Renormalised 3-point functions in CFT

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Outline

Introduction

- Conformal invariance imposes strong constraints on correlation functions.
- It determines two- and three-point functions of scalars, conserved vectors and the stress-energy tensor [Polyakov (1970)] ... [Osborn, Petkou (1993)]. For example,

$$\langle \mathcal{O}_1(\boldsymbol{x}_1)\mathcal{O}_2(\boldsymbol{x}_2)\mathcal{O}_3(\boldsymbol{x}_3)\rangle$$
$$= \frac{c_{123}}{|\boldsymbol{x}_1-\boldsymbol{x}_2|^{\Delta_1+\Delta_2-\Delta_3}|\boldsymbol{x}_2-\boldsymbol{x}_3|^{\Delta_2+\Delta_3-\Delta_1}|\boldsymbol{x}_3-\boldsymbol{x}_1|^{\Delta_3+\Delta_1-\Delta_2}}.$$

- It determines the form of higher point functions up to functions of cross-ratios.

Introduction

- These results (and many others) were obtained in position space.
- This is in stark contrast with general QFT were Feymnan diagrams are typically computed in momentum space.
- While position space methods are powerful, typically they
  - provide results that hold only at separated points ("bare" correlators).
  - are hard to extend beyond CFTs
- The purpose of this work is to provide a first principles analysis of CFTs in momentum space.

Introduction

- Momentum space results were needed in several recent applications:
  - Holographic cosmology [McFadden, KS](2010)(2011) [Bzowski, McFadden, KS (2011)(2012)] [Pimentel, Maldacena (2011)][Mata, Raju,Trivedi (2012)] [Kundu, Shukla,Trivedi (2014)][Arkani-Hamed, Maldacena (2015)].
  - Studies of 3d critical phenomena [Sachdev et al (2012)(2013)]

References

- Adam Bzowski, Paul McFadden, KS
  Implications of conformal invariance in momentum space
  1304.7760
- Adam Bzowski, Paul McFadden, KS
  Renormalized scalar 3-point functions
  15xx.xxxxx
- Adam Bzowski, Paul McFadden, KS
  Renormalized tensor 3-point functions
  15xx.xxxxx

Conformal invariance

- Conformal transformations consist of dilatations and special conformal transformations.
- Dilatations $\delta x^\mu = \lambda x^\mu$, are linear transformations, so their implications are easy to work out.
- Special conformal transforms, $\delta x^\mu = b^\mu x^2 - 2x^\mu b \cdot x$, are non-linear, which makes them difficult to analyse (and also more powerful).
- The corresponding Ward identities are partial differential equations which are difficult to solve.

Conformal invariance

- In position space one overcomes the problem by using the fact that special conformal transformations can be obtained by combining inversions with translations and then analyzing the implications of inversions.
- In momentum space we will see that one can actually directly solve the special conformal Ward identities.

Conformal Ward identities

- These are derived using the conformal transformation properties of conformal operators. For scalar operators:

$$\langle \mathcal{O}_1(\boldsymbol{x}_1) \cdots \mathcal{O}_n(\boldsymbol{x}_n) \rangle = \left| \frac{\partial x'}{\partial x} \right|_{x=x_1}^{\Delta_1/d} \cdots \left| \frac{\partial x'}{\partial x} \right|_{x=x_n}^{\Delta_n/d} \langle \mathcal{O}_1(\boldsymbol{x}'_1) \cdots \mathcal{O}_n(\boldsymbol{x}'_n) \rangle$$

- For (infinitesimal) dilatations this yields

$$0 = \left[\sum_{j=1}^{n} \Delta_j + \sum_{j=1}^{n} x_j^\alpha \frac{\partial}{\partial x_j^\alpha} \right] \langle \mathcal{O}_1(\boldsymbol{x}_1) \ldots \mathcal{O}_n(\boldsymbol{x}_n) \rangle.$$

- In momentum space this becomes

$$0 = \left[\sum_{j=1}^{n} \Delta_j - (n-1)d - \sum_{j=1}^{n-1} p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \right] \langle \mathcal{O}_1(\boldsymbol{p}_1) \ldots \mathcal{O}_n(\boldsymbol{p}_n) \rangle,$$

Special conformal Ward identity

- For (infinitesimal) special conformal transformations this yields

$$0 = \left[\sum_{j=1}^{n} \left(2\Delta_j x_j^\kappa + 2 x_j^\kappa x_j^\alpha \frac{\partial}{\partial x_j^\alpha} - x_j^2 \frac{\partial}{\partial x_{j\kappa}} \right) \right] \langle \mathcal{O}_1(\boldsymbol{x}_1) \dots \mathcal{O}_n(\boldsymbol{x}_n) \rangle$$

- In momentum space this becomes

$$0 = \mathcal{K}^\mu \langle \mathcal{O}_1(\boldsymbol{p}_1) \dots \mathcal{O}_n(\boldsymbol{p}_n) \rangle,$$

$$\mathcal{K}^\mu = \left[\sum_{j=1}^{n-1} \left(2(\Delta_j - d) \frac{\partial}{\partial p_j^\kappa} - 2 p_j^\alpha \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_j^\kappa} + (p_j)_\kappa \frac{\partial}{\partial p_j^\alpha} \frac{\partial}{\partial p_{j\alpha}} \right) \right]$$

Special conformal Ward identities

- To extract the content of the special conformal Ward identity we expand $\mathcal{K}^\mu$ in a basis of linear independent vectors, the $(n-1)$ independent momenta,

$$\mathcal{K}^\kappa = p_1^\kappa \mathcal{K}_1 + \ldots + p_{n-1}^\kappa \mathcal{K}_{n-1}.$$

- Special conformal Ward identities constitute $(n-1)$ scalar differential equations.

Conformal Ward identities

- Poincaré invariant n-point functions depends on $n(n-1)/2$ kinematic variables.
- Thus, after imposing $(n-1)+1$ conformal Ward identities the correlator should contain an arbitrary function of

$$\frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

variables.
- This number equals the number of conformal ratios in n variables in $d \geq n$ dimensions.

➠ It is not known however what do the cross ratios become in momentum space.

Outline

Outline

Scalar 2-point function

- The dilatation Ward identity reads

$$0 = \left[d - \Delta_1 - \Delta_2 + p \frac{\partial}{\partial p} \right] \langle O_1(\boldsymbol{p}) O_2(-\boldsymbol{p}) \rangle$$

- The 2-point function is a homogeneous function of degree $(\Delta_1 + \Delta_2 - d)$:

$$\langle O_1(\boldsymbol{p}) O_2(-\boldsymbol{p}) \rangle = c_{12} p^{\Delta_1 + \Delta_2 - d}.$$

where c_{12} is an integration constant.

Scalar 2-point function

- The special conformal Ward identity reads

$$0 = \mathcal{K}\langle O_1(\boldsymbol{p}) O_2(-\boldsymbol{p})\rangle, \qquad \mathcal{K} = \frac{d^2}{dp^2} - \frac{2\Delta_1 - d - 1}{p}\frac{d}{dp}$$

- Inserting the solution of the dilatation Ward identity we find that we need

$$\Delta_1 = \Delta_2$$

Scalar 2-point function

The general solution of the conformal Ward identities is:

$$\langle O_\Delta(\boldsymbol{p}) O_\Delta(-\boldsymbol{p})\rangle = c_{12} p^{2\Delta-d}.$$

- This solution is trivial when

$$\Delta = \frac{d}{2} + k, \qquad k = 0, 1, 2, \ldots$$

  because then correlator is local,

$$\langle O(\boldsymbol{p}) O(-\boldsymbol{p})\rangle = c p^{2k} \to \langle O(\boldsymbol{x}_1) O(\boldsymbol{x}_2)\rangle \sim \Box^k \delta(x_1 - x_2)$$

- Let ϕ_0 be the source of O. It has dimension $d-\Delta = d/2 - k$. The term

$$\phi_0 \Box^k \phi_0$$

  has dimension d and can act as a local counterterm.

Position space [Petkou, KS (1999)]

- In position space, it seems that none of these are an issue:

$$\langle \mathcal{O}(\boldsymbol{x})\mathcal{O}(0)\rangle = \frac{C}{x^{2\Delta}}$$

- This expression however is valid only at separated points, $x^2 \neq 0$.
- Correlation functions should be well-defined distributions and they should have well-defined Fourier transform.
- Fourier transforming we find:

$$\int d^d\boldsymbol{x}\; e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\,\frac{1}{x^{2\Delta}} = \frac{\pi^{d/2}2^{d-2\Delta}\Gamma\left(\frac{d-2\Delta}{2}\right)}{\Gamma(\Delta)}p^{2\Delta-d},$$

- This is well-behaved, except when $\Delta = d/2 + k$, where k is a positive integer.

Strategy

- Regularize the theory.
- Solve the Ward identities in the regulated theory.
- Renormalize by adding appropriate counterterms.

➠ The renormalised theory may be anomalous.

Regularization

- We use dimensional regularisation to regulate the theory

$$d \mapsto d + 2u\epsilon, \qquad \Delta_j \mapsto \Delta_j + (u+v)\epsilon$$

- In the regulated theory, the solution of the Ward identities is the same as before but the integration constant may depend on the regulator,

$$\langle O(\boldsymbol{p}) O(-\boldsymbol{p}) \rangle_{\text{reg}} = c(\epsilon, u, v) p^{2\Delta - d + 2v\epsilon}.$$

Regularization and Renormalization

$$\langle O(\boldsymbol{p})O(-\boldsymbol{p})\rangle_{\text{reg}} = c(\epsilon, u, v) p^{2\Delta - d + 2v\epsilon}.$$

- Now, in local CFTs:

$$c(\epsilon, u, v) = \frac{c^{(-1)}(u, v)}{\epsilon} + c^{(0)}(u, v) + O(\epsilon)$$

- This leads to

$$\langle O(\boldsymbol{p})O(-\boldsymbol{p})\rangle_{\text{reg}} = p^{2k}\left[\frac{c^{(-1)}}{\epsilon} + c^{(-1)} v \log p^2 + c^{(0)} + O(\epsilon)\right].$$

- We need to renormalise ....

Renormalization

- Let ϕ_0 the source that couples to O,

$$S[\phi_0] = S_0 + \int d^{d+2u\epsilon}\boldsymbol{x}\, \phi_0 O.$$

- The divergence in the 2-point function can be removed by the addition of the counterterm action

$$S_{\text{ct}} = a_{\text{ct}}(\epsilon, u, v) \int d^{d+2u\epsilon}\boldsymbol{x}\, \phi_0 \Box^k \phi_0 \mu^{2v\epsilon},$$

- Removing the cut-off we obtain the renormalised correlator:

$$\langle O(\boldsymbol{p}) O(-\boldsymbol{p})\rangle_{ren} = p^{2k}\left[C \log\frac{p^2}{\mu^2} + C_1\right]$$

Anomalies

- The counter term breaks scale invariance and as result the theory has a conformal anomaly.
- The 2-point function depends on a scale [Petkou, KS (1999)]

$$\mathcal{A}_2 = \mu \frac{\partial}{\partial \mu} \langle O(\boldsymbol{p}) O(-\boldsymbol{p}) \rangle = c p^{2\Delta - d},$$

- The integrated anomaly is Weyl invariant

$$A = \int d^d \boldsymbol{x} \; \phi_0 \Box^k \phi_0$$

On a curved background, $\Box^k$ is replaced by the "k-th power of the conformal Laplacian", P^k.

Outline

Scalar 3-point functions

We would now like to understand 3-point functions at the same level:

- What is the general solution of the conformal Ward identities?
- What is the analogue of the condition

$$\Delta = \frac{d}{2} + k, \qquad k = 0, 1, 2, \ldots$$

- Are there new conformal anomalies associated with 3-point functions and if yes what is their structure?

Conformal Ward identities

➢ Dilatation Ward identity

$$0 = \left[2d - \Delta_t + \sum_{j=1}^{3} p_j \frac{\partial}{\partial p_j} \right] \langle O_1(\boldsymbol{p}_1) O_2(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle$$

$\Delta_t = \Delta_1 + \Delta_2 + \Delta_3$

➠ The correlation is a homogenous function of degree $(2d - \Delta_t)$.

➢ The special conformal Ward identities give rise to two scalar 2nd order PDEs.

Special conformal Ward identities

- Special conformal WI

$$0 = \mathrm{K}_{12} \langle \mathcal{O}_1(\boldsymbol{p}_1) \mathcal{O}_2(\boldsymbol{p}_2) \mathcal{O}_3(\boldsymbol{p}_3) \rangle = \mathrm{K}_{23} \langle \mathcal{O}_1(\boldsymbol{p}_1) \mathcal{O}_2(\boldsymbol{p}_2) \mathcal{O}_3(\boldsymbol{p}_3) \rangle,$$

where

$$\begin{aligned} \mathrm{K}_{ij} &= \mathrm{K}_i - \mathrm{K}_j, \\ \mathrm{K}_j &= \frac{\partial^2}{\partial p_j^2} + \frac{d + 1 - 2\Delta_j}{p_j} \frac{\partial}{\partial p_j}, \ (i, j = 1, 2, 3). \end{aligned}$$

- This system of differential equations is precisely that defining Appell's F_4 generalised hypergeometric function of two variables. [Coriano, Rose, Mottola, Serino][Bzowski, McFadden, KS] (2013).

Scalar 3-point functions

- There are four linearly independent solutions of these equations.
- Three of them have unphysical singularities at certain values of the momenta leaving one physically acceptable solution.
- We thus recover the well-known fact that scalar 3-point functions are determined up to a constant.

Scalar 3-pt functions and triple-K integrals

- The physically acceptable solution has the following *triple-K integral* representation:

$$\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle = C_{123} p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \int_0^\infty dx\, x^{\frac{d}{2}-1} K_{\Delta_1-\frac{d}{2}}(p_1 x) K_{\Delta_2-\frac{d}{2}}(p_2 x) K_{\Delta_3-\frac{d}{2}}(p_3 x),$$

where $K_\nu(p)$ is a Bessel function and C_{123} is an constant.

- This is the general solution of the conformal Ward identities.

Triple K-integrals

- Triple-K integrals,

$$I_{\alpha\{\beta_1\beta_2\beta_3\}}(p_1, p_2, p_3) = \int_0^\infty dx\, x^\alpha \prod_{j=1}^{3} p_j^{\beta_j} K_{\beta_j}(p_j x),$$

are the building blocks of all 3-point functions.

- The integral converges provided

$$\alpha > \sum_{j=1}^{3} |\beta_j| - 1$$

- The integral can be defined by analytic continuation when

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 \neq -2k,$$

where k is any non-negative integer.

Renormalization and anomalies

- If the equality holds,

$$\alpha + 1 \pm \beta_1 \pm \beta_2 \pm \beta_3 = -2k,$$

the integral cannot be defined by analytic continuation.

- Non-trivial subtractions and renormalization may be required and this may result in conformal anomalies.
- Physically when this equality holds, there are new terms of dimension d that one can add to the action (counterterms) and/or new terms that can appear in T^μ_μ (conformal anomalies).

Scalar 3-pt function

- For the triple-K integral that appears in the 3-pt function of scalar operators the condition becomes

$$\frac{d}{2} \pm (\Delta_1 - \frac{d}{2}) \pm (\Delta_2 - \frac{d}{2}) \pm (\Delta_3 - \frac{d}{2}) = -2k$$

- There are four cases to consider, according to the signs needed to satisfy this equation. We will refer to the 4 cases as the $(- - -)$, $(- - +)$, $(- + +)$ and $(+ + +)$ cases.
- Given Δ_1, Δ_2 and Δ_3 these relations may be satisfied with more than one choice of signs and k.

Procedure

- To analyse the problem we will proceed by using dimensional regularisation

$$d \mapsto d + 2u\epsilon, \qquad \Delta_j \mapsto \Delta_j + (u+v)\epsilon$$

- In the regulated theory the solution of the conformal Ward identity is given in terms of the triple-K integral but now the integration constant C_{123} in general will depend on the regulator ϵ, u, v.
- We need to understand the singularity structure of the triple-K integrals and then renormalise the correlators.
- We will discuss each case in turn.

The $(- - -)$ case

$$\Delta_1 + \Delta_2 + \Delta_3 = 2d + 2k$$

- This the analogue of the $\Delta = d/2 + k$ case in 2-point functions.
- There are possible counterterms

$$S_{\text{ct}} = a_{\text{ct}}(\epsilon, u, v) \int d^d\boldsymbol{x}\, \Box^{k_1}\phi_1 \Box^{k_2}\phi_2 \Box^{k_3}\phi_3$$

  where $k_1 + k_2 + k_3 = k$. The same terms may appear in T^μ_μ as new conformal anomalies.
- After adding the contribution of the countertrems one may remove the regulator to obtain the renormalised correlator.

Example: $\Delta_1 = \Delta_2 = \Delta_3 = 2$, $d = 3$

- The source ϕ for an operator of dimension 2 has dimension 1, so ϕ^3 has dimension 3.
- Regularizing:

$$\langle \mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = C_{123}\left(\frac{\pi}{2}\right)^{3/2}\int_0^\infty dx\, x^{-1+\epsilon} e^{-x(p_1+p_2+p_3)}$$

$$= C_{123}\left(\frac{\pi}{2}\right)^{3/2}\left[\frac{1}{\epsilon} - (\gamma_E + \log(p_1+p_2+p_3)) + O(\epsilon)\right].$$

Renormalization and anomalies

- We add the counterterm

$$S_{ct} = -\frac{C_{123}}{3!\epsilon}\left(\frac{\pi}{2}\right)^{3/2}\int d^{3+2\epsilon}\boldsymbol{x}\,\phi^3\mu^{-\epsilon}$$

- This leads to the renormalized correlator,

$$\langle\mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = -C_{123}\left(\frac{\pi}{2}\right)^{3/2}\log\frac{p_1+p_2+p_3}{\mu}$$

- The renormalized correlator is not scale invariant

$$\mu\frac{\partial}{\partial\mu}\langle\mathcal{O}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = C_{123}\left(\frac{\pi}{2}\right)^{3/2}$$

- There is a new conformal anomaly:

$$\langle T\rangle = -\phi\langle\mathcal{O}\rangle + \frac{1}{3!}C_{123}\left(\frac{\pi}{2}\right)^{3/2}\phi^3.$$

The $(- - +)$ case

$$\Delta_1 + \Delta_2 - \Delta_3 = d + 2k$$

- In this case the new local term one can add to the action is

$$S_{\rm ct} = a_{\rm ct} \int d^d x \Box^{k_1} \phi_1 \Box^{k_2} \phi_2 O_3$$

  where $k_1 + k_2 = k$.
- In this case we have renormalization of sources,

$$\phi_3 \to \phi_3 + a_{\rm ct} \Box^{k_1} \phi_1 \Box^{k_2} \phi_2$$

- The renormalised correlator will satisfy a Callan-Symanzik equation with beta function terms.

Callan-Symanzik equation

- The quantum effective action $\mathcal{W}$ (the generating functional of renormalised connected correlators) obeys the equation

$$\mu \frac{d}{d\mu}\mathcal{W} = \Big(\mu \frac{\partial}{\partial \mu} + \sum_i \int d^d\vec{x}\, \beta_i \frac{\delta}{\delta \phi_i(\vec{x})}\Big)\mathcal{W} = \int d^d\vec{x}\, \mathcal{A},$$

- This implies that for 3-point functions we have

$$\mu \frac{\partial}{\partial \mu}\langle O_i(p_1) O_j(p_2) O_j(p_3)\rangle =$$

$$\beta_{j,ji}\big(\langle O_j(p_2)O_j(-p_2)\rangle + \langle O_j(p_3)O_j(-p_3)\rangle\big) + \mathcal{A}^{(3)}_{ijj},$$

$$\beta_{i,jk} = \frac{\delta^2 \beta_i}{\delta\phi_j \delta\phi_k}\Big|_{\{\phi_l\}=0}, \quad \mathcal{A}^{(3)}_{ijk}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = -\frac{\delta^3}{\delta\phi_i(\vec{x}_1)\delta\phi_j(\vec{x}_2)\delta\phi_k(\vec{x}_3)} \int d^d\vec{x}\, \mathcal{A}(\{\phi_l(\vec{x})\})$$

Example: $\Delta_1 = 4, \Delta_2 = \Delta_3 = 3$ in $d = 4$

- $\Delta_1 + \Delta_2 + \Delta_3 = 10 = 2d + 2k$, which satisfies the $(- - -)$-condition with $k = 1$.
- There is an anomaly

$$\int d^d x \phi_0 \phi_1 \Box \phi_1$$

- $\Delta_1 + \Delta_2 - \Delta_3 = 4 = d + 2k$, which satisfies the $(- - +)$ condition with $k = 0$. The following counterterm is needed,

$$\int d^4 x \phi_0 \phi_1 O_3$$

- There is a beta function for ϕ_1.

$\langle O_4 O_3 O_3 \rangle$

$$
\begin{aligned}
\langle O_4(\boldsymbol{p}_1) O_3(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle &= \alpha \left(2 - p_1 \frac{\partial}{\partial p_1} \right) I^{\text{(non-local)}} \\
+ \ \frac{\alpha}{8} \Bigg[& (p_2^2 - p_3^2) \log \frac{p_1^2}{\mu^2} \left(\log \frac{p_3^2}{\mu^2} - \log \frac{p_2^2}{\mu^2} \right) \\
& - (p_2^2 + p_3^2) \log \frac{p_2^2}{\mu^2} \log \frac{p_3^2}{\mu^2} \\
& (p_1^2 - p_2^2) \log \frac{p_3^2}{\mu^2} + (p_1^2 - p_3^2) \log \frac{p_2^2}{\mu^2} + p_1^2 \Bigg]
\end{aligned}
$$

$\langle O_4 O_3 O_3 \rangle$

$$
\begin{aligned}
I^{(\text{non-local})} &= \langle O_2 O_2 O_2 \rangle \\
&= -\frac{1}{8}\sqrt{-J^2}\left[\frac{\pi^2}{6} - 2\log\frac{p_1}{p_3}\log\frac{p_2}{p_3} + \log\left(-X\frac{p_2}{p_3}\right)\log\left(-Y\frac{p_1}{p_3}\right)\right. \\
&\qquad \left. - Li_2\left(-X\frac{p_2}{p_3}\right) - Li_2\left(-Y\frac{p_1}{p_3}\right)\right],
\end{aligned}
$$

$$
J^2 = (p_1 + p_2 - p_3)(p_1 - p_2 + p_3)(-p_1 + p_2 + p_3)(p_1 + p_2 + p_3),
$$

$$
X = \frac{p_1^2 - p_2^2 - p_3^2 + \sqrt{-J^2}}{2p_2p_3}, \qquad Y = \frac{p_2^2 - p_1^2 - p_3^2 + \sqrt{-J^2}}{2p_1p_3}.
$$

Callan-Symanzik equation

- It satisfies

$$\mu \frac{\partial}{\partial \mu} \langle O_4(\boldsymbol{p}_1) O_3(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle =$$
$$\frac{\alpha}{2} \left(p_2^2 \log \frac{p_2^2}{\mu^2} + p_3^2 \log \frac{p_3^2}{\mu^2} - p_1^2 + \frac{1}{2}(p_2^2 + p_3^2) \right).$$

- This is indeed the correct Callan-Symanzik equation. (Recall that $\langle O_3(\boldsymbol{p}) O_3(\boldsymbol{p}) \rangle = p^2 \log \frac{p^2}{\mu^2}$)

The $(+++)$ and $(-++)$ cases

- In these cases it is the representation of the correlator in terms of the triple-K integral that is singular, not the correlator itself,

$$\langle \mathcal{O}_1(\boldsymbol{p}_1)\mathcal{O}_2(\boldsymbol{p}_2)\mathcal{O}_3(\boldsymbol{p}_3)\rangle = C_{123} p_1^{\Delta_1-\frac{d}{2}} p_2^{\Delta_2-\frac{d}{2}} p_3^{\Delta_3-\frac{d}{2}} \times I_{d/2-1,\{\Delta_1-d/2,\Delta_3-d/2,\Delta_3-d/2\}}$$

  Taking the integration constant $C_{123} \sim \epsilon^m$ for appropriate m and sending $\epsilon \to 0$ results in an expression that satisfies the original (non-anomalous) Ward identity.
- In other words, the Ward identities admit a solution that is finite.

The $(-++)$ case

$$\Delta_1 - \Delta_2 - \Delta_3 = 2k$$

- For $k = 0$, there are "extremal correlators" . In position, the 3-point function is a product of 2-point functions

$$\langle O_1(\boldsymbol{x}_1) O_2(\boldsymbol{x}_2) O_3(\boldsymbol{x}_3) \rangle = \frac{c_{123}}{|\boldsymbol{x}_2 - \boldsymbol{x}_1|^{2\Delta_2} |\boldsymbol{x}_3 - \boldsymbol{x}_1|^{2\Delta_3}}$$

- In momentum space, the finite solution to the Ward identities is

$$\langle O_1(\boldsymbol{p}_1) O_2(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle = c p_2^{(2\Delta_2 - d)} p_3^{(2\Delta_3 - d)}$$

- When the operators have these dimensions there are "multi-trace" operators of dimension Δ_1

$$\mathcal{O} = \Box^{k_2} O_2 \Box^{k_3} O_3$$

where $k_2 + k_3 = k$.

The $(+++)$ case

$$\Delta_1 + \Delta_2 + \Delta_3 = d - 2k$$

- For example, for $k = 0$ the finite solution to the Ward identities is

$$\langle O_1(\boldsymbol{p}_1) O_2(\boldsymbol{p}_2) O_3(\boldsymbol{p}_3) \rangle = c p_1^{(\Delta_1 - \Delta_2 - \Delta_3)} p_2^{(\Delta_2 - \Delta_1 - \Delta_3)} p_3^{(\Delta_3 - \Delta_1 - \Delta_2)}$$

- When the operators have these dimensions there are "multi-trace" operators which are classically marginal

$$\mathcal{O} = \Box^{k_1} O_1 \Box^{k_2} O_2 \Box^{k_3} O_3$$

where $k_1 + k_2 + k_3 = k$.

Outline

$\Delta_1 + \Delta_2 + \Delta_3 = d$

- It turns out that these 3-point functions are also invariant under conformal transformation but now acting in momenta: dual conformal invariance.
- In terms of $\vec{p}_1 = \vec{y}_{23}, \vec{p}_2 = \vec{y}_{31}, \vec{p}_3 = \vec{y}_{12},$, where $\vec{y}_{ij} = \vec{y}_i - \vec{y}_j$, the 3-point function becomes

$$\langle O_{\Delta_1} O_{\Delta_2} O_{\Delta_3} \rangle = \frac{C_{123}}{|y_{23}|^{\Delta_2+\Delta_3-\Delta_1} |y_{31}|^{\Delta_3+\Delta_1-\Delta_2} |y_{12}|^{\Delta_1+\Delta_2-\Delta_3}}.$$

- This is invariant under usual conformal transformations acting on $\vec{y}$.

Dual conformal invariance

- Scattering amplitudes in planar $N = 4$ SYM exhibit dual (super)conformal symmetry [Drummond et al (2008)].
- This is believed to be linked with the integrability of the theory.
- Is there a relation?

Outline

Tensorial correlators

- New issues arise for tensorial correlation functions, such as those involving stress-energy tensors and conserved currents.
- Lorentz invariance implies that the tensor structure will be carried by tensors constructed from the momenta p^μ and the metric $\delta_{\mu\nu}$.
- After an appropriate parametrisation, the analysis becomes very similar to the one we discussed here.
- In particular, these correlator are also given in terms of triple-K integrals.

Diffeomorphism and Weyl Ward identities

- The fact that classically a current and the stress-energy tensor are conserved implies that n-point functions involving insertions of $\partial_\alpha J^\alpha$ or $\partial_\alpha T^{\alpha\beta}$ can be expressed in terms of lower-point functions without such insertions.
- The same holds for correlation functions with insertions of the trace of the stress-energy tensor.
- The first step in our analysis is to implement these Ward identities. We do this by providing reconstruction formulae that yield the full 3-point functions involving stress-energy tensors/currents/scalar operators starting from expressions that are exactly conserved/traceless.

Example: $\langle T^{\mu\nu}\mathcal{O}\mathcal{O}\rangle$

- Ward identities

$$p_1^\nu \langle T_{\mu\nu}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = p_{3\mu}\langle \mathcal{O}(\boldsymbol{p}_3)\mathcal{O}(-\boldsymbol{p}_3)\rangle + (\boldsymbol{p}_2 \leftrightarrow \boldsymbol{p}_3)$$
$$\langle T^\mu_\mu(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = -\Delta\langle \mathcal{O}(\boldsymbol{p}_3)\mathcal{O}(-\boldsymbol{p}_3)\rangle + (\boldsymbol{p}_2 \leftrightarrow \boldsymbol{p}_3)$$

- Reconstruction formula

$$\langle T^{\mu\nu}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle = \langle t^{\mu\nu}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle$$
$$+ \left[p_2^\alpha \mathscr{T}^{\mu\nu}_\alpha(\boldsymbol{p}_1) - \frac{\Delta}{d-1}\pi^{\mu\nu}(\boldsymbol{p}_1)\right] \langle \mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(-\boldsymbol{p}_2)\rangle + (\boldsymbol{p}_2 \leftrightarrow \boldsymbol{p}_3),$$

where $\langle t^{\mu\nu}(\boldsymbol{p}_1)\mathcal{O}(\boldsymbol{p}_2)\mathcal{O}(\boldsymbol{p}_3)\rangle$ is transverse-traceless and

$$\pi^{\mu\nu}(\boldsymbol{p}) = \delta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}, \quad \mathscr{T}^{\mu\nu}_\alpha(\boldsymbol{p}) = \frac{1}{p^2}\left[2p^{(\mu}\delta^{\nu)}_\alpha - \frac{p_\alpha}{d-1}\left(\delta^{\mu\nu} + (d-2)\frac{p^\mu p^\nu}{p^2}\right)\right]$$

Tensorial decomposition

- It remains to determine the transverse-traceless part of the correlator which is undetermined by the Weyl and diffeomorphism Ward identities.
- We now use Lorentz invariance to express the transverse-traceless correlator in terms of scalar form factors.

Example: $\langle T^{\mu\nu} \mathcal{O}\mathcal{O} \rangle$

- The tensorial decomposition of $\langle t^{\mu\nu}(\boldsymbol{p}_1) \mathcal{O}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle$ involves one form factor:

$$\langle t^{\mu\nu}(\boldsymbol{p}_1) \mathcal{O}(\boldsymbol{p}_2) \mathcal{O}(\boldsymbol{p}_3) \rangle = \Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}_1) A_1(p_1, p_2, p_3) p_2^\alpha p_2^\beta,$$

where $\Pi^{\mu\nu}_{\alpha\beta}(\boldsymbol{p}_1)$ is a projection operator into transverse traceless tensors.

- The state-of-the-art decomposition of $\langle T_{\mu_1\nu_1} T_{\mu_2\nu_2} T_{\mu_3\nu_3} \rangle$ [Cappelli et al (2001)] prior to this work involved 13 form factors, while the method described here requires only 5.

Dilatation and special conformal Ward identities

It remains to impose the dilatation and special conformal Ward identities.

- The dilatation Ward identity imply that the form factors are homogeneous functions of the momenta of specific degree.
- The special conformal Ward identities (CWI) imply that that the form factors satisfy certain differential equations.
- These split into two categories:
  1. The primary CWIs. Solving these determines the form factors up to constants (primary constants).
  2. The secondary CWIs. These impose relation among the primary constants.

Example: $\langle T^{\mu\nu} \mathcal{O} \mathcal{O} \rangle$ – primary CWI

- The primary CWI is

$$\mathrm{K}_{ij} \, A_1 = 0, \qquad i, j = 1, 2, 3.$$

- This is precisely the same equation we saw earlier in the analysis of $\langle \mathcal{O} \mathcal{O} \mathcal{O} \rangle$.
- The general solution in given in terms of a triple-K integral

$$A_1 = \alpha_1 I_{d/2+1\{\Delta-d/2,\Delta-d/2,\Delta-d/2\}},$$

where α_1 is constant (primary constant).

Example: $\langle T^{\mu\nu}\mathcal{O}\mathcal{O}\rangle$ – secondary CWI

- The secondary CWI is

$$\left(c_1(p)\frac{\partial}{\partial p_1} + c_2(p)\frac{\partial}{\partial p_2} + c_3(p)\right) A_1 \sim \langle \mathcal{O}\mathcal{O}\rangle$$

  where $c_1(p), c_2(p), c_3(p)$ are specific polynomials of the momenta.

- This equation then determines the primary constant α_1 in terms of the normalization of the 2-point function of $\mathcal{O}$.
- $\langle T^{\mu\nu}\mathcal{O}\mathcal{O}\rangle$ is completely determined, including constants, by conformal invariance.

The general case

- If we have n form factors then the structure of the primary CWI is

$$
\begin{aligned}
&\mathrm{K}_{12}\, A_1 = 0, && \mathrm{K}_{13}\, A_1 = 0,\\
&\mathrm{K}_{12}\, A_2 = c_{21} A_1, && \mathrm{K}_{13}\, A_2 = d_{21} A_1,\\
&\mathrm{K}_{12}\, A_3 = c_{31} A_1 + c_{32} A_2, && \mathrm{K}_{13}\, A_3 = d_{31} A_1 + d_{32} A_2,\\
&\ldots && \ldots\\
&\mathrm{K}_{12}\, A_n = \sum_{j=1}^{n-1} c_{nj} A_j && \mathrm{K}_{13}\, A_n = \sum_{j=1}^{n-1} d_{nj} A_j
\end{aligned}
$$

  where c_{ij}, d_{ij} are lower triangular matrices with constant matrix elements.
- These equations can be solved in terms of triple-K integrals.
- The solution depends on n primary constants, one for each form factor.

Example: $\langle T_{\mu_1\nu_1} T_{\mu_2\nu_2} T_{\mu_3\nu_3} \rangle$

- In $d > 3$ there are 5 form factors and thus 5 primary constants.
- The secondary CWI impose additional constraints and we are left with the normalization c_T of the 2-point function of $T_{\mu\nu}$ and two additional constants.
- In $d = 4$ the normalization of the 2-point function and one constant can be traded for the conformal anomaly coefficients, c and a.
- Thus, in $d = 4$ the conformal anomaly determines $\langle T_{\mu_1\nu_1} T_{\mu_2\nu_2} T_{\mu_3\nu_3} \rangle$ up to one constant.

Example: $\langle T_{\mu_1\nu_1} T_{\mu_2\nu_2} T_{\mu_3\nu_3} \rangle$ in $d = 3$

- In $d = 3$ there are only 2 form factors and thus 2 primary constants.
- The secondary Ward identities relates one of them with the normalization of the 2-point function.

Remarks

- In the same manner one can obtain all three-point functions involving the stress-energy tensor $T_{\mu\nu}$, conserved currents J^a_μ and scalar operators.
  - In odd dimensions the triple-K integrals reduce to elementary integrals and can be computed by elementary means.
  - In even dimensions the evaluation of the triple-K integrals is non-trivial.
- In special cases, which include all 3-point functions of $T_{\mu\nu}$ and J^a_μ in even dimensions, non-trivial renormalization is needed.

Higher-point functions?

- The complexity of the analysis increases because the number of possible tensor structure and thus form factors increases.
- The form factors now depend also on the number of independent scalar products

$$p_{ij} = \boldsymbol{p}_i \cdot \boldsymbol{p}_j, \quad i,j = 1,2,\ldots,n, \quad i<j$$

- The number of independent such scalar products is $n(n-3)/2$.
- This is equal to the number of independent cross-ratios.

Outline

Conclusions

- We obtained the implications of conformal invariance for three-point functions working in momentum space.
- We discussed renormalization and anomalies.
- The presence of "beta function" terms in the Callan - Symanzik equation for CFT correlators is new.
- We found that certain 3-point functions enjoy dual conformal invariance.

Outlook

- It would be interesting to understand how to extend the analysis to higher point functions. What is the momentum space analogue of cross-ratios?
- What are the implications of the dual conformal symmetry?
- Bootstrap in momentum space?