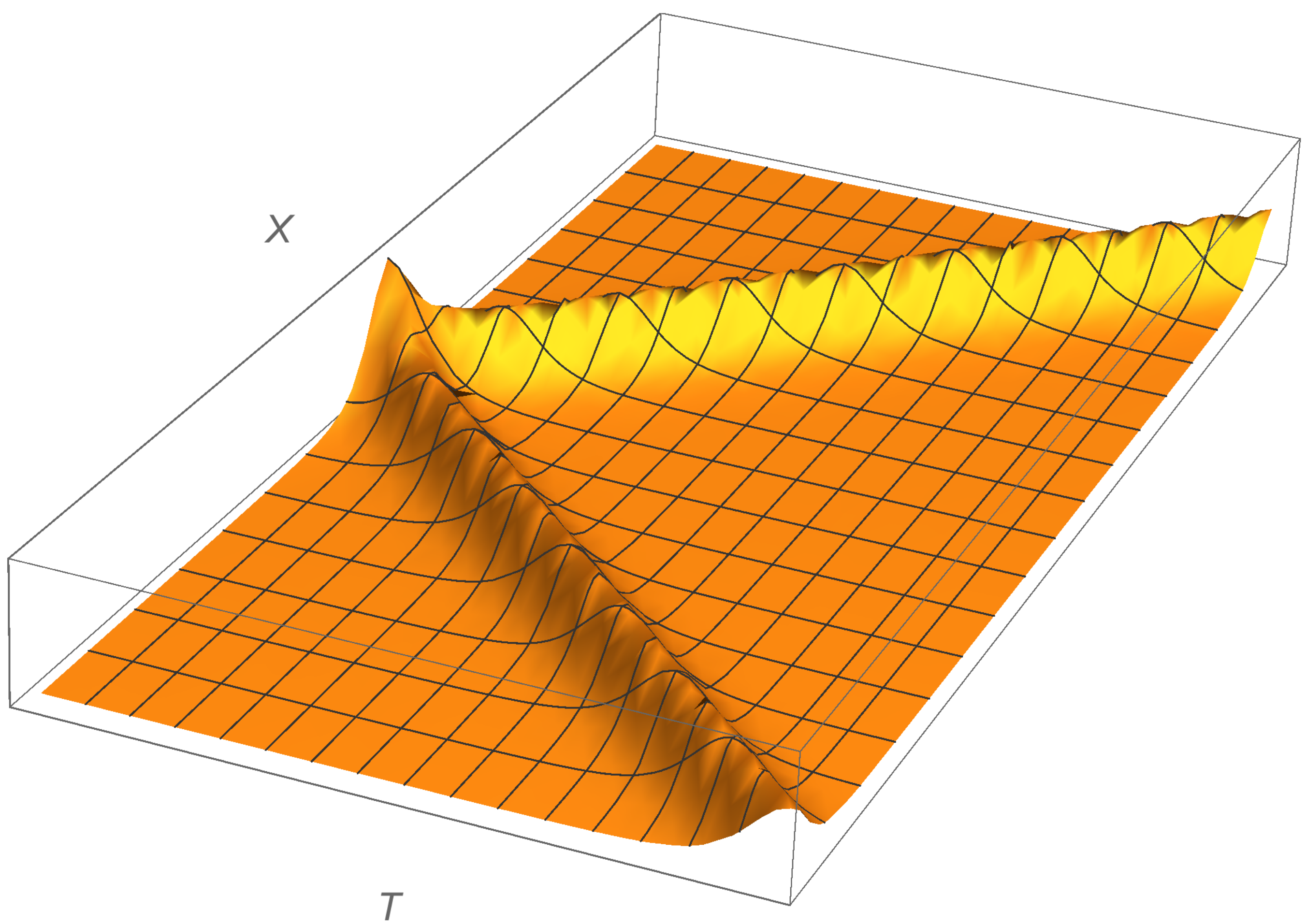


similar to a local quench

Energy density

$$\langle T_{tt} \rangle = \frac{\langle O^\dagger(w_2, \bar{w}_2) T_{tt}(x,x) O(w_1, \bar{w}_1) \rangle}{\langle O^\dagger(w_2, \bar{w}_2) O(w_1, \bar{w}_1) \rangle} = \Delta_O \epsilon^2 \left[\frac{1}{((x+l-t)^2+\epsilon^2)^2} + \frac{1}{((x+l+t)^2+\epsilon^2)^2} \right]$$

$$w_1 = i(\epsilon - it) - l, \quad w_2 = -i(\epsilon + it) - l,$$
$$\bar{w}_1 = -i(\epsilon - it) - l, \quad \bar{w}_2 = i(\epsilon + it) - l.$$

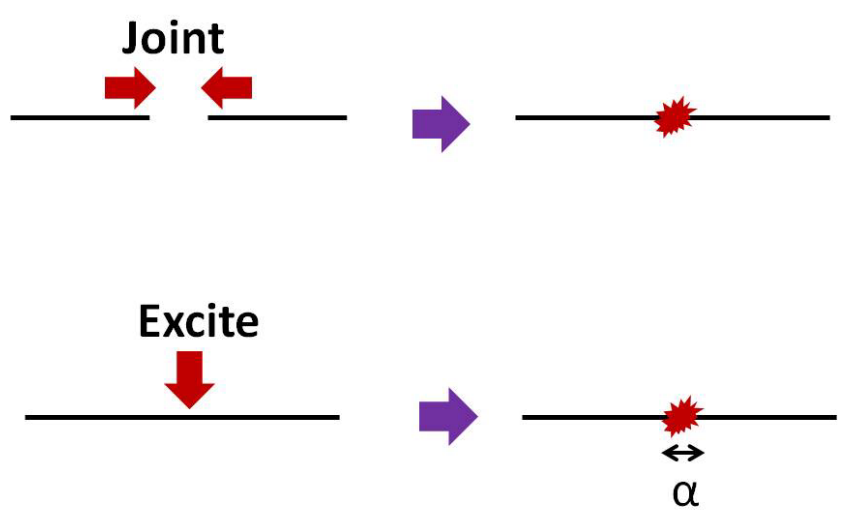


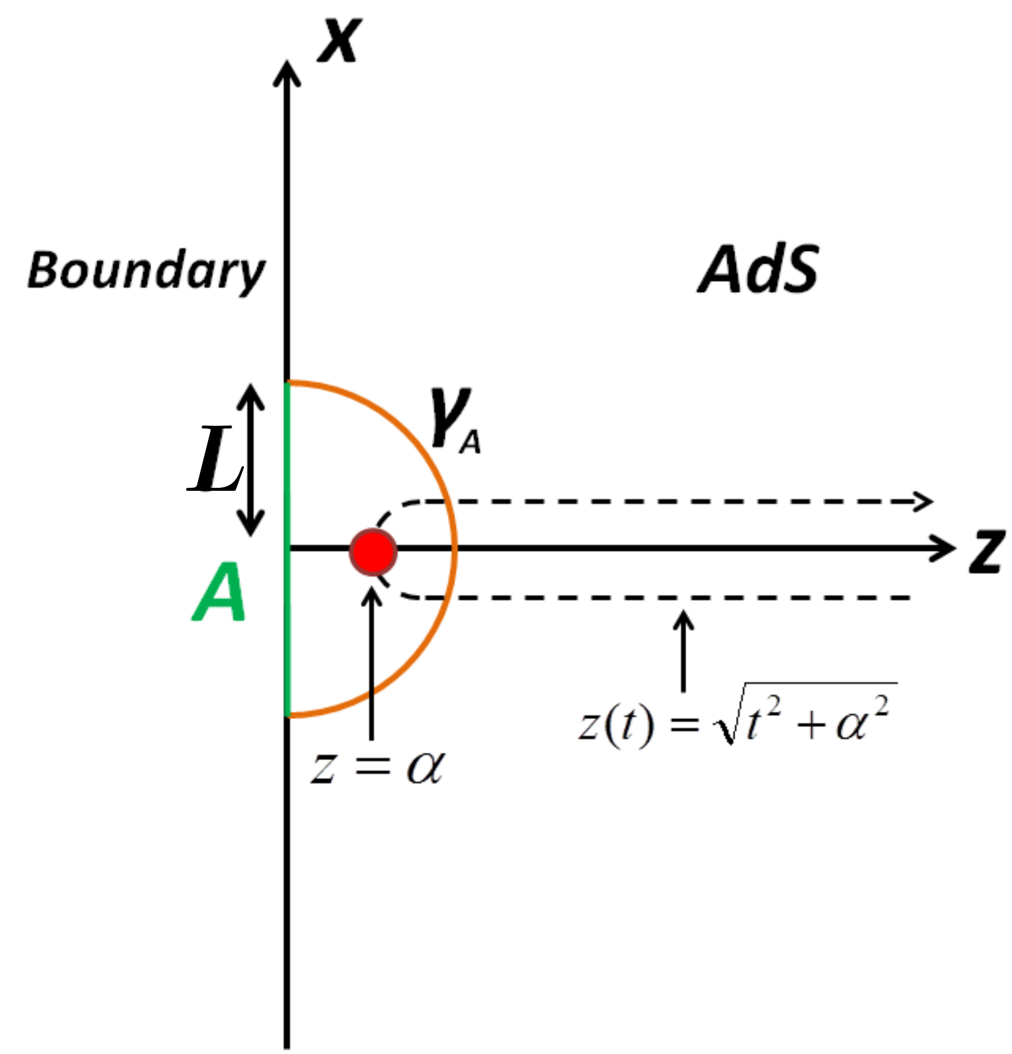
$$E \sim \frac{\Delta_O}{\epsilon}$$

Falling particle of mass m in AdS

[Nozaki,Numasawa,Takayanagi'13]
[PC,Nozaki,Takayanagi'14]

CFT (chain)





In our setup:

$$\Delta S^{(1)} \sim \frac{c}{6} \log \left[\frac{\sin \pi a}{a} \frac{t(L-t)}{\epsilon L} \right] \to \Delta S^{(1)} \sim \frac{c}{6} \log \left[\frac{t}{\epsilon} \right] + \frac{c}{6} \log \left[\frac{\sin \pi a}{a} \right]$$

$$a = \sqrt{1 - \frac{\mu}{R^2}}$$

Twist operators

[Bernamonti et al.'14]

$$\rho(t) = Ne^{-iHt}O(x_4, \bar{x}_4)\,|0\rangle\,\langle 0|\,O(x_1, \bar{x}_1)e^{iHt}$$

$$Tr\rho_A^n = \frac{\langle O(x_1, \bar{x}_1)\sigma(x_2, \bar{x}_2)\tilde{\sigma}(x_3, \bar{x}_3)O(x_4, \bar{x}_4)\rangle_{CFT^n/Z_n}}{\langle O(x_1, \bar{x}_1)O(x_4, \bar{x}_4)\rangle^n}$$

$$\bar{x}_1 = i\epsilon, \quad \bar{x}_4 = -i\epsilon$$
$$x_1 = -i\epsilon, \quad x_4 = i\epsilon$$

$$x_2 = l_1 - t, \quad x_3 = l_2 - t$$
$$\bar{x}_2 = l_1 + t, \quad \bar{x}_3 = l_2 + t$$

$$Tr\rho_A^n = |x_{23}|^{-4\Delta_n}|1 - z|^{4\Delta_n}G_n(z, \bar{z})$$

Large c limit of conformal blocks

[Zamolodchikov….]

$$G(z,\bar{z}) \sim e^{f(z,\bar{z})} \qquad\qquad c \to \infty$$

Two-heavy and two light operators

[Fitzpatrick et al.'14]

$$h/c \to 0 \qquad \Delta_O/c \ - \text{fixed}$$

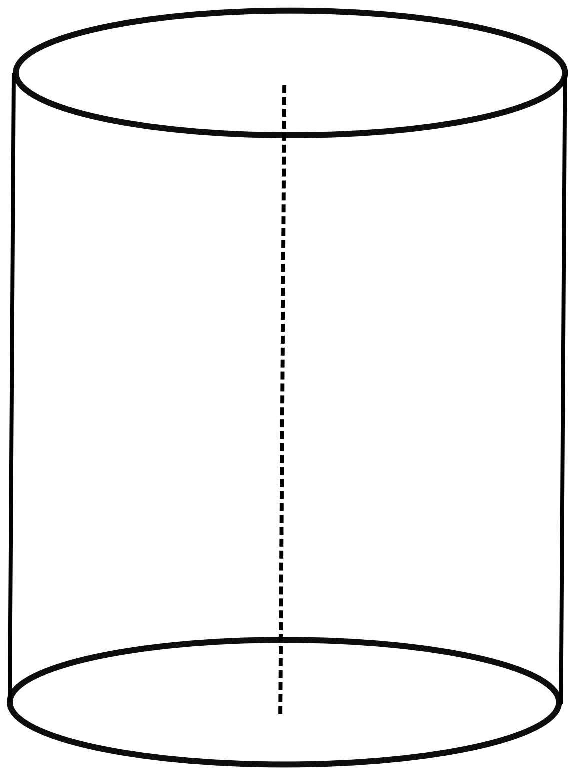
$$G(z,\bar{z}) \simeq \left(\frac{z^{\frac{1-\alpha}{2}}(1-z^{\alpha})\bar{z}^{\frac{1-\alpha}{2}}(1-\bar{z}^{\alpha})}{\alpha^2} \right)^{-2h} \qquad\qquad \alpha = \sqrt{1 - \frac{24\Delta_O}{c}}$$

Using this we can compute

$$\Delta S^{(1)} \sim \frac{c}{6} \log\left[\frac{\sin \pi\alpha}{\alpha} \frac{t(L-t)}{\epsilon L} \right] \qquad t < L$$

Back-reaction from a point particle in AdS

[Horowitz,Itzhaki'99]



$$ds^2 = -\left(r^2 + R^2 - \frac{M}{r^{d-2}}\right) d\tau^2 + \frac{R^2 dr^2}{R^2 + r^2 - M/r^{d-2}} + r^2 d\Omega_{d-1}^2$$

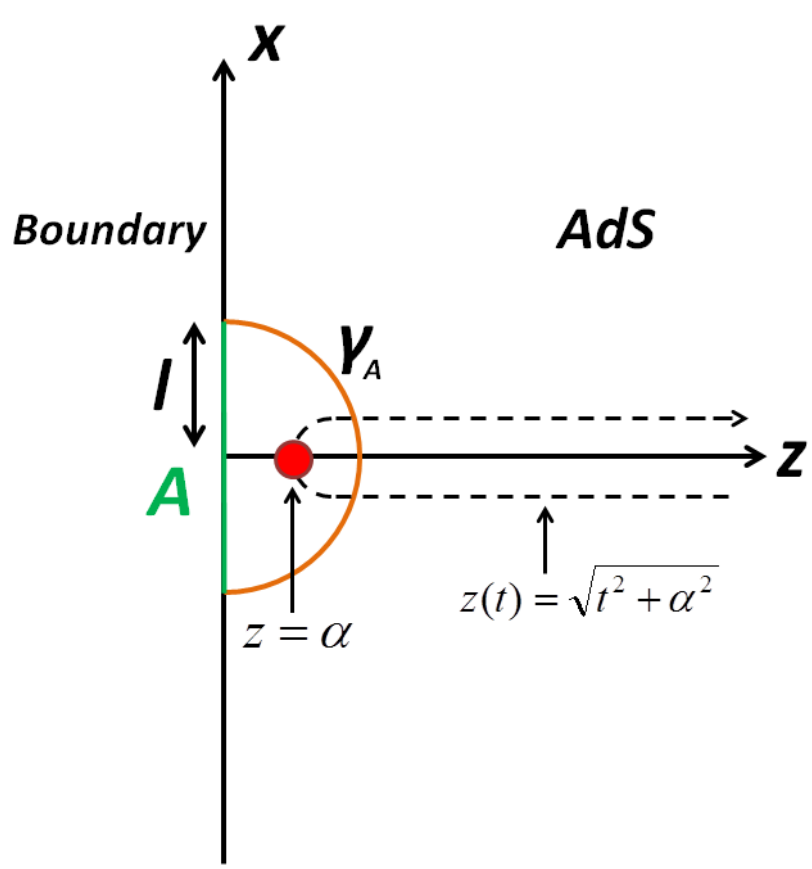
$$m = \frac{(d-1)\pi^{d/2-1}}{8\Gamma(d/2)} \cdot \frac{M}{G_N R^2}$$

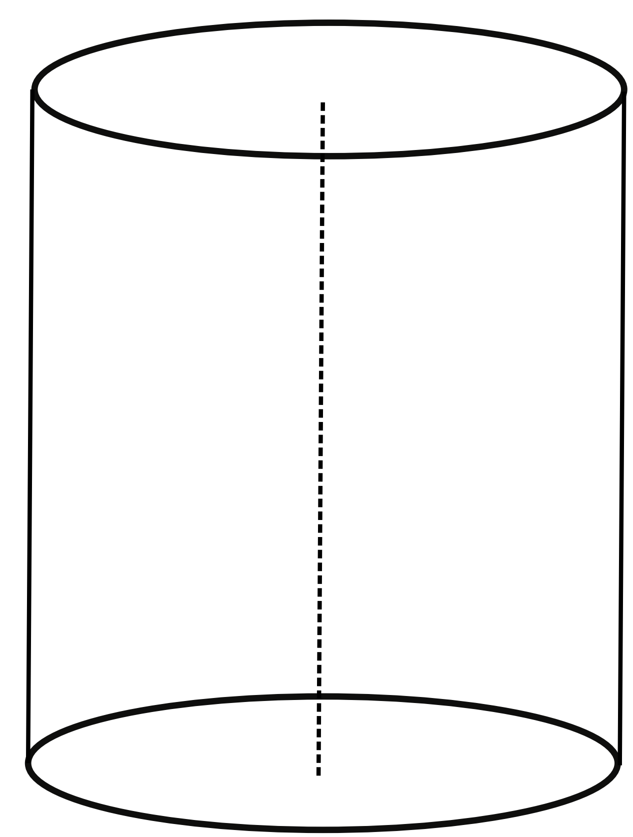
In order to find a back-reaction from a particle in AdS we "just" have to find the map to the r=0 solution in global AdS and insert to the above metric

Details:

$$ds^2 = R^2 \left(\frac{dz^2 - dt^2 + \sum_{i=1}^{d-1} dx_i^2}{z^2} \right)$$

$$S = -mR \int dt \frac{\sqrt{1 - \dot{z}(t)^2}}{z(t)}. \qquad z(t) = \sqrt{(t - t_0)^2 + \alpha^2}$$





Map:

$$\sqrt{R^2+r^2}\cos\tau = \frac{R^2e^{\beta}+e^{-\beta}(z^2+x^2-t^2)}{2z},$$
$$\sqrt{R^2+r^2}\sin\tau = \frac{Rt}{z},$$
$$r\Omega_i = \frac{Rx_i}{z} \quad (i=1,2,\cdots,d-1),$$
$$r\Omega_d = \frac{-R^2e^{\beta}+e^{-\beta}(z^2+x^2-t^2)}{2z}.$$

$$\boxed{\alpha = \epsilon = Re^{\beta}}$$

Back reacted metric after inserting:

$$r = \frac{1}{2z}\sqrt{R^4e^{2\beta}+e^{-2\beta}(z^2+x_i^2-t^2)^2-2R^2(z^2-x^2-t^2)},$$
$$d\tau^2 = d(\cos\tau)^2+d(\sin\tau)^2, \quad d\Omega_{d-1}^2 = \sum_{i=1}^{d}(d\Omega_i)^2.$$

we can check that we get the appropriate energy density

Entanglement Entropy (d=2)

$$S_A = \frac{c}{6}\left[\log\left(r_\infty^{(1)}\cdot r_\infty^{(2)}\right) + \log\frac{2\cos\left(|\Delta\tilde{\tau}_\infty|\frac{\sqrt{R^2-\mu}}{R}\right) - 2\cos\left(|\Delta\phi_\infty|\frac{\sqrt{R^2-\mu}}{R}\right)}{R^2-\mu}\right]$$

where

$$\tan\tau_\infty^{(i)} = \frac{2Rt}{R^2e^\beta + e^{-\beta}((l^{(i)})^2 - t^2)},$$

$$\tan\theta_\infty^{(i)} = -\frac{2Rl^{(i)}}{e^{-\beta}((l^{(i)})^2 - t^2) - R^2e^\beta},$$

$$r_\infty^{(i)} = \frac{1}{z_\infty}\sqrt{R^2(l^{(i)})^2 + \frac{1}{4}\left(e^{-\beta}((l^{(i)})^2 - t^2) - R^2e^\beta\right)^2}.$$

$$\Delta S^{(1)} \sim \frac{c}{6}\log\left[\frac{\sin\pi a}{a}\frac{t(L-t)}{\epsilon L}\right] \to \Delta S^{(1)} \sim \frac{c}{6}\log\left[\frac{t}{\epsilon}\right] + \frac{c}{6}\log\left[\frac{\sin\pi a}{a}\right]$$

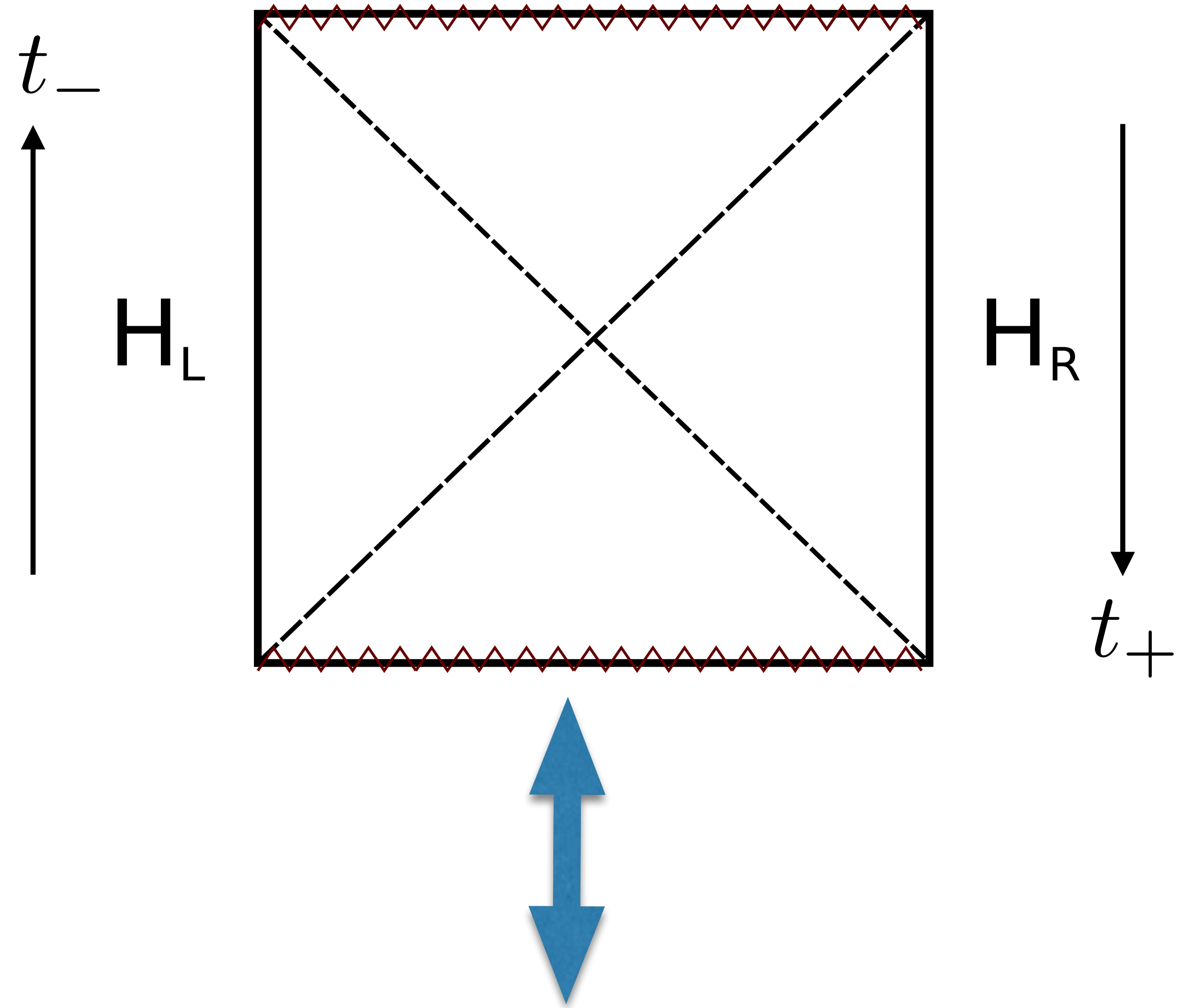
Finite Temperature

$$I_{A:B} = S_A + S_B - S_{A\cup B}$$

Eternal BH-TFD duality

[Maldacena'01]

Eternal BH



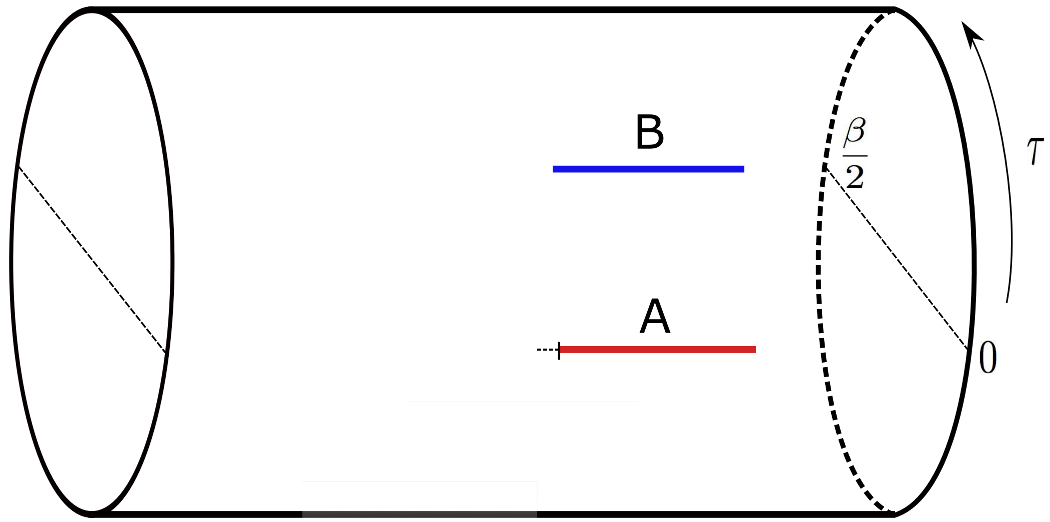
TFD

$$|\Psi_\beta\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\frac{\beta}{2}E_n} |n\rangle_L |n\rangle_R$$

Evolution of EE in TFD

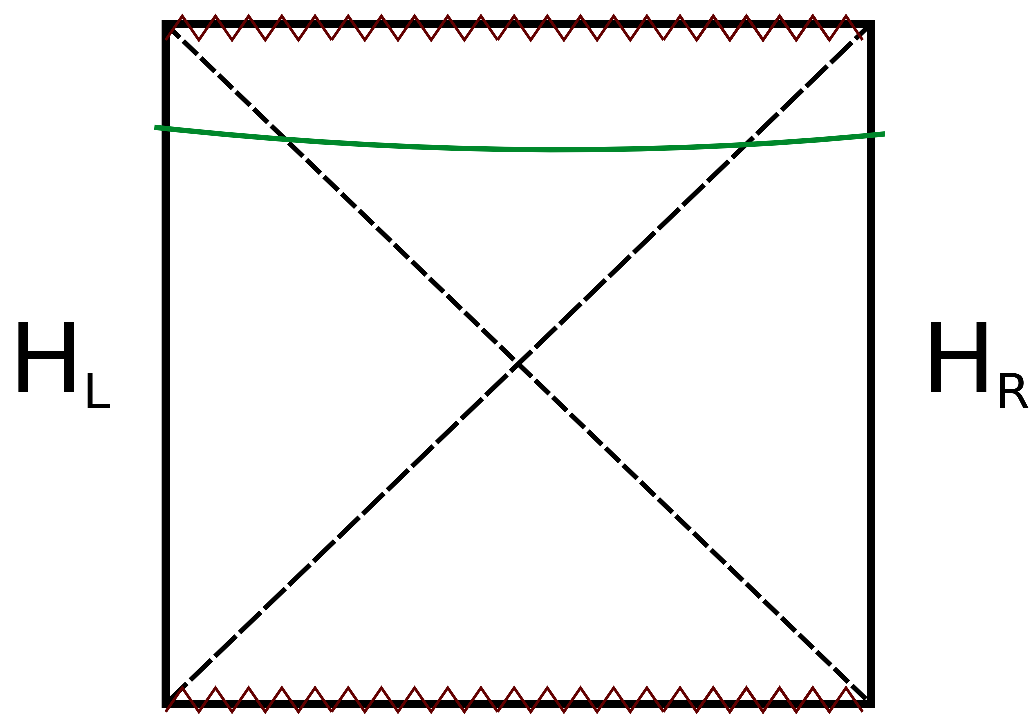
[Maldacena Hartman]

[Morrison,Roberts]



$$S_{A\cup B} \simeq t \qquad t < L/2$$

$$S_{A\cup B} \simeq 2S_{th} \qquad t > L/2$$

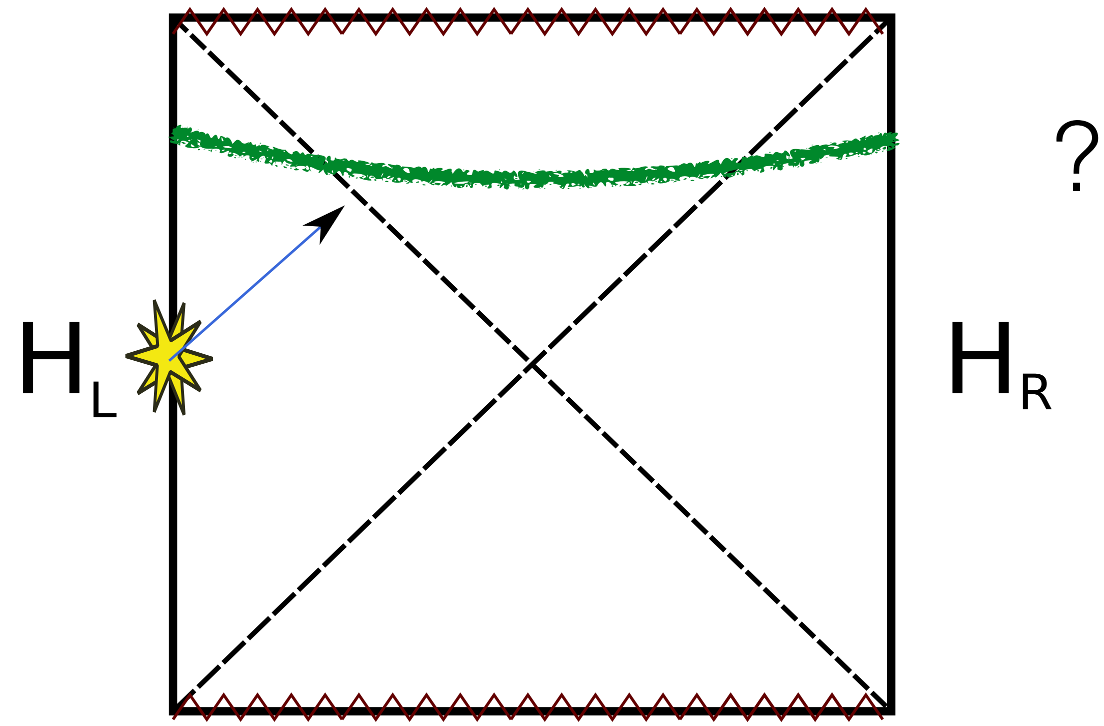


$$I_{A:B} = S_A + S_B - S_{A\cup B}$$

Operator Insertion to TFD

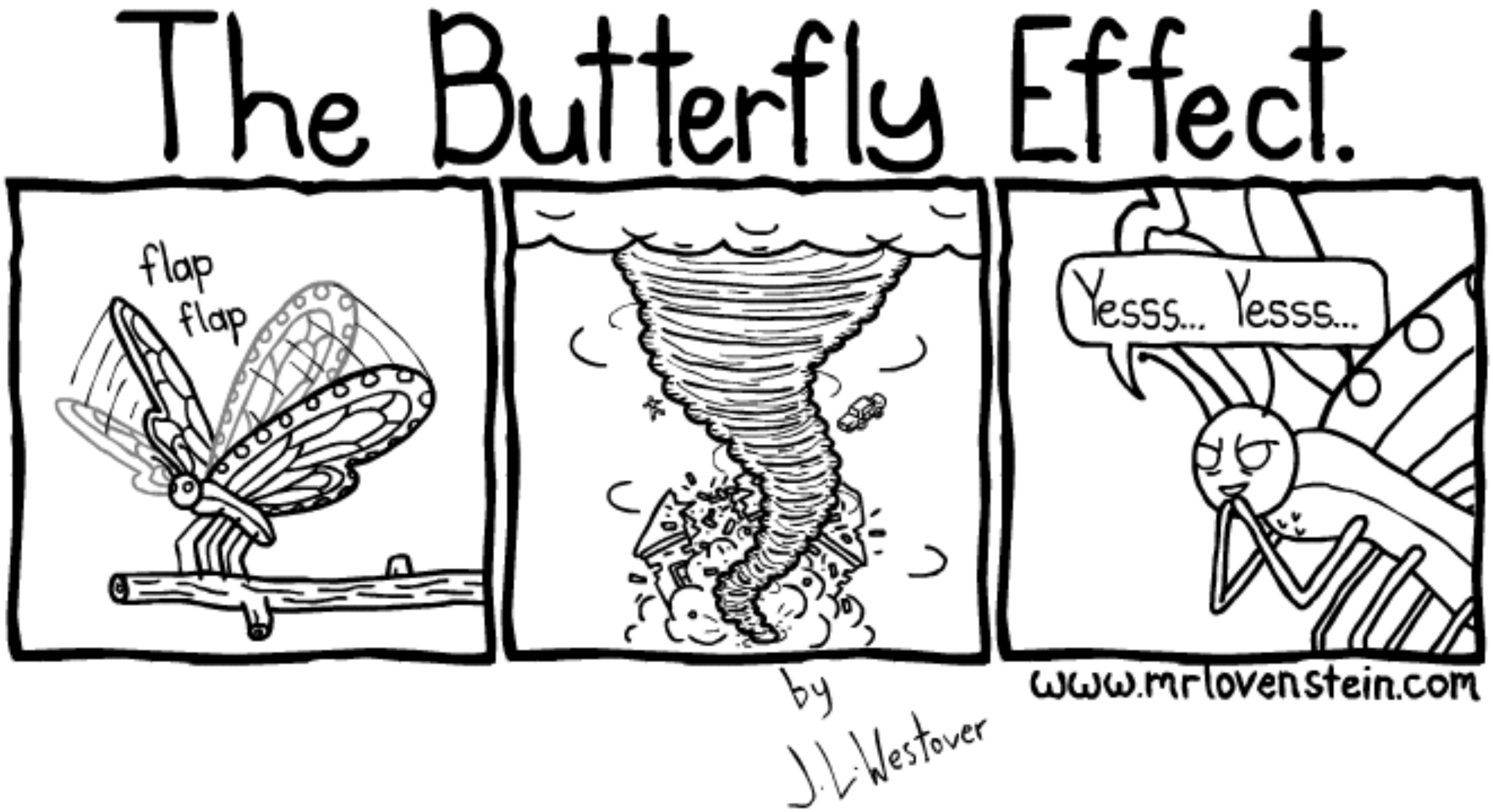
[P.C,Simon,Stikonas,Takayanagi’14]

Eternal BH



TFD

$$O_L|\psi_\beta>$$



$$|\psi'\rangle = e^{-iH_L t_w} O(x) e^{iH_L t_w} |\psi\rangle$$

[Shenker,Stanford]

[Roberts,Stanford]

[+ Susskind]

$$I_{A:B}(t_w) = 0?$$

$$t_w \sim \beta \log c \sim \beta \log S$$

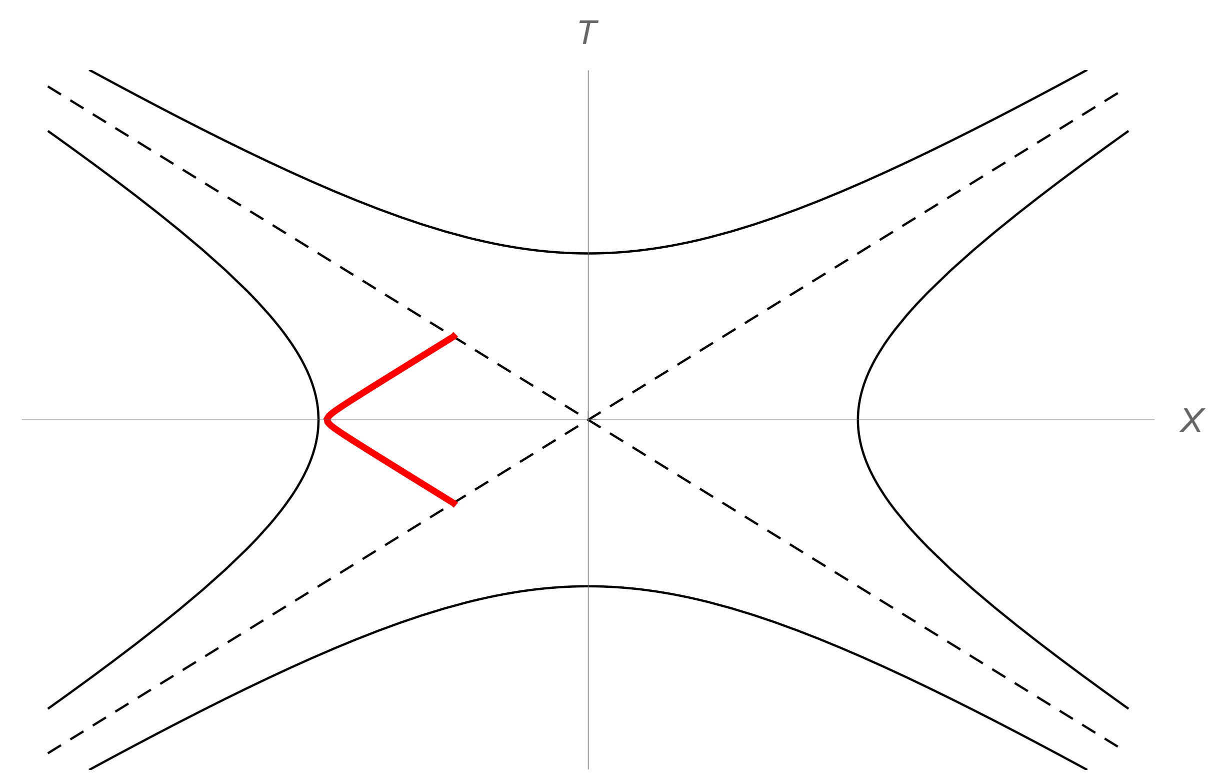
Point particle in BTZ

[PC,Simon,Stikonas,Takayanagi'14]

$$ds^2 = \frac{R^2}{z^2}\left(-\left(1 - Mz^2\right)dt^2 + \frac{dz^2}{(1 - Mz^2)} + dx^2\right)$$

$$S_p = -mR\int \frac{d\tau}{z(\tau)}\sqrt{1 - Mz(\tau)^2 - \frac{\dot{z}(\tau)^2}{1 - Mz(\tau)^2}}$$

$$z(\tau) = \frac{\beta}{2\pi}\sqrt{1 - \left(1 - \left(\frac{2\pi\epsilon}{\beta}\right)^2\right)\left(1 - \tanh^2\left(\frac{2\pi\tau}{\beta}\right)\right)}$$



Check:

Entanglement Entropy gravity

$$\Delta S_A \simeq \frac{c}{6}\log\left[\frac{\beta}{\pi\epsilon}\frac{\sin a}{a}\frac{\sinh\frac{\pi(t+t_w)}{\beta}\sinh\frac{\pi(L-t-t_w)}{\beta}}{\sinh\frac{\pi L}{\beta}}\right]$$

CFT large c

$$\mathrm{Tr}\rho_A^n(t) = \frac{\langle\psi(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\tilde{\sigma}(x_3,\bar{x}_3)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_n}}{\left(\langle\psi(x_1,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n} \qquad w(x) = e^{\frac{2\pi}{\beta}x}$$

$$\mathcal{O} \equiv \psi$$

$$\Delta S_A = \frac{c}{6}\log\left[\frac{\beta}{\pi\epsilon}\frac{\sin\pi\alpha_\psi}{\alpha_\psi}\frac{\sinh\left(\frac{\pi(L-t-t_w)}{\beta}\right)\sinh\left(\frac{\pi(t+t_w)}{\beta}\right)}{\sinh\left(\frac{\pi L}{\beta}\right)}\right]$$

Point particle in Kruskal coordinates

$$ds^2 = R^2 \frac{-4dudv + (-1+uv)^2 d\phi^2}{(1+uv)^2} = R^2 \frac{-4dT^2 + 4dX^2 + (1 - T^2 + X^2)^2 d\phi^2}{(1 + T^2 - X^2)^2}$$

$$t_- = \tilde{\tau}, \quad \theta = 0, \quad 1 - Mz^2 = (1 - M\epsilon^2) \cosh^{-2}\left(\sqrt{M}(\tilde{\tau} + t_\omega)\right)$$

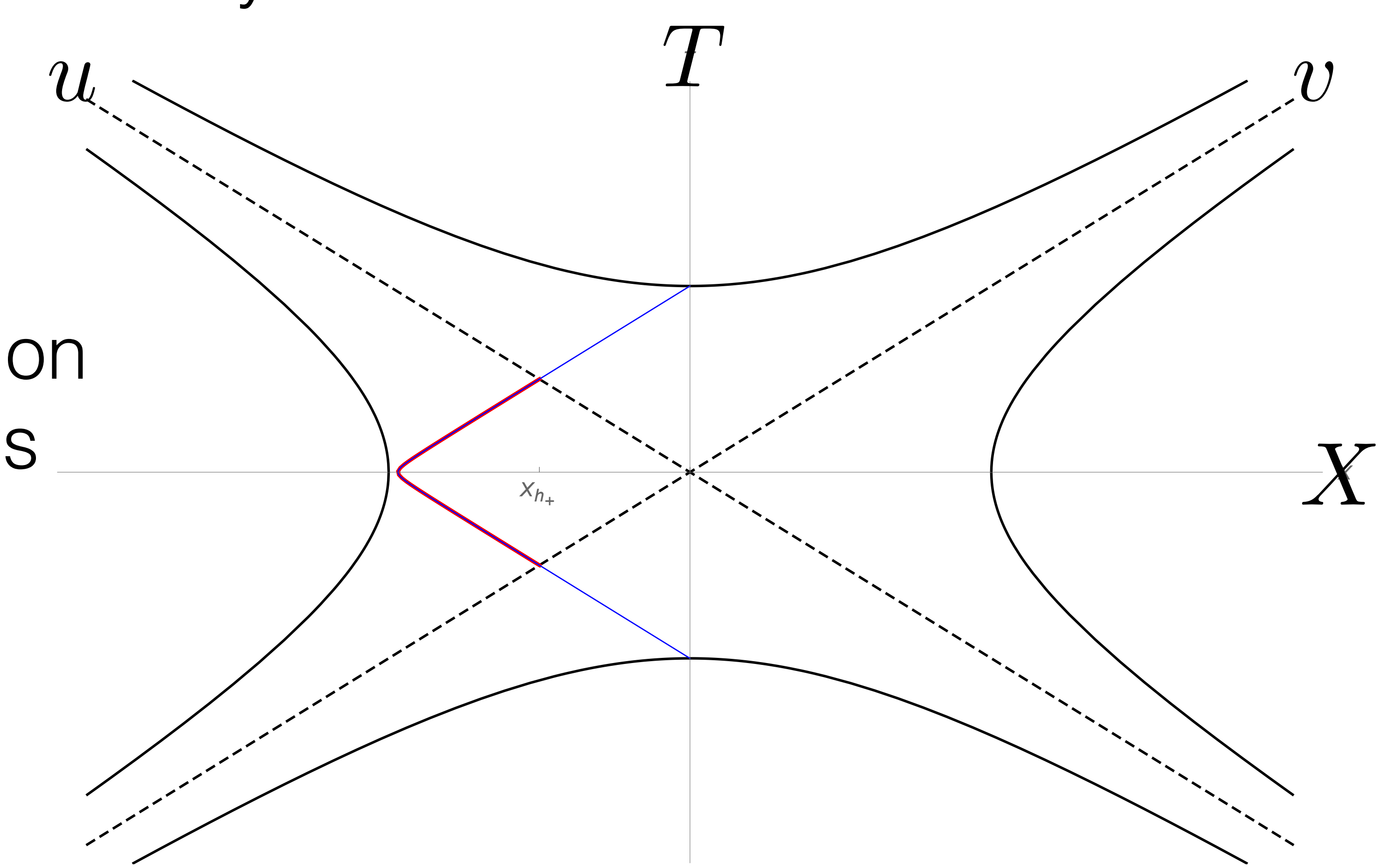
our solution in v(u) or T(X) is valid everywhere

$$v(u) = -\frac{a_1 u - 1}{u + a_2}$$

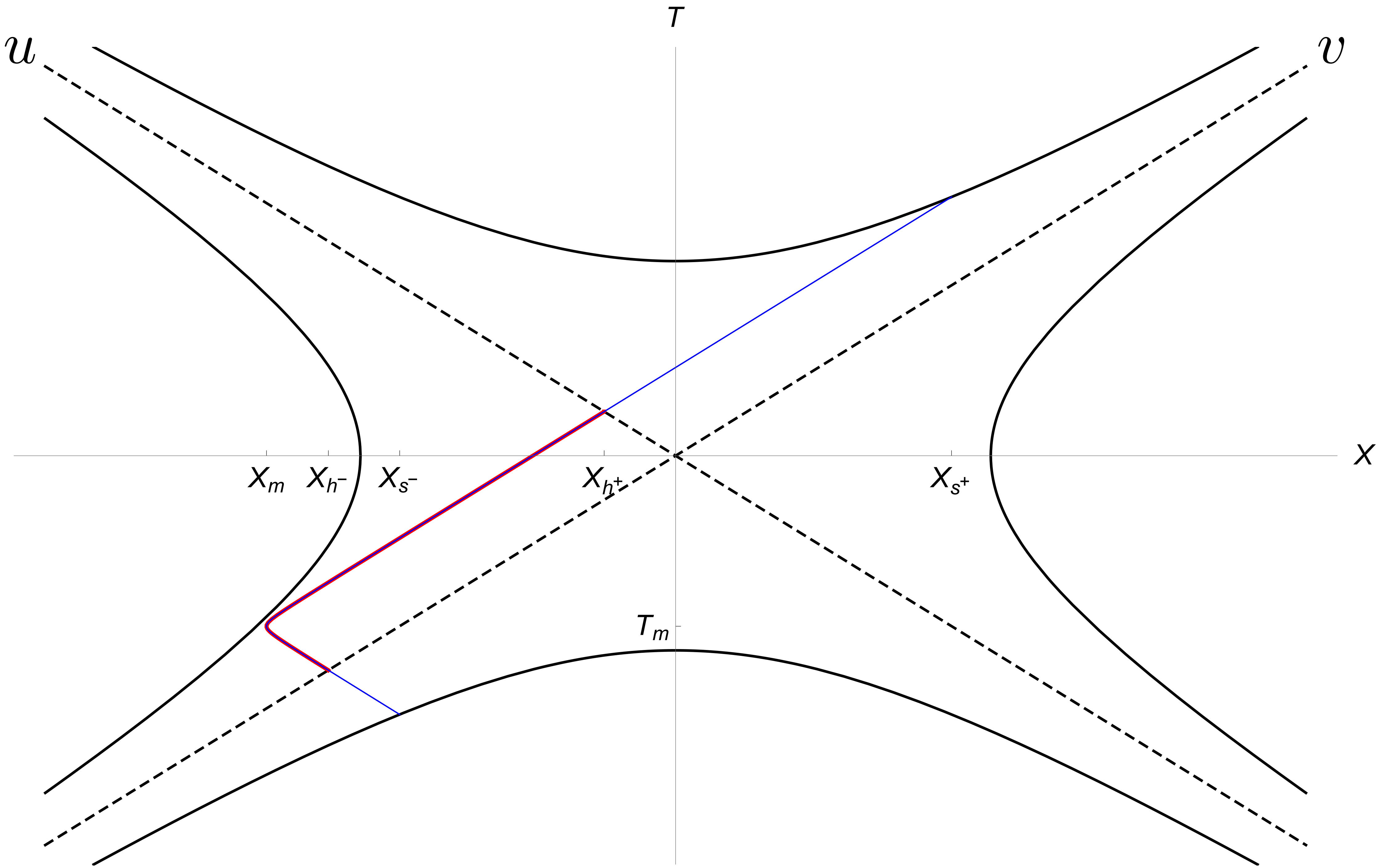
we can compute the back reaction using a map with two parameters

$$\lambda_1 = \sqrt{M} t_w$$

$$\tanh \lambda_2 = \sqrt{1 - M\epsilon^2}$$



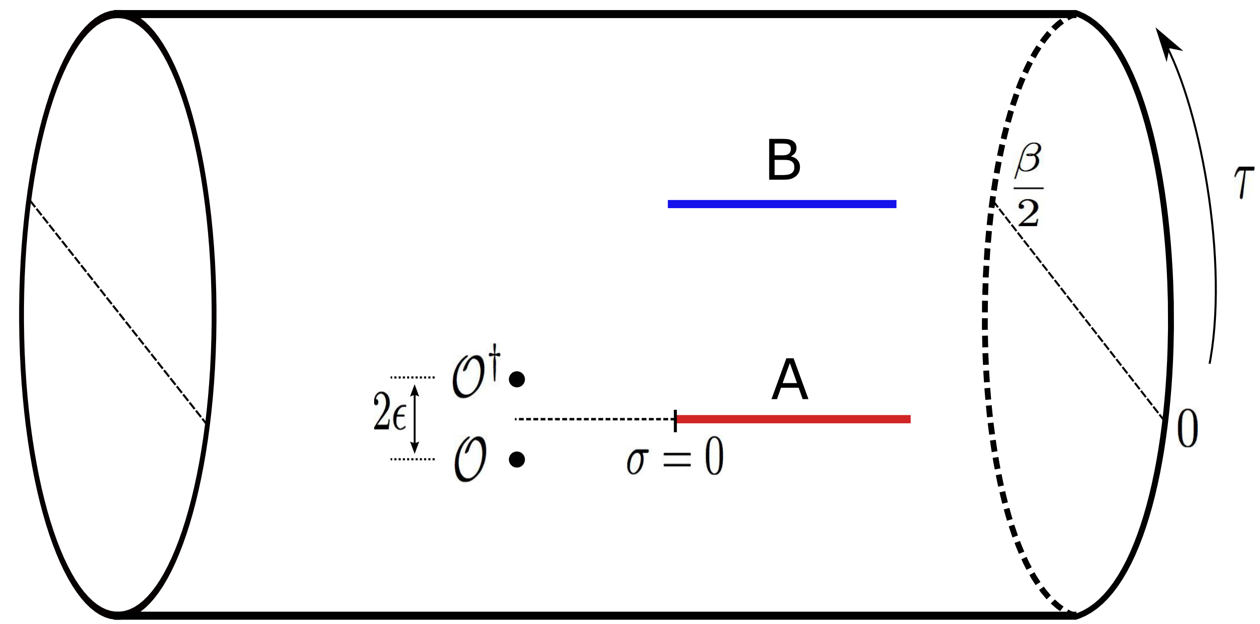
Large t_w



Mutual Information CFT

[PC,Simon,Stikonas,Takayanagi,Watanabe]

$S_{A\cup B}$



$\mathcal{O} \equiv \psi$

$$\mathrm{Tr}\rho_A^n(t) = \frac{\langle \psi(x_1,\bar{x}_1)\sigma(x_2,\bar{x}_2)\tilde{\sigma}(x_3,\bar{x}_3)\sigma(x_5,\bar{x}_5)\tilde{\sigma}(x_6,\bar{x}_6)\psi^\dagger(x_4,\bar{x}_4)\rangle}{\left(\langle \psi(x_1,\bar{x}_1)\psi^\dagger(x_4,\bar{x}_4)\rangle_{C_1}\right)^n}$$

Mutual Information results

[PC,Simon,Stikonas,Takayanagi,Watanabe]

$$I_{A:B}(t_-, t_+, t_w, L, a) = I_{A:B}(t_-, t_+, t_w, L, \alpha)$$

$$I_{A:B}(t_w^*) = 0?$$

$$\boxed{t_\omega^\star = f(L,\beta) + \frac{\beta}{2\pi}\log\frac{S}{\pi E_\psi}}$$

$$\frac{\beta}{4\pi\epsilon}\frac{\sin\left(\pi\sqrt{1-\frac{24\Delta_O}{c}}\right)}{\sqrt{1-\frac{24\Delta_O}{c}}} \simeq \frac{3\beta\Delta_O}{c\epsilon} = \frac{\pi E_O}{S}$$

Scrambling time and two-point functions

$$C_4 = \frac{\langle O_{h_w}(x_1,\bar{x}_1) O_h(x_2,\bar{x}_2) O_h(x_3,\bar{x}_3) O_{h_w}(x_4,\bar{x}_4)\rangle}{\langle O_{h_w}(x_1,\bar{x}_1) O_{h_w}(x_4,\bar{x}_4)\rangle}$$

$$w(x) = e^{\frac{2\pi}{\beta}x}$$

$$C_4 \simeq \left(\frac{\beta}{2\pi z_\infty}\right)^{-4h} \exp\left[-\frac{4\pi h}{\beta}\left(t_w + \frac{\beta}{2\pi}\log\left(\frac{\beta}{\pi\epsilon}\frac{\sin(\pi\alpha)}{\alpha}\right)\right)\right]$$

$$\sim \beta \log S$$

[see also Roberts,Stanford'15]

Conclusions

- Local excitations are exciting !
- Entanglement Entropy (and MI) is the right tool to explore
- We have a model for studying local excitations in AdS/CFT
- Perfect agreement with CFT
- Scrambling time from AdS and CFT
- ….