A Caveat for Applied Holography: Spacetime Reconstruction from a Non-Relativistic Boundary

1308.5689, 1404.4877

with G. Knodel and J. Liu

and 1504.xxxxx also with K. Sun

Anti de Sitter space in (d+1)+1 dimensions

• Maximally symmetric space with negative curvature that solves Einstein equations with negative cosmological constant

• Symmetry group SO(2,d+1)

• Conformally maps to half of Einstein static universe

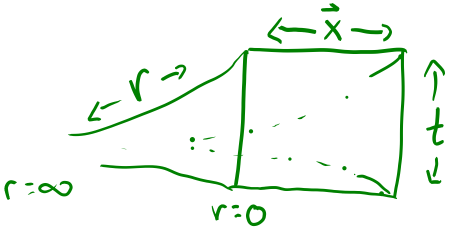
• Not globally hyperbolic: data at a constant time surface must be augmented with data at spatial boundary to have a well-defined initial value problem

Global coordinates:

$$ds^2 = R^2(-\cosh^2\rho \; d\tau^2 + d\rho^2 + \sinh^2\rho \; d\Omega_d^2)$$

Poincare coordinates:

$$ds^2_{d+2} = -\left(\frac{L}{r}\right) dt^2 + \left(\frac{L}{r}\right)^2 (d\vec{x}_d^2 + dr^2).$$



What is the boundary data?
Sources for a CFT!

AdS/CFT correspondence relates:

(type IIB string theory on) AdS in (d+1)+1 dimensions
to
($\mathcal{N}$=4 SYM at large SU(N)) a Conformal Field Theory in (d+1) dimensions.

$$\langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})}\rangle_{CFT} = \mathcal{Z}_{string}\left[\phi(\vec{x},z)\Big|_{z=0} = \phi_0(\vec{x})\right]$$

In words: field perturbations on AdS need extra boundary data to be a well-defined problem.
Pick extra boundary data on AdS ~ Pick sources to add in CFT
Solve field equations on AdS ~ Solve CFT to find VEVs due to sources

CFT has same symmetry group: SO(2,d+1). Consider scaling symmetry:

$t \to \Lambda t$, $\vec{x} \to \Lambda \vec{x}$. in gravity add: $r \to \Lambda r$

(Also important: strong coupling in CFT is weak coupling in gravity)

Scalar Perturbations on AdS

Study the Klein-Gordon equation for a scalar living in AdS:

$$(\Box - m^2)\phi = 0$$

We use Poincare coordinates.
Conformal boundary of spacetime is at r=0
Interior of spacetime, "IR" regime, is where r goes to infinity.

Expand in Fourier modes for t, x dependence.
KG equation is a second order ODE so two boundary conditions suffice.
Also there are two near-boundary (r=0) behaviors:

$$\phi \sim \hat{A}\left(\frac{r}{L}\right)^{\Delta_-} + \hat{B}\left(\frac{r}{L}\right)^{\Delta_+}, \qquad \Delta_\pm = \frac{d+1}{2} \pm \sqrt{(mL)^2 + \left(\frac{d+1}{2}\right)^2}.$$

1) Pick non-normalizable boundary data A ("sources" in CFT)
2) Insist on regularity in the interior or "bulk" of spacetime
3) find response= normalizable data B ("vevs" in CFT)

Bulk Scalar Profile Reconstruction via Smearing Function

(Balasubramanian, Kraus, Lawrence, Trivedi, Giddings; Freivogel, Bousso, Leichenauer, Rosenhaus, Zukowski)

- Use boundary position space operator to find profile throughout the bulk, via K:

$$\phi\left(t, \vec{x}, r\right) = \int dt' d^d x' K\left(t, \vec{x}, r | t', \vec{x}'\right) O\left(t', \vec{x}'\right)$$

- relation among normalizable modes: not equal to bulk-boundary propagator

- Expand in Fourier modes and invert to actually compute K:

$$K\left(t, \vec{x}, r | t', \vec{x}'\right) = \int dE d^d p \phi_{E,p}\left(t, \vec{x}, r\right) \varphi^*_{E,p}\left(t', \vec{x}'\right)$$

Once we know the kernel K, we can then turn normalizable boundary information into full bulk profile information.

K is also known as the "smearing function".

Applied Holography: Non-relativistic Systems

Many condensed matter problems are strong-coupling problems.
Can we build a gravitational system dual to these strong-coupling field theories?

First we want different symmetries, e.g. Lifshitz scale symmetry:

$$t \to \Lambda^z t, \quad \vec{x} \to \Lambda \vec{x} \quad ; \qquad r \to \Lambda r.$$

We will study the Lifshitz geometry first introduced by Kachru, Liu, and Mulligan:

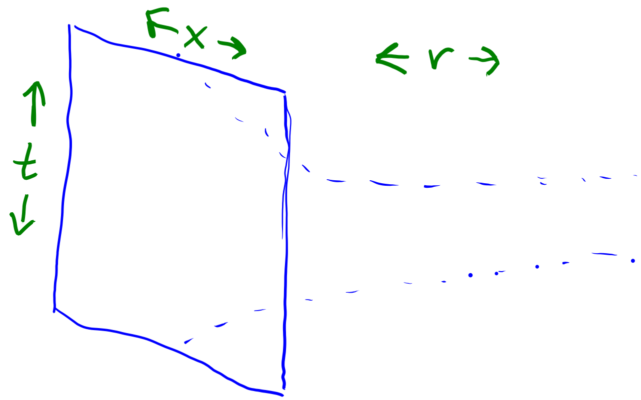
$$ds^2_{d+2} = -\left(\frac{L}{r}\right)^{2z} dt^2 + \left(\frac{L}{r}\right)^2 (d\vec{x}^2_d + dr^2).$$

If z=1 we recover AdS in Poincare coordinates.

(no large N, no SUSY.. but proceeding anyhow)

A Caveat for Applied Holography

• Does reconstruction of bulk information from boundary data proceed differently in nonrelativistic dual spacetimes from AdS?



A "nonrelativistic-dual spacetime" is a spacetime whose constant-radius slices have a nonrelativistic scaling symmetry:

$$t \to \Lambda^z t, \quad x \to \Lambda x, \quad r \to \Lambda r$$

One way of extracting bulk information from boundary data: Smearing Function.

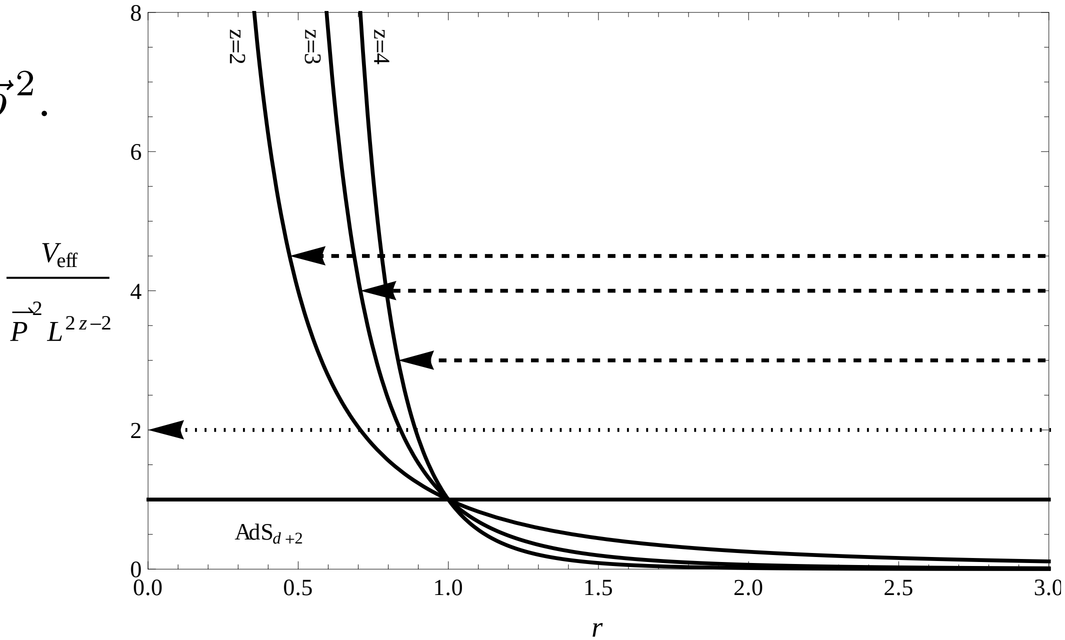
Lifshitz in a Geometric Optics Limit

Let us study the null geodesics of Lifshitz spacetime:

$$ds^2_{d+2} = -\left(\frac{L}{r}\right)^{2z} dt^2 + \left(\frac{L}{r}\right)^2 (d\vec{x}^2_d + dr^2).$$

Light rays with transverse momentum do not reach the boundary at r=0. Instead they feel a potential barrier:

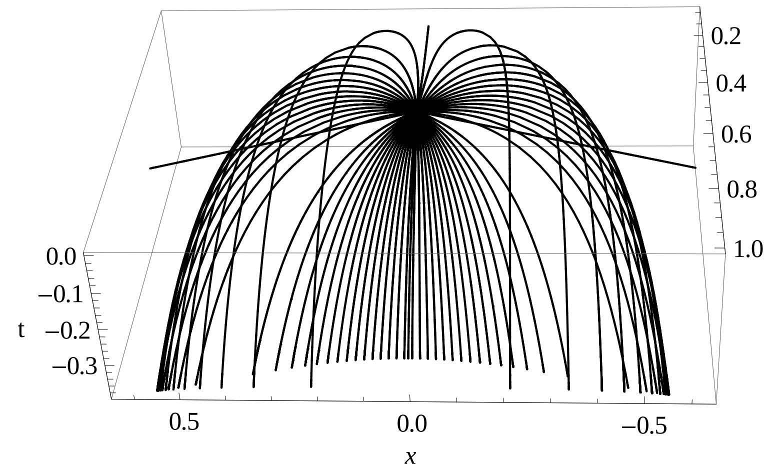
$$V_{\rm eff}(r) = \left(\frac{L}{r}\right)^{2z}\kappa + \left(\frac{L}{r}\right)^{2(z-1)} \vec{p}^{\,2}.$$



Plots of Lifshitz z=3 null geodesics

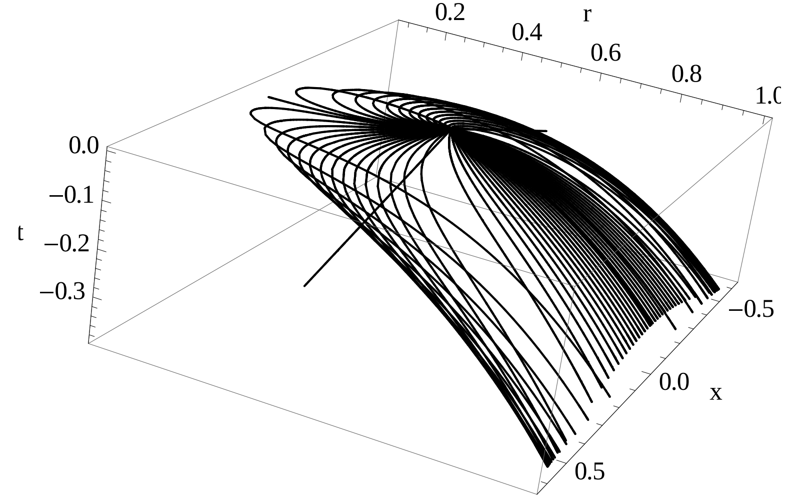
• Plots show null geodesics which pass through the point

$$t = 0,\ \vec{x} = 0,\ r = 1/2$$



• These plots depict the "causal past" of that point.

• Note only the p=0 geodesic actually reaches the r=0 boundary.



But does this (very naive) classical intuition have any quantum effect?

Lifshitz z=2 Klein-Gordon Equation as Schroedinger Potential

Rewrite $(\Box - m^2)\phi = 0$ as an effective Schroedinger equation by introducing

$$\phi = e^{iEt + i\vec{p}\cdot\vec{x}} \left(\frac{2\rho}{L}\right)^{d/4} \psi(\rho)$$

Using ' to denote derivatives with respect to ρ , we find

$$-\psi'' + U\psi = 0$$ with the potential $$U = \frac{\nu_2^2 - 1/4}{\rho^2} - E^2 + \frac{L}{2\rho}\vec{p}^2$$

and $$\nu_2 = \frac{1}{2}\sqrt{(mL)^2 + \left(\frac{d}{2} + 1\right)^2} .$$

Here the coordinate $\rho = r^2/2L$ is chosen so there are no single derivative terms, and so the effective "Schroedinger" energy E^2 is a constant in U .

(still classical wave function solutions, really)

Comparing Lifshitz to AdS: the effective potential U

Lifshitz z=2:

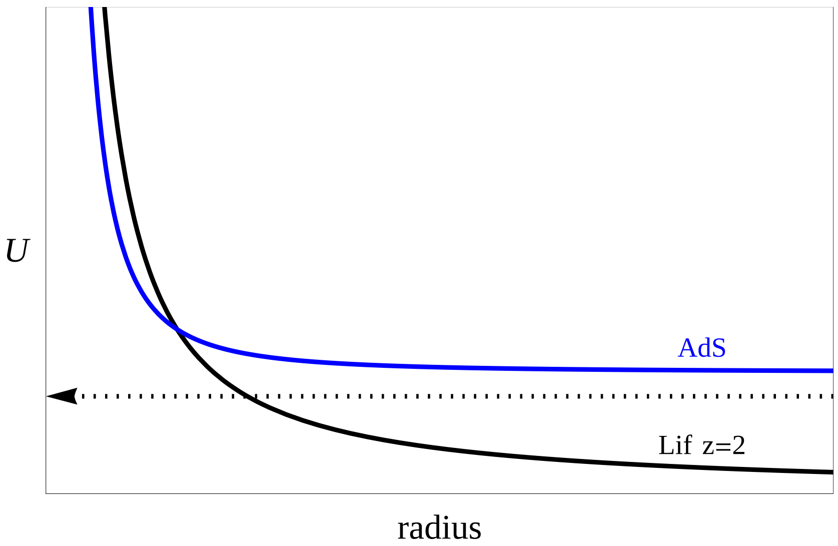
$$U = \frac{\nu_2^2 - 1/4}{\rho^2} - E^2 + \frac{L}{2\rho}\vec{p}^2$$

$$\nu_2 = \frac{1}{2}\sqrt{(mL)^2 + \left(\frac{d}{2} + 1\right)^2}$$

AdS:

$$U = \frac{\nu^2 - 1/4}{r^2} - E^2 + \vec{p}^2$$

$$\nu = \sqrt{(mL)^2 + \left(\frac{d+1}{2}\right)^2}$$



Exploring the effective potential U

Lifshitz z=2

$$U = \frac{\nu_2^2 - 1/4}{\rho^2} - E^2 + \frac{L}{2\rho}\vec{p}^2$$

AdS

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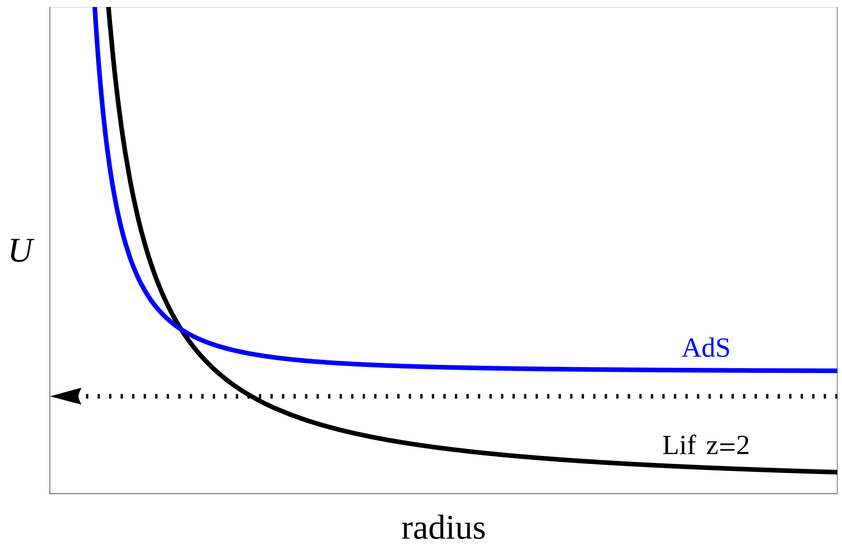

- Near boundary behavior is 1/radius^2 for both cases. Thus we get polynomial falloffs near the boundary:

$$\phi \sim \hat{A}\left(\frac{r}{L}\right)^{\Delta_-} + \hat{B}\left(\frac{r}{L}\right)^{\Delta_+}$$

The conformal dimensions are dependent on z:

$$\Delta_{\pm} = \frac{d+z}{2} \pm \sqrt{(mL)^2 + \left(\frac{d+z}{2}\right)^2}.$$

Exploring the effective potential U

Lifshitz z=2

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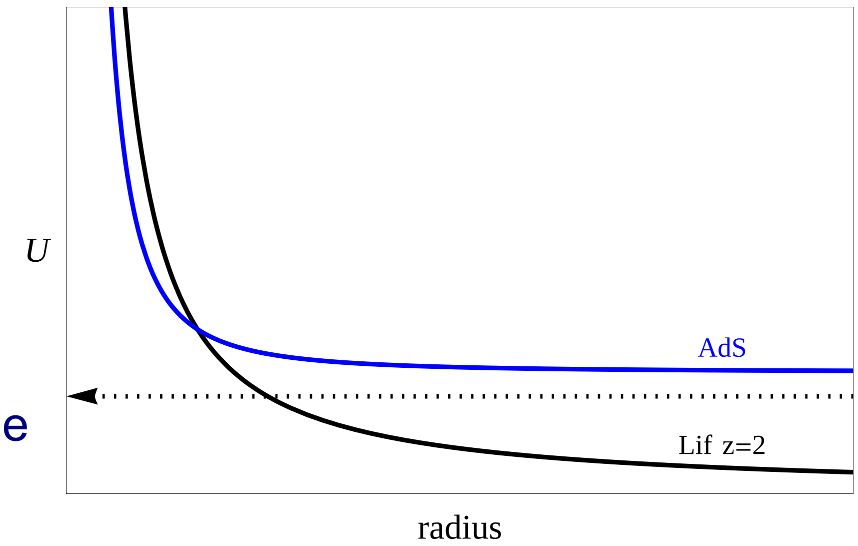
$$U = \frac{\nu^2 - 1/4}{r^2} - E^2 + \vec{p}^2$$

- Near boundary behavior is 1/radius^2
- Far IR behavior is constant.



Thus in the IR, the wavefunction oscillates.

For AdS, when E^2<p^2, no mode available
For Lifshitz, modes available above E^2=0

So in the IR we have

$$\phi \sim a\left(\frac{r}{L}\right)^{d/2} \exp\left(i\frac{EL}{z}\left(\frac{r}{L}\right)^z\right) + b\left(\frac{r}{L}\right)^{d/2} \exp\left(-i\frac{EL}{z}\left(\frac{r}{L}\right)^z\right)$$

But the dotted line represents a mode present in Lifshitz but not allowed in AdS.

Exploring the effective potential U

Lifshitz z=2

$$U = \frac{\nu_2^2 - 1/4}{\rho^2} - E^2 + \frac{L}{2\rho}\vec{p}^2$$

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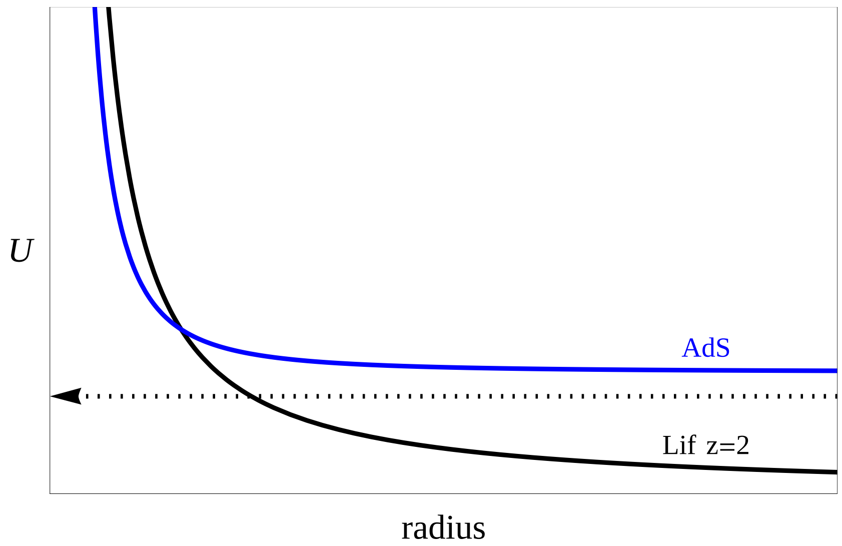

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- In Lifshitz, there is a 1/radius term.

This term represents a "wide" barrier between the UV and IR behavior; modes such as the dotted line must tunnel through the broad 1/radius term.

This extra tunneling will cause exponential suppression of A, B coeffs. in the UV when compared to the a, b coeffs. in the IR region.

Suppression in UV coefficients

Lifshitz z=2 can be solved exactly.

Finding the connection formulae to set A,B in terms of a,b
and turning off A (non normalizable mode) at the boundary:

$$\frac{B}{b} = 2^{-i\alpha/2}\left(\frac{2}{i}\right)^{\frac{1}{2}+\nu}\frac{\Gamma(\frac{1}{2}+\nu+\frac{i\alpha}{2})}{\Gamma\left(1+2\nu\right)}e^{-\pi\alpha/4}. \text{ with } \alpha = \vec{p}^{\,2}L/2E$$

Two interesting limits:

• small p for fixed E, small α: constant behavior

$$\frac{|B|}{|b|} \approx \frac{2^{\nu+\frac{1}{2}}\Gamma\left(\frac{1}{2}+\nu\right)}{\Gamma\left(1+2\nu\right)}$$

• large p for fixed E, big α : suppressed exponentially in α

$$\frac{|B|}{|b|} \approx \frac{\sqrt{4\pi}e^{-\left(\nu+\frac{1}{2}\right)}}{\Gamma\left(1+2\nu\right)}\alpha^{\nu}e^{-\pi\alpha/2}$$

Bulk Scalar Profile Reconstruction via Smearing Function

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• Use boundary position space operator to find profile throughout the bulk, via K:

$$\phi\left(t,\vec{x},r\right)=\int dt'd^dx'K\left(t,\vec{x},r|t',\vec{x}'\right)O\left(t',\vec{x}'\right)$$

• relation among normalizable modes: not equal to bulk-boundary propagator

• Expand in Fourier modes and invert to actually compute K:

$$K\left(t,\vec{x},r|t',\vec{x}'\right)=\int dEd^dp\phi_{E,p}\left(t,\vec{x},r\right)\varphi^*_{E,p}\left(t',\vec{x}'\right)$$

However this inversion is **not defined** for Lifshitz spacetimes, due to the exponential suppression.

Physically, as we integrate to very large p, the "wide" barrier becomes infinitely insurmountable, and thus hides IR information from the UV boundary.

Can we fix the smearing function?

- What if we fix the boundary?

Doesn't help. Via WKB approximation, for any fixed E, p can be set large enough to cause suppression; the smearing function integral is still not defined.

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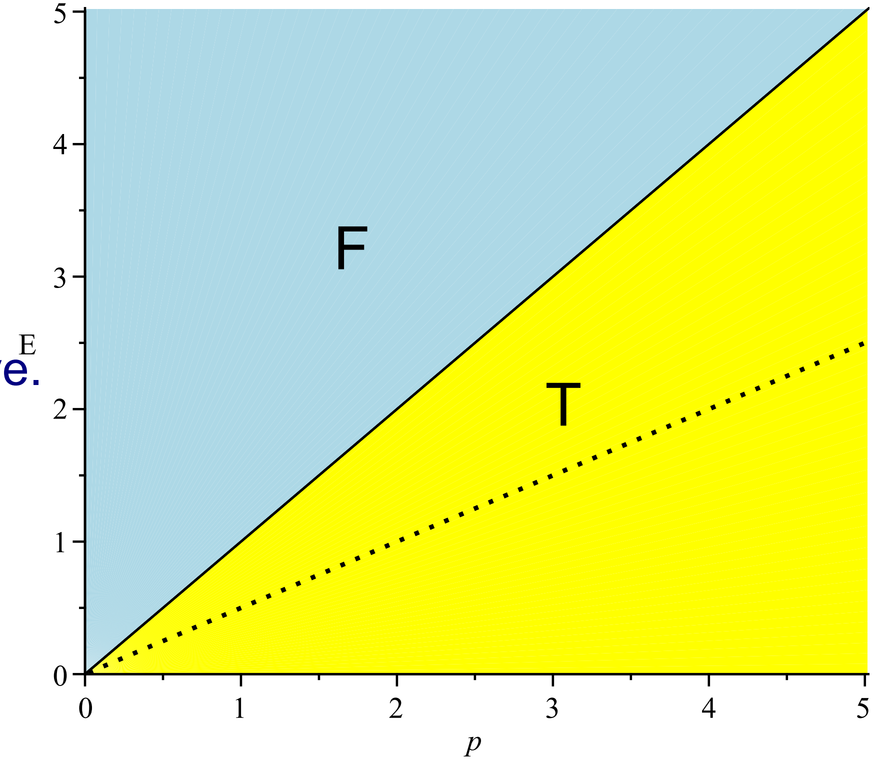
• What if we fix the IR?
Doesn't help.

This sketch of free (F) and trapped (T) modes shows deforming the geometry in the IR may introduce a cutoff (dotted line), but this line will always remain below the solid line, and some trapped modes survive.



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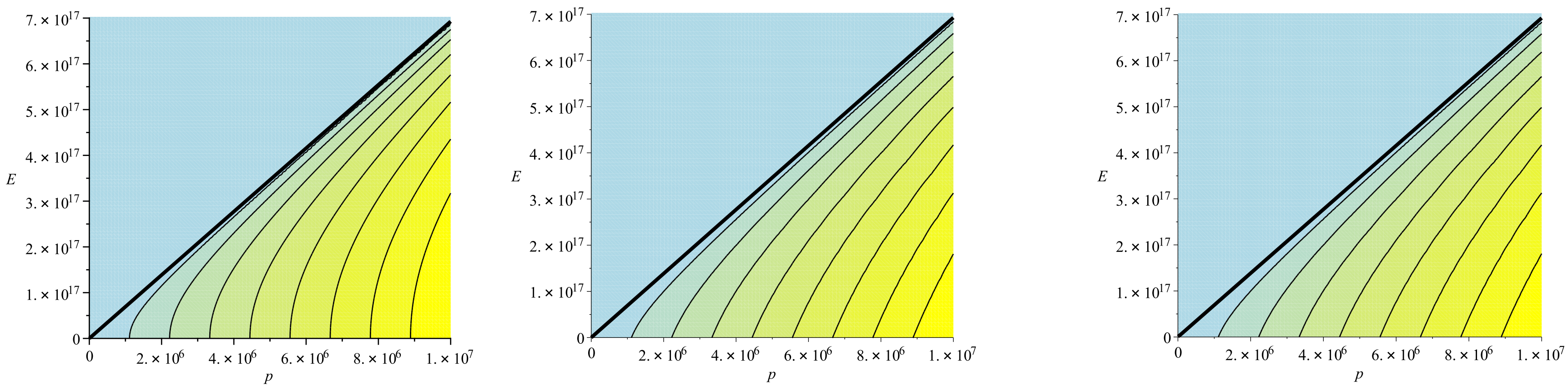
• What if we fix the IR?
Doesn't help. IR deformation can't remove all suppressed modes.

• What if we fix both?
Doesn't help.
From numerical solution of AdS4-Lifshitz-AdS2 via J. Liu and G. Knodel, even for bulk points within a "fixed" region, the smearing function integral fails.
(blue areas are "free" modes, yellow are "trapped")



point in AdS4 region | point in Lifshitz region | point in AdS2xR2 region

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• What if we fix the IR?
Doesn't help. IR deformation can't remove all suppressed modes.

• What if we fix both?
Doesn't help. Numerics show the smearing function integral still fails.

• What if we impose a strict cutoff at large p?
Yes, this will work.

However, a strict cutoff at large p means we cannot fully localize when we return to position space. Perfect local reconstruction from one-point boundary information in a spacetime with a Lifshitz-like region does not appear possible.

Green's function reconstruction

Still shows echoes of same behavior. Green's function is B/A with b=0.

AdS Green's function, with small change in boundary condition in IR:

$$G(E,\vec{p}) = -\frac{\pi}{2^{2\nu}\Gamma(\nu)\Gamma(\nu+1)}(E^2-\vec{p}^2)^\nu L^{2\nu}$$

$$\times \left[\cot\nu\pi - i(1+2e^{-i(2\nu+1)\frac{\pi}{2}}\epsilon)\right]$$

Lif z=2 Green's function, in $\alpha \to \infty$ (i.e. $p^2 \gg E$) limit, and small IR change:

$$G(E,p) \sim \left(\frac{pL}{4}\right)^{4\nu} \frac{\Gamma(-2\nu)}{\Gamma(2\nu)} \left[1 - \frac{2\left(\frac{\alpha}{4}\right)^{i\alpha} e^{-\pi\alpha} b/a}{1+i\left(\frac{\alpha}{4}\right)^{i\alpha} e^{-i\pi\alpha} b/a}\right]$$

- Scaling behavior at large p and E
- Im[G]=Spectral function suppressed for Lif

Other Non-Relativistic-Dual Spacetimes

- Lifshitz z>1

smearing function doesn't exist
Spectral function suppressed

- Any spacetime with a Lifshitz-like region (and transverse symmetry), for z>1

smearing function doesn't exist
Spectral function suppressed

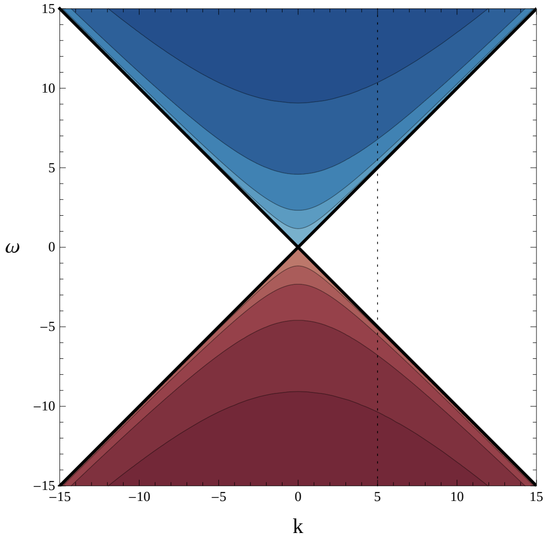
- Schroedinger with z=2

acts like AdS, except effective ν depends on null momentum
smearing function ok
Spectral function looks conformal

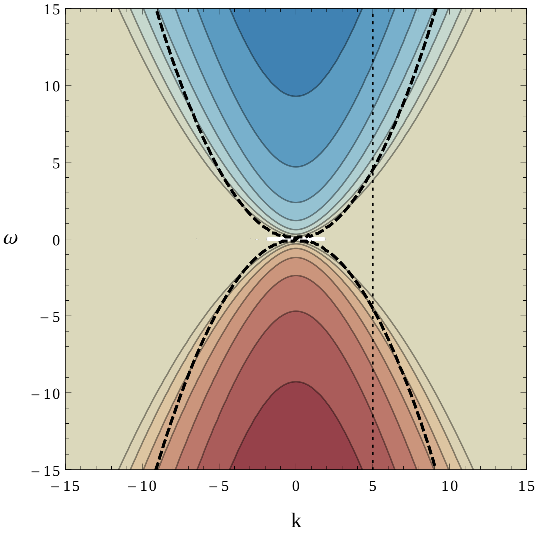
- Schroedinger with 1<z<2.

superselection sectors in null direction momentum mean:
smearing function is ok within a superselection sector
Spectral function suppressed

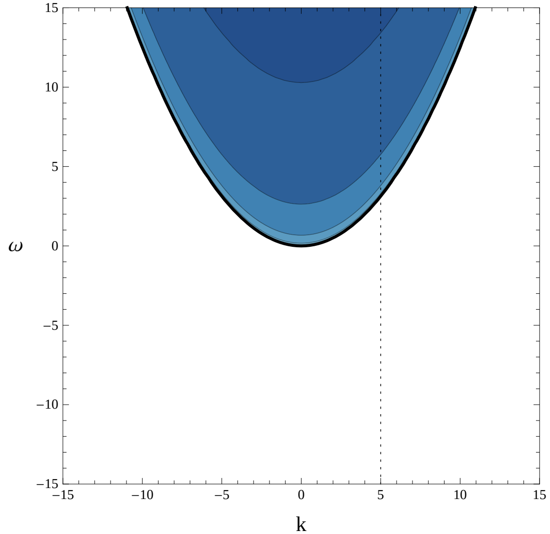
Spectral function suppression



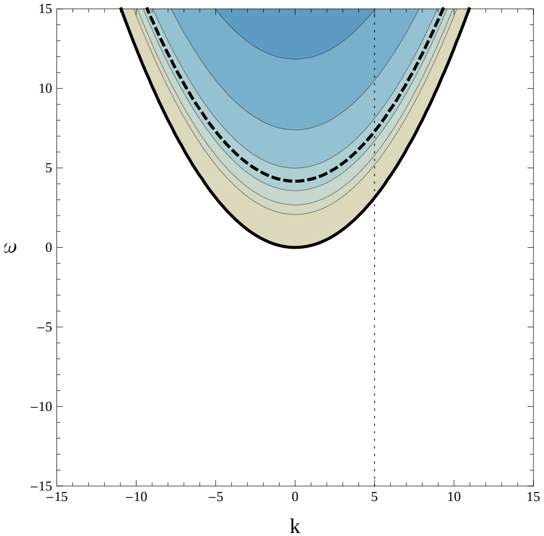
AdS spectral function with nu=1.1.
Contour steps are logarithmic.



Lifshitz z=2 spectral function, nu=1.1
Dashed line is omega << k^2/2nu.



Schroedinger z=2 spectral function.



Schroedinger z=3/2 spectral function.

Current and Future Considerations…