Integrability and magnon kinematics in the AdS/CFT correspondence†

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(See also the talk by E. López)

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Introduction

The AdS/CFT correspondence:

The large N limit of $\mathcal{N} = 4$ Yang-Mills is dual to type IIB string theory on $AdS_5 \times S^5$ $\Rightarrow$ Spectra of both theories should agree

$\longrightarrow$ Difficult to test, because the correspondence is a strong/weak coupling duality: we can not use perturbation theory on both sides

String energies expanded at large λ

$$E(\lambda) = \lambda^{1/4} E_0 + \lambda^{-1/4} E_1 + \lambda^{-3/4} E_2 + \dots$$

Scaling dimensions of gauge operators at small λ

$$\Delta(\lambda) = D_0 + \lambda D_1 + \lambda^2 D_2 + \dots$$

$$\boxed{E(\lambda) \leftrightarrow \Delta(\lambda)}$$

$\longrightarrow$ **Integrability illuminates both sides of the correspondence**

$\longrightarrow$ S_{string} **should interpolate to** S_{gauge}

Integrability in the AdS/CFT correspondence

A complete formulation of the AdS/CFT correspondence $\Rightarrow$ Precise identification of **string states** with local **gauge invariant operators**

$$\Rightarrow E\sqrt{\alpha'} = \Delta$$

Strong evidence in the **supergravity regime**, $R^2 \gg \alpha'$ $(R^4 = 4\pi g_{YM}^2 N \alpha'^2)$

Difficulties:
- String quantization in $AdS_5 \times S^5$
- Obtaining the whole spectrum of $\mathcal{N} = 4$ is involved

An insight: There is a maximally supersymmetric **plane-wave background** for the IIB string [Blau et al]

$$\Downarrow$$

Plane-wave geometry $\Rightarrow$ **Penrose limit**

The Penrose limit shows up on the field theory side [Berenstein, Maldacena, Nastase]

$\Downarrow$

Operators carrying **large charges**, $\boxed{\mathrm{tr}\,(X_1^J \ldots), J \gg 1}$

$\rightarrow$ Dual description in terms of **small closed strings** whose center moves **with angular momentum** J along a circle in S^5 [Gubser, Klebanov, Polyakov]

Generalization: Operators of the form $\mathrm{tr}\,(X_1^{J_1} X_2^{J_2} X_3^{J_3})$ are dual to strings with angular momenta J_i [Frolov, Tseytlin]

$\Rightarrow$ The energy of these **semiclassical strings** admits an analytic expansion in λ/J^2

$$E = J\left[1 + c_1\left(\frac{J_i}{J}\right)\frac{\lambda}{J^2} + \ldots\right]$$

$\Rightarrow$ Comparison with anomalous dimensions of large Yang-Mills operators:

- Bare dimension $\Delta_0 \rightarrow J$
- **One-loop** anomalous dimension $\rightarrow \frac{\lambda}{J} c_1 \left(\frac{J_i}{J}\right)$

Verifying AdS/CFT in **large** spin sectors $\Longrightarrow$ Computation of the anomalous dimensions of **large operators**

(Difficult problem due to **operator mixing**)

Insightful solution:

$\longrightarrow$ The **one-loop planar dilatation operator** of $\mathcal{N}=4$ Yang-Mills leads to an integrable spin chain ($SO(6)$ in the scalar sector [Minahan,Zarembo] or $PSU(2,2|4)$ in the complete theory [Beisert,Staudacher])

Single trace operators can be mapped to states in a closed spin chain $\Rightarrow$ BMN **impurities**: **magnon** excitations

$$\mathrm{tr}(XXXYYX\ldots) \leftrightarrow |\uparrow\uparrow\uparrow\downarrow\downarrow\uparrow\ldots\rangle$$

The Bethe ansatz

→ The rapidities u_j parameterizing the momenta of the magnons satisfy a set of **one-loop Bethe equations**

$$e^{ip_j J} \equiv \left(\frac{u_j + i/2}{u_j - i/2}\right)^J = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} \equiv \prod_{k\neq j}^{M} S(u_j, u_k)$$

Thermodynamic limit: **integral equations**

→ **Assuming integrability** an **asymptotic long-range Bethe ansatz** has been proposed [Beisert,Dippel,Staudacher]

$$\left(\frac{x_j^+}{x_j^-}\right)^J = \prod_{k\neq j}^{M} \frac{u_j - u_k + i}{u_j - u_k - i} = \prod_{k\neq j}^{M} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+} \frac{1 - \lambda/16\pi^2 x_j^+ x_k^-}{1 - \lambda/16\pi^2 x_j^- x_k^+}$$

where $x_j^\pm$ are generalized rapidities

$$x_j^\pm \equiv x(u_j \pm i/2) \ , \quad x(u) \equiv \frac{u}{2} + \frac{u}{2}\sqrt{1 - 2\frac{\lambda}{8\pi^2}\frac{1}{u^2}}$$

The quantum string Bethe ansatz

String non-linear sigma model on the coset

$$\frac{PSU(2,2|4)}{SO(4,1)\times SO(5)}$$

Integrable

[Mandal,Suryanarayana,Wadia] [Bena,Polchinski,Roiban]

Admits a Lax representation: there is a family of flat connections $A(z)$, $dA(z)-A(z)\wedge A(z)=0$

$\Rightarrow$ Classical solutions of the sigma model are parameterized by an integral equation [Kazakov,Marshakov,Minahan,Zarembo]

$$\boxed{-\frac{x}{x^2-\frac{\lambda}{16\pi^2 J^2}}\frac{\Delta}{J}+2\pi k = 2\,\mathcal{P}\int_{\mathcal{C}} dx'\,\frac{\rho(x')}{x-x'} \qquad x\in\mathcal{C}}$$

Reminds of the **thermodynamic Bethe equations** for the spin chain ...

In fact, it **leads to the spin chain equations** when $\lambda/J^2\to 0$

The previous string integral equations are **classical/thermodynamic** equations

$\Downarrow$

Assuming integrability survives at the quantum level, a discretization would provide a **quantum string Bethe ansatz** [Arutyunov,Frolov,Staudacher]

The **quantum string Bethe ansatz** is [Arutyunov,Frolov,Staudacher]

$$\left(\frac{x_j^+}{x_j^-}\right)^J = \prod_{k\neq j}^{M} \frac{x_j^+ - x_k^-}{x_j^- - x_k^+}\,\frac{1-\lambda/16\pi^2 x_j^+ x_k^-}{1-\lambda/16\pi^2 x_j^- x_k^+}\, e^{2i\theta(x_j,x_k)}$$

The string and gauge theory ansätze **differ by a dressing phase factor!!**
The phase factor is given by

$$\theta_{12} = 2\sum_{r=2}^{\infty} c_r(\lambda)\Big(q_r(x_1)q_{r+1}(x_2) - q_{r+1}(x_1)q_r(x_2)\Big)$$

$\Big($ $q_r(p_i)$ are the conserved magnon charges

$$q_r(x^\pm) = \frac{i}{r-1}\left(\frac{1}{(x^+)^{r-1}} - \frac{1}{(x^-)^{r-1}}\right) \Big)$$

$\longrightarrow$ **The dressing phase coefficients $c_r(\lambda)$ should interpolate from the string to the gauge theory** (strong/weak-interpolation)

$\longrightarrow$ To recover the integrable structure of the classical string the coefficients must satisfy $c_r(\lambda) \to 1$ as $\lambda \to \infty$

$$\Downarrow$$

Explicit form of $c_r(\lambda)$ ...

To constrain **the string Bethe ansatz** and find the structure of the dressing phase we can compare with **one-loop corrections** to semiclassical strings

$$\Downarrow$$

The classical limit $c_r(\infty) = 1$ needs to be modified in order to include quantum corrections to the string

The S-matrix of AdS/CFT

The S-matrices of the (discrete) quantum string and the long-range gauge Bethe ansätze differ simply **by a phase** [Arutyunov,Frolov,Staudacher]

$$\boxed{S_{\text{string}}(p_1,p_2) = e^{i\,\theta(p_1,p_2)} S_{\text{gauge}}(p_1,p_2)}$$

The S-matrix can be determined explicitly

$\Downarrow$

The spin chain vacuum breaks the $PSU(2,2|4)$ symmetry algebra down to $(PSU(2|2)\times PSU(2|2)')\ltimes\mathbb{R}$, with $\mathbb{R}$ a shared central charge

The S-matrix is determined up to a scalar (dressing phase) factor

[Beisert] [Klose,McLoughlin,Roiban,Zarembo]

$$\mathcal{S}_{12} = S^0_{12}\; S^{SU(2|2)}_{12}\; S^{SU(2|2)'}_{12}$$

$$S^0_{12} = \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\,\frac{1-1/x_1^- x_2^+}{1-1/x_1^+ x_2^-}\, e^{2i\theta_{12}}$$

Symmetries of the scattering matrix

One-loop corrections to semiclassical strings

→ **One-loop corrections** are obtained from the spectrum of **quadratic fluctuations** around a classical solution [Frolov,Tseytlin] [Frolov,Park,Tseytlin]

→ They amount to empirical constraints on the string Bethe ansatz

→ Careful analysis of the one-loop sums over bosonic and fermionic frequencies [Schäfer-Nameki,Zamaklar,Zarembo] [Beisert,Tseytlin] [RH,López] [Freyhult,Kristjansen]

provides a compact form of the **first quantum correction**

[RH,López] [Gromov,Vieira]

$$\boxed{\begin{aligned} c_{r,s} &= \delta_{r+1,s} + \frac{1}{\sqrt{\lambda}} a_{r,s} \\ a_{r,s} &= -8\,\frac{(r-1)(s-1)}{(r+s-2)(s-r)} \end{aligned}}$$

Crossing symmetry and the dressing phase factor

Crossing symmetry

The structure of the complete S-matrix is [Beisert]

$$\mathcal{S}_{12} = S^0_{12}\left[\, S_{12}^{SU(2|2)}\, S_{12}^{SU(2|2)'}\,\right]$$

- Term in the bracket: determined by the **symmetries** (Yang-Baxter)
- The scalar coefficient is the dressing factor: constrained by unitarity and crossing ($\rightarrow$ **dynamics**) [Janik], which implies

$$\theta(x_1^\pm, x_2^\pm) + \theta(1/x_1^\pm, x_2^\pm) = -2i \log h(x_1^\pm, x_2^\pm)$$

with

$$h(x_1, x_2) = \frac{x_2^-}{x_2^+}\,\frac{x_1^- - x_2^+}{x_1^+ - x_2^+}\,\frac{1 - 1/x_1^- x_2^-}{1 - 1/x_1^+ x_2^-}$$

An expansion of both sides has been shown to agree, using the explicit form of the one-loop correction in $\theta(x_1, x_2)$ [Arutyunov,Frolov]

$$\Downarrow$$

The $\theta^{\text{one-loop}}(\lambda)$ phase is a solution of the crossing equations

Higher corrections

Idea: Search for coefficients to fit the expansion of the crossing function $h(x_1, x_2)$

This provides a **strong-coupling expansion** [Beisert,RH,López]

$$\boxed{c_{r,s} = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}}$$

for the coefficients in the dressing phase $\left(g \equiv \sqrt{\lambda}/4\pi\right)$

The **all-order** proposal is

$$\boxed{c_{r,s}^{(n)} = (r-1)(s-1)\,\mathrm{B}_n\,\mathcal{A}(r,s,n)}$$

with

$$\mathcal{A}(r,s,n) = \frac{\left((-1)^{r+s}-1\right)}{4\cos(\frac{1}{2}\pi n)\,\Gamma[n+1]\,\Gamma[n-1]} \times \frac{\Gamma[\frac{1}{2}(s+r+n-3)]}{\Gamma[\frac{1}{2}(s+r-n+1)]}\,\frac{\Gamma[\frac{1}{2}(s-r+n-1)]}{\Gamma[\frac{1}{2}(s-r-n+3)]}$$

$\longrightarrow$ Includes the **classical** and **one-loop** terms

$\longrightarrow$ An expansion dressed with the Bernoulli numbers

$\longrightarrow$ The **one-loop contribution** alone satisfies part of the crossing relation (odd crossing)

$\longrightarrow$ The remaining piece of the crossing condition is satisfied by the ***n*-loop contribution**, with n even

(The **solution** is however **not unique**: it is possible to include additional **homogeneous solutions** to the crossing constraints)

$\longrightarrow$ The phase shows **agreement with perturbative string theory** (semiclassical scattering of giant magnons [Hofman,Maldacena])

Weak-coupling expansion

→ The previous (strong-coupling) asymptotic expansion

$$c_{r,s} = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n}$$

agrees with the string theory regime

→ The **weak-coupling regime** is constrained by perturbative computations:

- Up to three-loops the phase $\theta(x_1, x_2)$ should remain zero
- A recent **four-loop** computation requires a **first non-vanishing piece** in the dressing phase [Bern,Czakon,Dixon,Kosower,Smirnov]

→ The four-loop result can be recovered from a **long-range Bethe ansatz computation** [Beisert,Eden,Staudacher] (See also [Benna,Benvenuti,Klebanov, Sardicchio] [Alday,Arutyunov,Benna,Eden,Klebanov] [Beccaria,DeAngelis,Forini] ...)

Quantum–deformed magnon kinematics

The previous succesful interpolation from the strong to the weak-coupling regime relies strongly on the long-range Bethe ansatz of [Beisert,Dippel,Staudacher]

We will now try to address two questions

$\longrightarrow$ What is the magnon kinematics underlying the long-range ansatz?

$\longrightarrow$ Is the gauge theory (the correspondence) really integrable?

Clarifying the **features of magnon kinematics** is indeed of great importance

$\Downarrow$

$\longrightarrow$ In $1+1$ relativistic theories physical conditions are used to constrain the S-matrix: **unitarity, bootstrap principle, crossing symmetry**

$\longrightarrow$ The remaining traditional condition is **Lorentz covariance** $\Rightarrow$ Forces dependence on the diference of rapidities

Let us briefly recall the way the long-range Bethe ansatz is constructed

→ The (one-loop) Heisenberg chain has dispersion relation

$$E = 4\sin^2\left(\frac{p}{2}\right)$$

→ The Bethe ansatz can be **deformed** to include the magnon dispersion relation for planar $\mathcal{N} = 4$ Yang-Mills,

$$E^2 = 1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)$$

The extension/deformation is the long-range Bethe ansatz

[Beisert,Dippel,Staudacher]

We will now try to uncover the **magnon kinematics** underlying planar $\mathcal{N} = 4$ Yang-Mills

→ In a $1+1$-dimensional relativistic theory particles transform in irreps of the Poincaré algebra, $E(1,1)$,

$$[J, P] = E \ , \quad [J, E] = P \ , \quad [E, P] = 0$$

An irrep is specified by a value of the Casimir operator

$$m^2 = E^2 - p^2$$

→ The dispersion relation in planar $\mathcal{N} = 4$ is a deformation of the usual relativistic relation ⇒ **There is an algebra whose Casimir has the adequate form!!**

⇓

It is a quantum deformation of the $1+1$ Poincaré algebra, $E_q(1,1)$

[Gómez,RH] [Young]

$E_q(1,1)$ is the algebra

$$\begin{aligned} KEK^{-1} &= E \ , \quad KJK^{-1} = J + iaE \ , \\ KK^{-1} &= \mathbb{1} \ , \quad JE - EJ = \frac{K - K^{-1}}{2ia} \end{aligned}$$

with **deformation parameter** $q = e^{ia}$ and $K = e^{iaP}$ (the limit $a \to 0$ corresponds to the usual Poincaré algebra)

Furthermore, the boost generator J can be used to introduce a uniformizing rapidity through $J = \frac{\partial}{\partial z}$. Then the algebra implies

$$\frac{\partial p}{\partial z} = \sqrt{1 + \frac{\lambda}{\pi^2} \sin^2\left(\frac{p}{2}\right)}$$

which provides a **elliptic uniformization** in terms of Jacobi functions

[Beisert] [RH,Gómez] [Kostov,Serban,Volin]

$$\sin\left(\frac{p}{2}\right) = k' \mathrm{sd}(z) \ , \quad E(z) = \frac{1}{2\mathrm{dn}(z)}$$

(The relativistic uniformization is $p(z) = m \sinh z$, $E(z) = m \cosh z$)

Semi-continuum limit in the Ising model

The quantum-deformed Poincaré, or the dispersion relation in planar $\mathcal{N}=4$ Yang-Mills, can in fact be obtained from the Ising model

→ The lattice spacings a_x and a_t can be mapped to the Ising couplings K and L ($*$ stands for the Kramers-Wannier dual)

$$\sinh 2L \sinh 2K^* = \left(\frac{a_x}{a_t}\right)^2 \quad , \quad 2\sinh(L-K^*) = \mu a_x$$

→ Define $\gamma \equiv p a_x$, $\omega \equiv E a_t$. Then **Onsager´s hypergeometric relation** becomes

$$\cosh\gamma = \cosh 2L \cosh 2K^* - \sinh 2L \sinh 2K^* \cos\omega$$
$$\Downarrow$$
$$a_t^2\big(\cosh p a_x - 1\big) + a_x^2\big(\cos E a_t - 1\big) = \tfrac{1}{2}\mu^2 a_t^2 a_x^2$$

$\rightarrow$ In the continuum limit $a_x, a_t \rightarrow 0$ we get $p^2 + E^2 = \mu^2$

The **semi-continuum limit** $a_t \rightarrow 0$ leads to the dispersion relation in planar $\mathcal{N} = 4$ Yang-Mills (after analytical continuation of p and the introduction of an effective scale through a_x)

In fact, the uniformization in planar $\mathcal{N} = 4$ is the same as that in the Ising model (cf [Baxter])

The Boltzmann weights are indeed made out from $x^{\pm}$

$$x^{\pm} = e^{2L} e^{\mp 2K}$$

(map by [Kostov,Serban,Volin])

and **integrability from the star-triangle relation** implies Beisert´s algebraic constraint

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{1}{4ig}$$

Conclusions

- Testing AdS/CFT in large spin sectors $\Rightarrow$ Integrability in the planar limit of $\mathcal{N}=4$ Yang-Mills: Precision tests of the correspondence
- Quantum corrections constrain the string Bethe ansatz
  - Simple form of the first correction
  - A crossing-symmetric phase has been suggested to higher orders
- A proof of the the AdS/CFT correspondence requires identification of spectra, together with interpolation as the coupling evolves
  - The dressing factor interpolates from the string to the gauge theory, and strong to weak-coupling

$$S_{st}(p_j, p_k) = e^{i\theta(p_j,p_k)} S_g(p_j, p_k)$$

- The quantum–deformed plane of magnon kinematics in planar $\mathcal{N}=4$ Yang-Mills has been identified

Open questions

- **Algebraic origin** of the structure of the dressing phase factor

  $\Downarrow$

  Underlying quantum group symmetry pattern organizing the gauge coupling evolution

  [Gómez,RH] [Plefka,Spill,Torrielli] [Arutyunov,Frolov,Plefka,Zamaklar] [Torrielli] [Beisert] [Moriyama,Torrielli] ...

- Is the AdS/CFT correspondence really **integrable**?

  $\Downarrow$

  What is the **origin** and **meaning** of integrability in the correspondence?