Posets of alternating sign matrices and totally symmetric self-complementary plane partitions

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May 18, 2015

Some failed attempts to find a certain bijection, and what has come of it

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Outline

1. Alternating sign matrices and totally symmetric self-complementary plane partitions
2. Poset structures
3. Toggle group dynamics
4. A permutation case bijection

Outline

Alternating sign matrix definition

Definition

Alternating sign matrices (ASMs) are square matrices with the following properties:

- entries $\in \{0, 1, -1\}$
- each row and each column sums to 1
- nonzero entries alternate in sign along a row/column

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Examples of alternating sign matrices

- All seven of the 3×3 ASMs.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

- Two of the forty-two 4×4 ASMs.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Enumeration - How many?

- In 1983, W. Mills, D. Robbins, and H. Rumsey conjectured that $n \times n$ ASMs are counted by:

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!} = \frac{1!4!7!\cdots(3n-2)!}{n!(n+1)!\cdots(2n-1)!}.$$

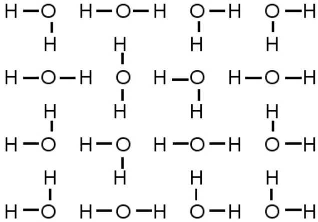
  1, 2, 7, 42, 429, 7436, 218348, 10850216, ...

- This was proved by Zeilberger (1996) and Kuperberg (1996). Kuperberg's proof relied on the connection to the six-vertex model.

Physics connection - Square ice

Alternating sign matrices are in bijection with configurations of the six-vertex model with domain wall boundary conditions.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$



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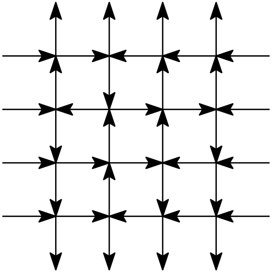

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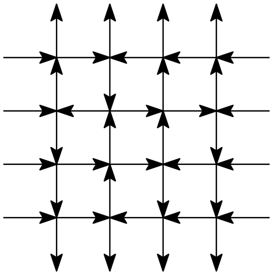

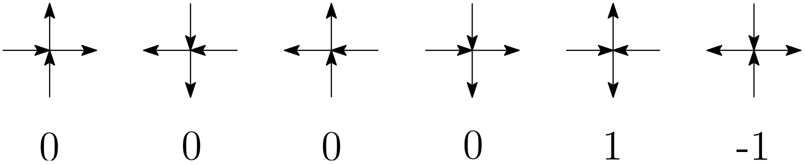

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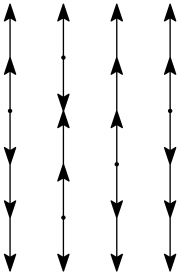

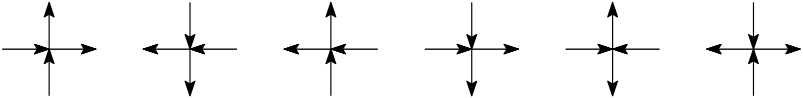

0 0 0 0 1 -1

Physics connection - Square ice

Alternating sign matrices are in bijection with configurations of the six-vertex model with domain wall boundary conditions.

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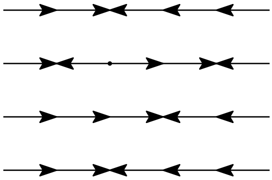

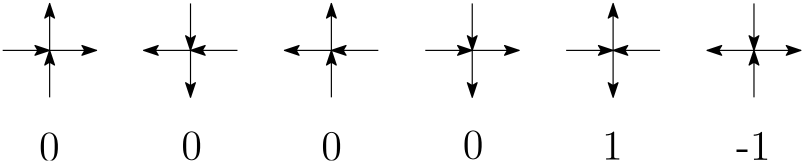

Physics connection - Square ice

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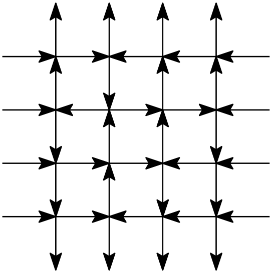
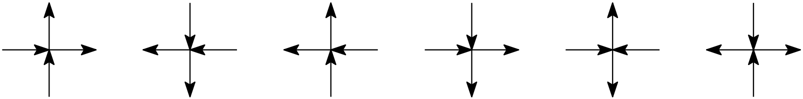
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$





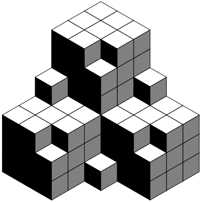
0 0 0 0 1 -1

Totally symmetric self–complementary plane partitions

Definition

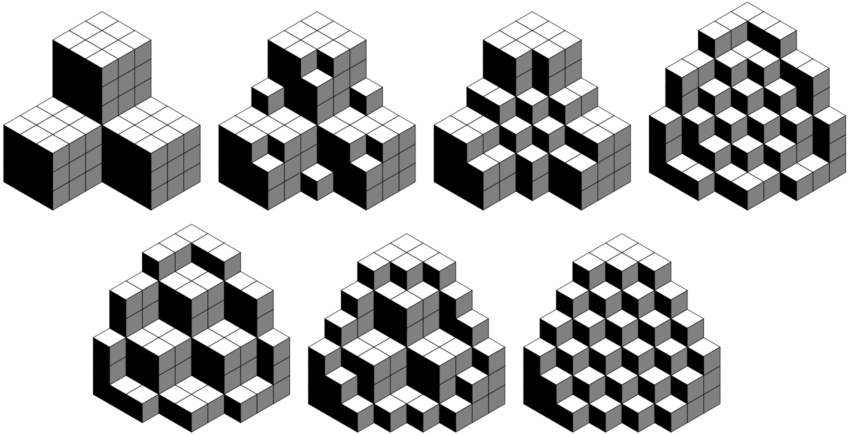
A totally symmetric self–complementary plane partition (TSSCPP) in a cube of side length $2n$ is:

- PP: A corner-justified stack of unit cubes
- TS: Invariant under all permutations of the axes
- SC: Equal to its complement inside the box



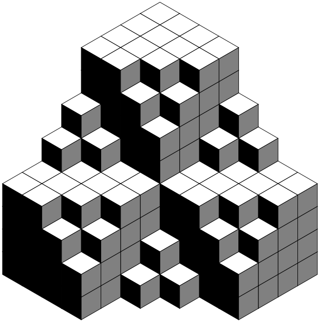
Examples of TSSCPPs

All seven of the TSSCPPs inside a $6 \times 6 \times 6$ box.



A missing bijection

Totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box are also counted by $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ (Andrews 1994), but **no explicit bijection is known**.



?

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Known alternating sign matrix bijections

ASM

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

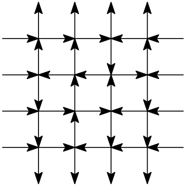
Monotone triangle

$$\begin{array}{ccccccc} & & & 3 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

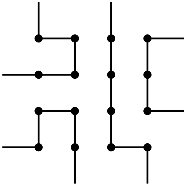
Height function

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 2 & 3 \\ 2 & 1 & 2 & 3 & 2 \\ 3 & 2 & 3 & 2 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{pmatrix}$$

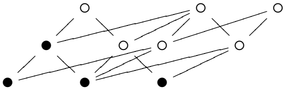
Six-vertex model



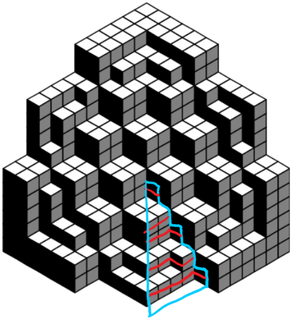
Fully-packed loop



Order ideal



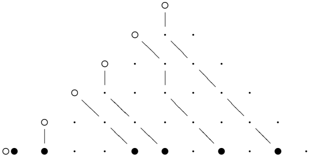
Known TSSCPP bijections



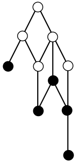
Magog triangle

```
          5
        5   6
      3   4   5
    2   3   4   6
  1   2   3   4   5
1   2   3   4   5   6
```

NILP



Order ideal



Progress on the 'missing bijection' problem

Bijections on ASM-TSSCPP subclasses:

- ASM $\cap$ TSSCPP / 132–avoiding ASMs (Ayyer, Cori, Goyou-Beauchamps 2011, S. 2008/2011)
- Two-diagonal case (Biane–Cheballah 2011)
- Permutation case (S. 2013)

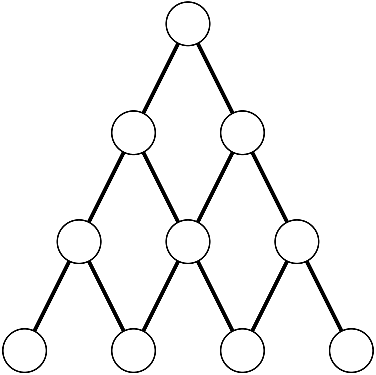
Outline

Posets

A **poset** is a **p**artially **o**rdered **set**.

Definition

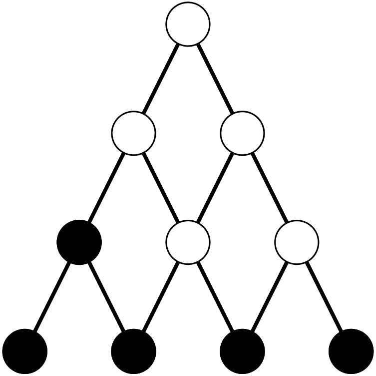
A *poset* is a set with a partial order " $\leq$ " that is reflexive, antisymmetric, and transitive.



Order ideals

Definition

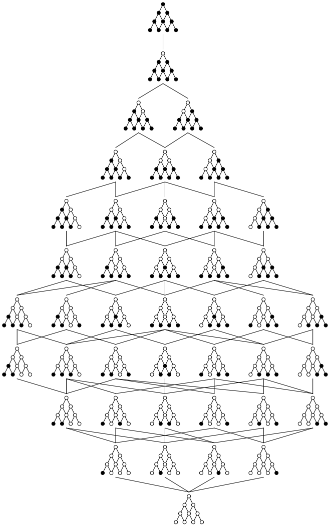
An *order ideal* of a poset P is a subset $X \subseteq P$ such that if $y \in X$ and $z \leq y$, then $z \in X$. The set of order ideals of P is denoted $J(P)$.



Order ideals

Ordered by inclusion, order ideals form a *distributive lattice*, denoted $J(\mathcal{P})$.

The distributive lattice of order ideals $J(P)$



Alternating sign matrix poset

Theorem (Elkies, Kuperberg, Larsen, Propp 1992)

Let a partial order on alternating sign matrices be given by componentwise comparison of the corresponding monotone triangles (or corner sum matrices or height functions). This is a distributive lattice (that is, a lattice of order ideals) with a particularly nice structure.

ASM height functions

All seven of the height functions of order 3.

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

Alternating sign matrix poset

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Alternating sign matrix poset

$$
\begin{tikzcd}
 & \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \arrow[dr, dash] & & \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \arrow[dl, dash] \\
 & \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \arrow[dl, dash] \arrow[dr, dash] & \\
\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \arrow[dr, dash] & & \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \arrow[dl, dash] \\
 & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} &
\end{tikzcd}
$$

Alternating sign matrix poset

$$\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$$

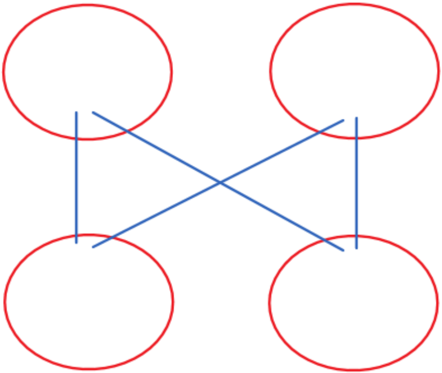
$$\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \qquad \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

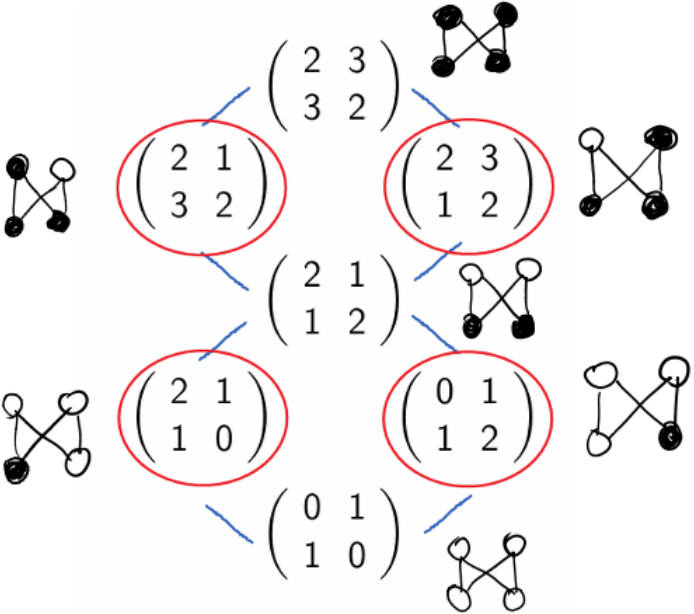
$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

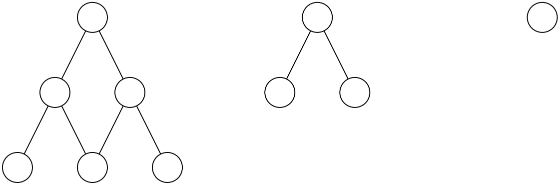
Alternating sign matrix poset



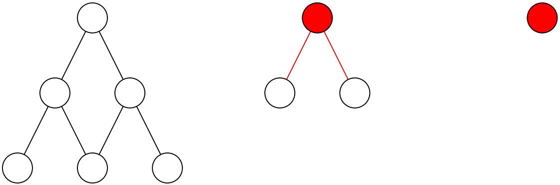
Alternating sign matrix poset



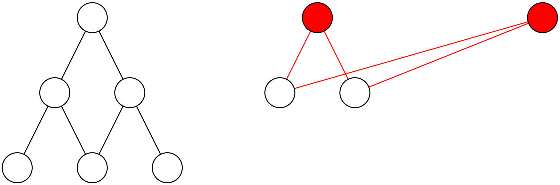
Alternating sign matrix poset



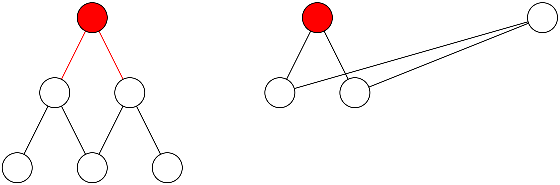
Alternating sign matrix poset



Alternating sign matrix poset



Alternating sign matrix poset



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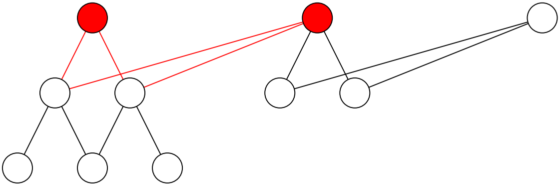

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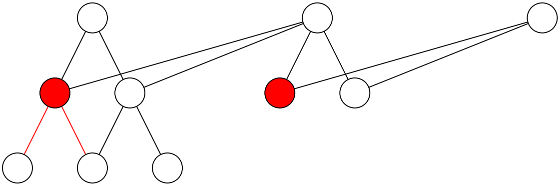

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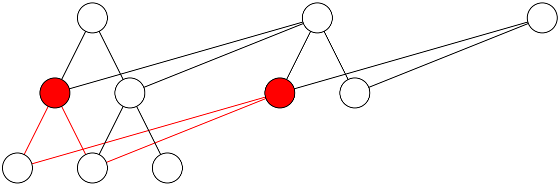

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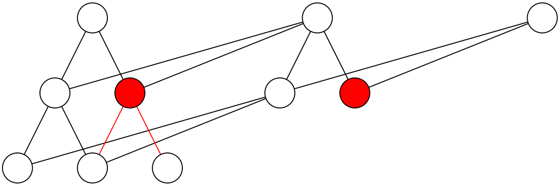

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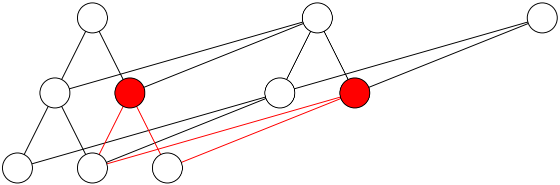

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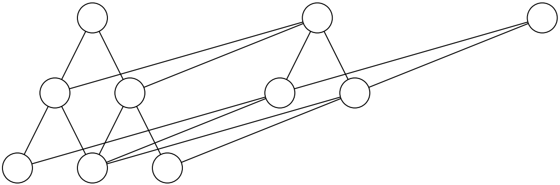

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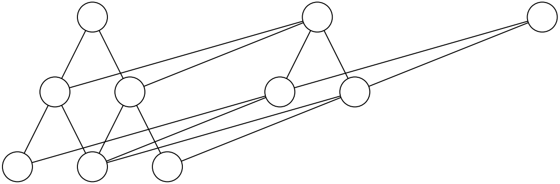
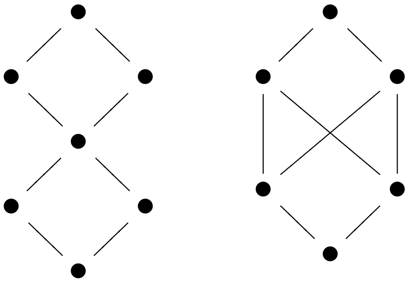


$n \times n$ ASMs are in bijection with order ideals in this poset with $n-1$ layers, as constructed above.

Alternating sign matrix poset

Theorem (Lascoux and Schützenberger 1996)

The restriction of the ASM poset to permutations is the Bruhat order. In fact, is the smallest lattice containing the Bruhat order on the symmetric group as a subposet (i.e. the MacNeille completion).

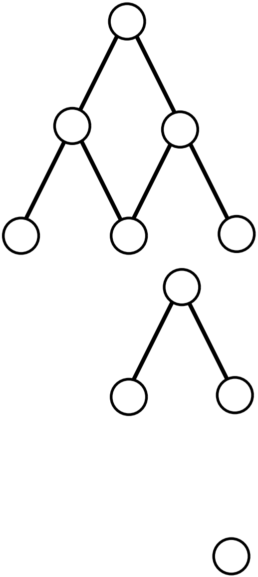


TSSCPP poset

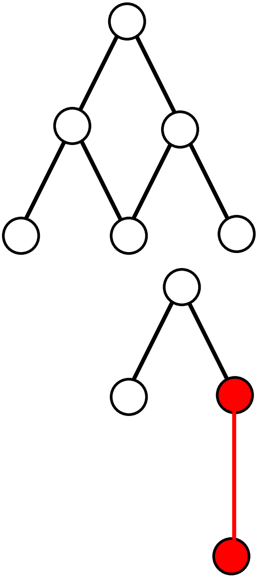
Theorem (S. 2011)

Let a partial order on TSSCPPs be given by componentwise comparison of the corresponding magog triangles. This is a distributive lattice (that is, a lattice of order ideals) with a particularly nice structure.

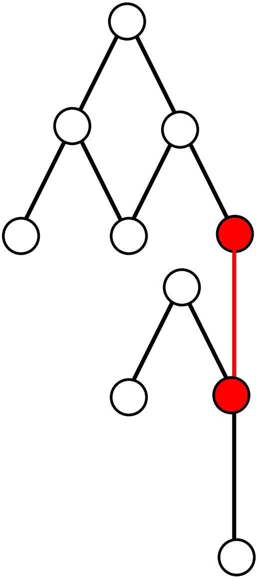
TSSCPP poset



TSSCPP poset



TSSCPP poset



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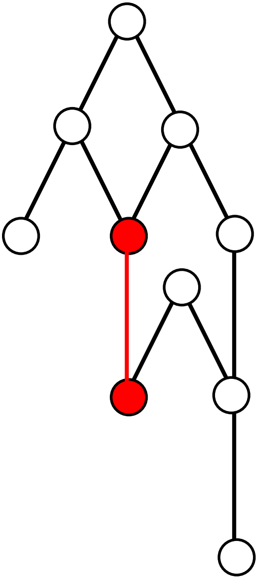

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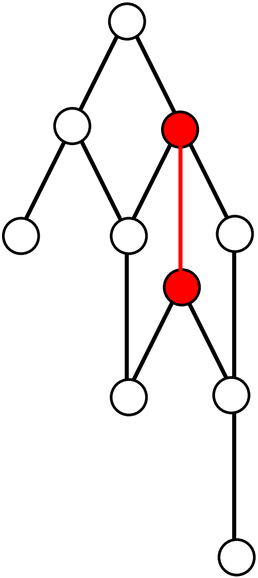

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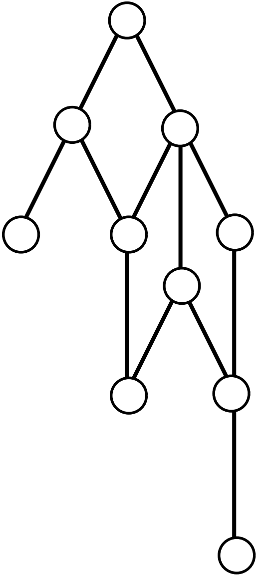
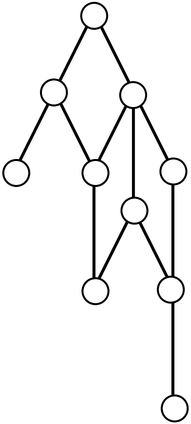
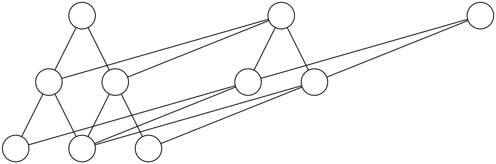


TSSCPP poset

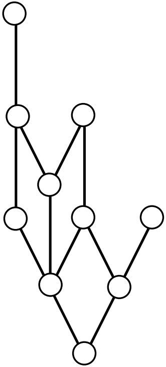


TSSCPPs inside a $2n \times 2n \times 2n$ box are in bijection with order ideals in this poset with $n-1$ layers, as constructed above.

ASM and TSSCPP posets (S. 2011)



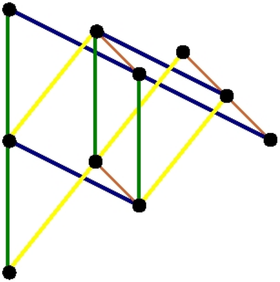
ASM



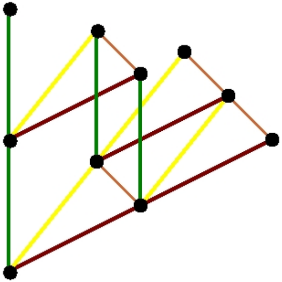
TSSCPP

ASM and TSSCPP posets (S. 2011)

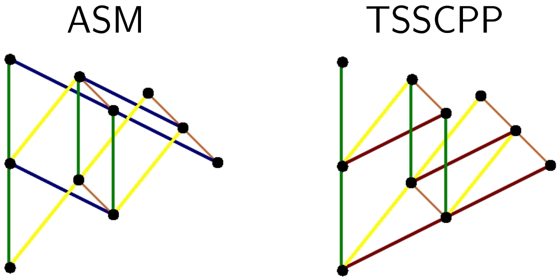
ASM



TSSCPP

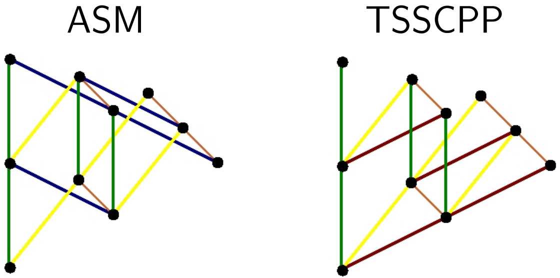


ASM and TSSCPP posets (S. 2011)



Idea: Transform one poset into the other while preserving the number of order ideals

ASM and TSSCPP posets (S. 2011)

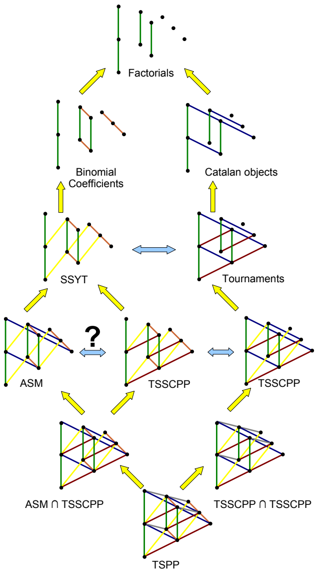


Idea: Transform one poset into the other while preserving the number of order ideals

Problem: Doesn't work

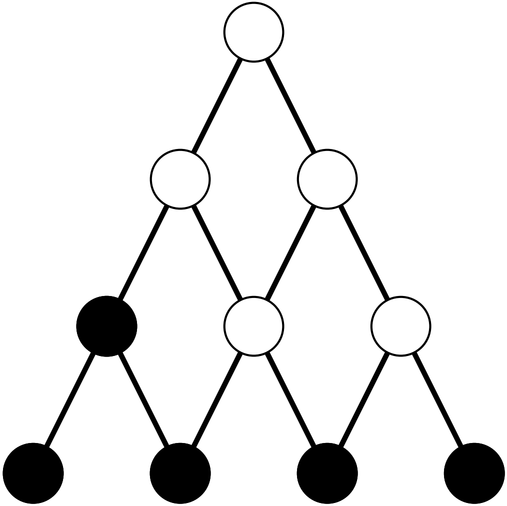
What came of it:

Tetrahedral poset family (S. 2011)

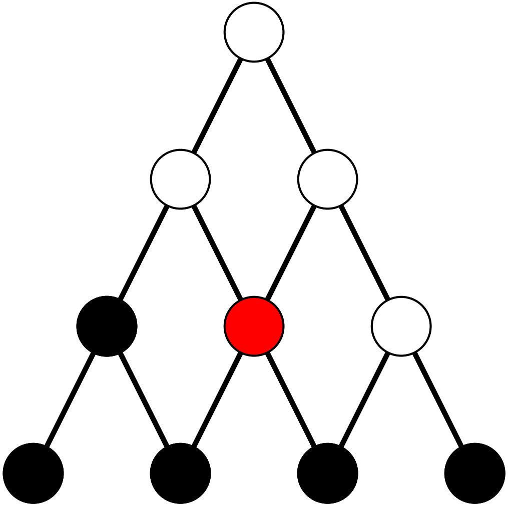


Outline

Toggles act on order ideals

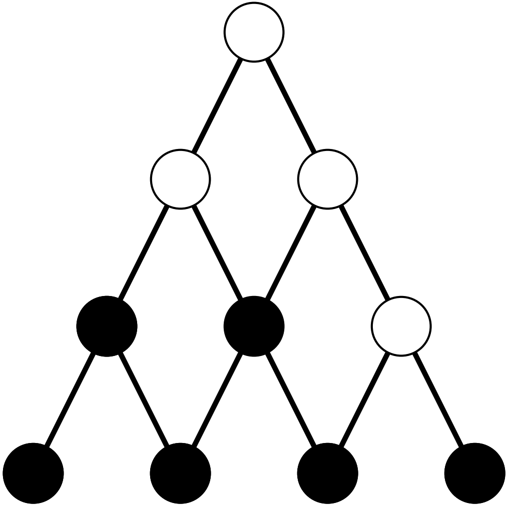


Define a toggle, t_e, for each $e \in P$.



Toggles t_e add e when possible.

Toggles act on order ideals



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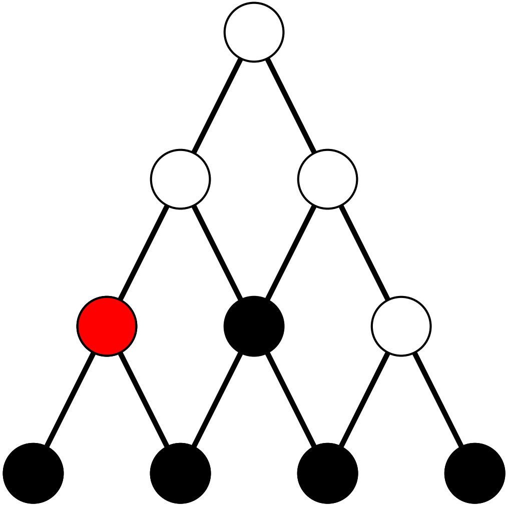

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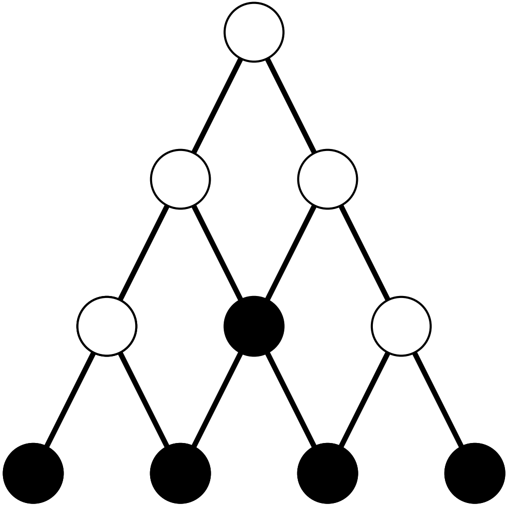

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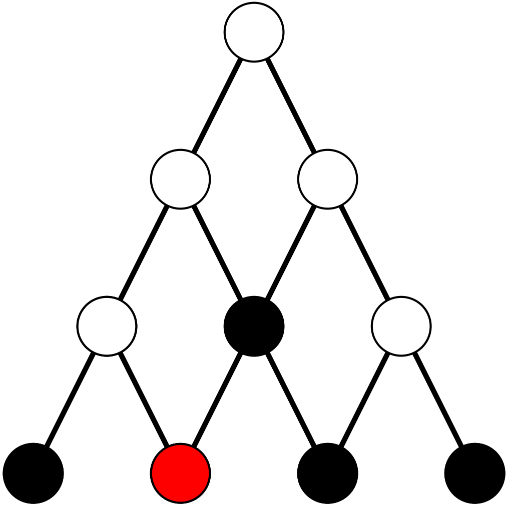
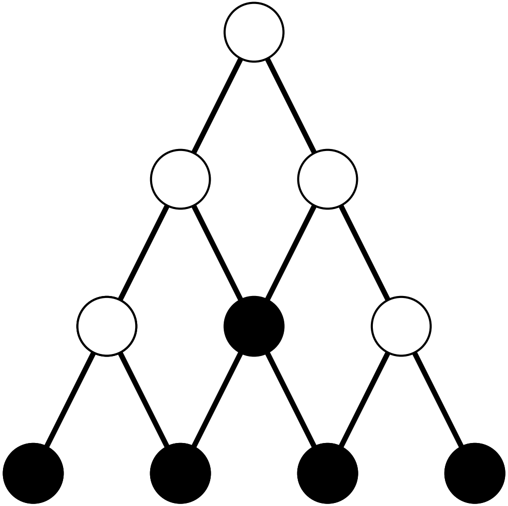


Toggles t_e do nothing otherwise.

Toggles act on order ideals



Toggles t_e do nothing otherwise.

Toggles act on order ideals

Let P be a poset and $J(P)$ its set of order ideals.

Definition

For each element $e \in P$ define its *toggle* $t_e : J(P) \to J(P)$ as follows.

$$t_e(X) = \begin{cases} X \cup \{e\} & \text{if } e \notin X \text{ and } X \cup \{e\} \in J(P) \\ X \setminus \{e\} & \text{if } e \in X \text{ and } X \setminus \{e\} \in J(P) \\ X & \text{otherwise} \end{cases}$$

Toggles generate a group

Definition (Cameron and Fon-der-Flaass 1995)

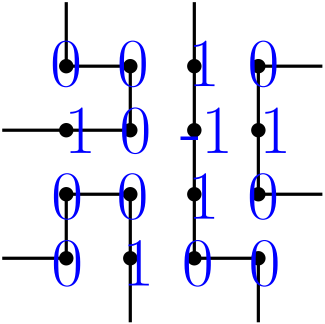
The *toggle group* $T(J(P))$ is the subgroup of the symmetric group $\mathfrak{S}_{J(P)}$ generated by $\{t_e\}_{e \in P}$.

Toggle group actions are compositions of toggles that act on order ideals.

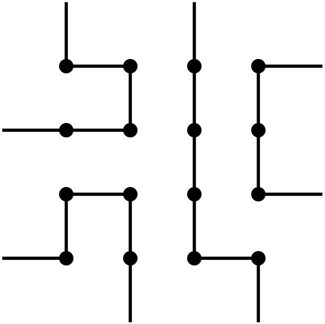
Alternating sign matrix

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Alternating sign matrix ↔ fully-packed loop

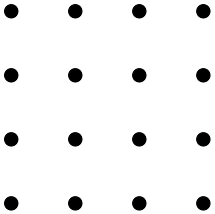


Fully-packed loop

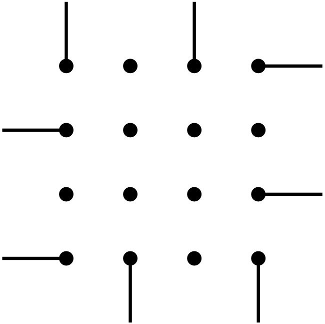


Fully-packed loops

Start with an $n \times n$ grid.

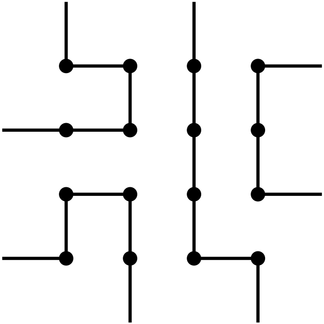


Add boundary conditions.



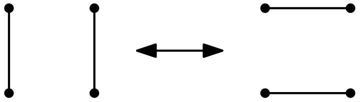
Fully-packed loops

Interior vertices adjacent to 2 edges.

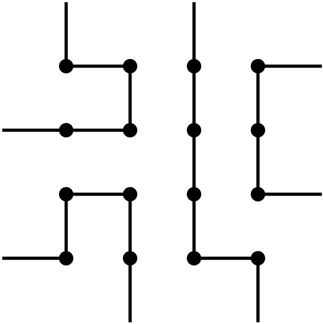


Gyration on fully-packed loops

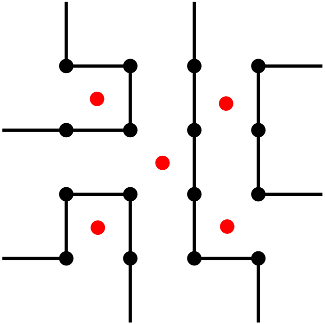
The nontrivial local move.



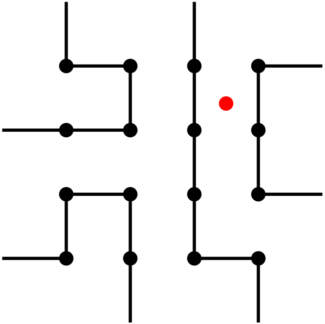
Gyration on fully-packed loops



Start with the even squares.



Apply the nontrivial local move.



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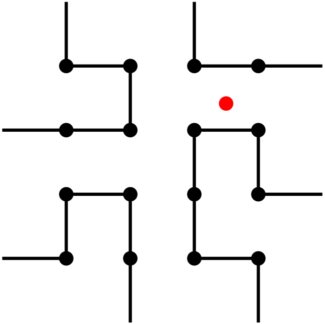
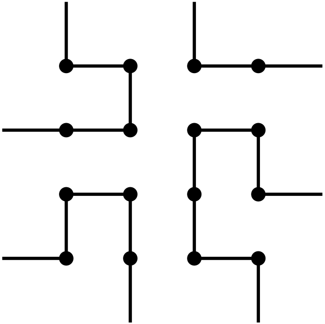
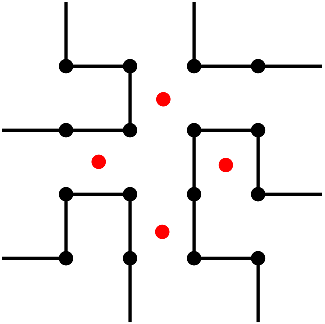


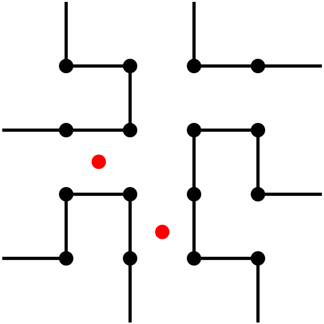
Apply the nontrivial local move.



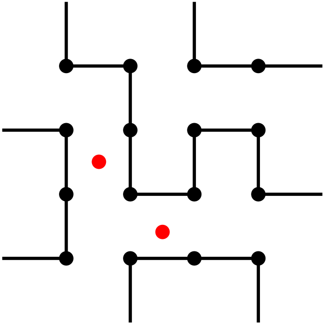
Now consider the odd squares.



Apply the nontrivial local move.



Apply the nontrivial local move.



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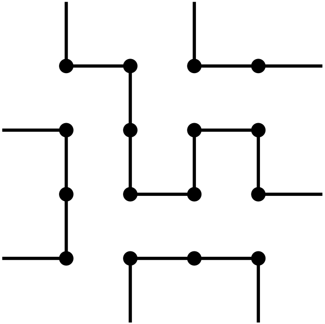
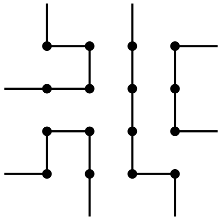
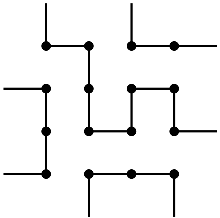


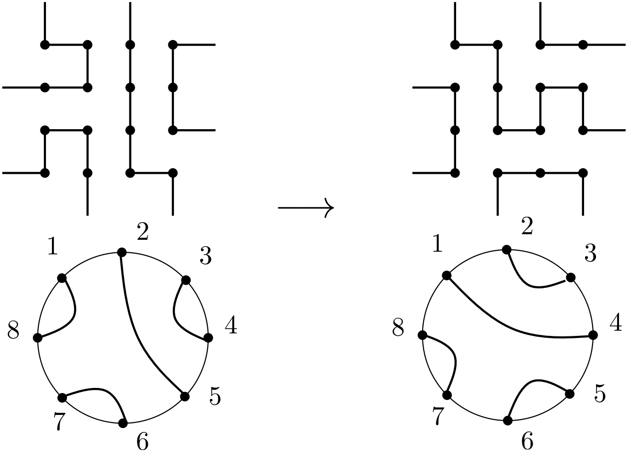
Gyration on fully-packed loops



$\longrightarrow$



Gyration on fully-packed loops

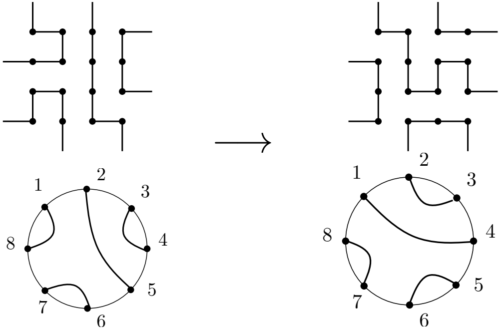


The square is a circle

Theorem (B. Wieland 2000)

Gyration on an order n fully-packed loop rotates the link pattern by a factor of $2n$.

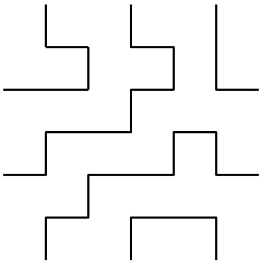
Gyration exhibits *resonance* with pseudo-period $2n$.



Gyration as a toggle group action

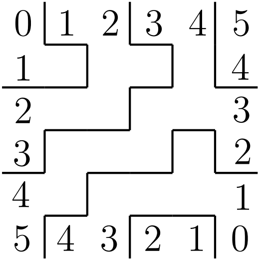
How does this relate to the toggle group?

Start with a fully-packed loop



Gyration as a toggle group action

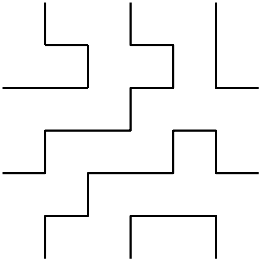
Biject to a height function



Biject to a height function

$$\begin{array}{cccccc}
0 & 1 & 2 & 3 & 4 & 5 \\
1 & 2 & 3 & 4 & 5 & 4 \\
2 & 1 & 2 & 3 & 4 & 3 \\
3 & 2 & 1 & 2 & 3 & 2 \\
4 & 3 & 2 & 1 & 2 & 1 \\
5 & 4 & 3 & 2 & 1 & 0
\end{array}$$

Gyration as a toggle group action



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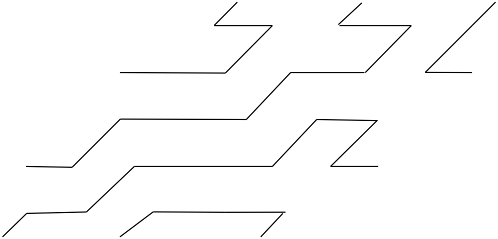

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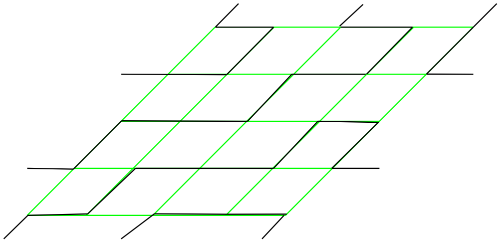

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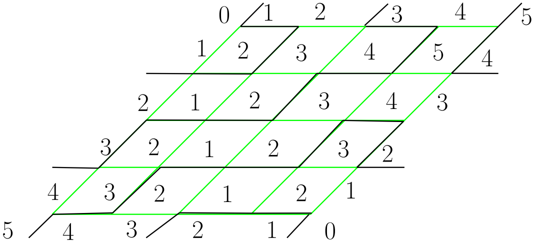

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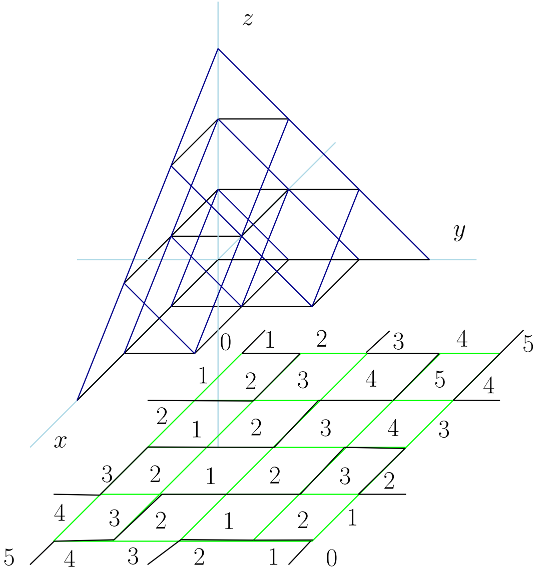

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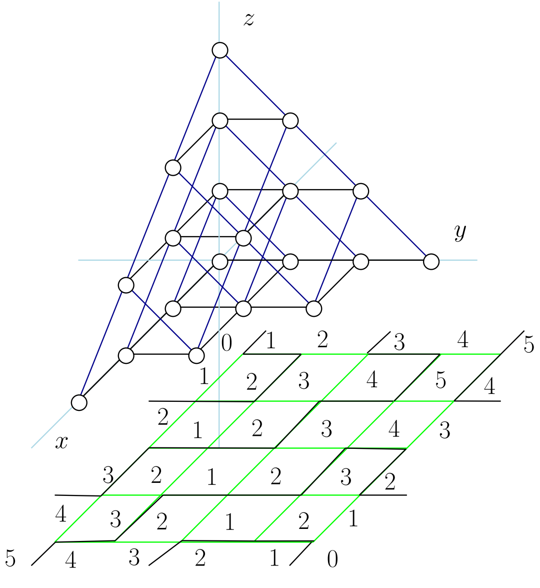

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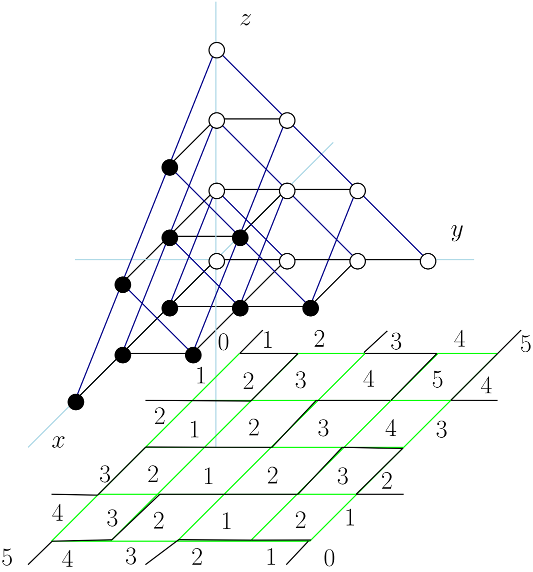

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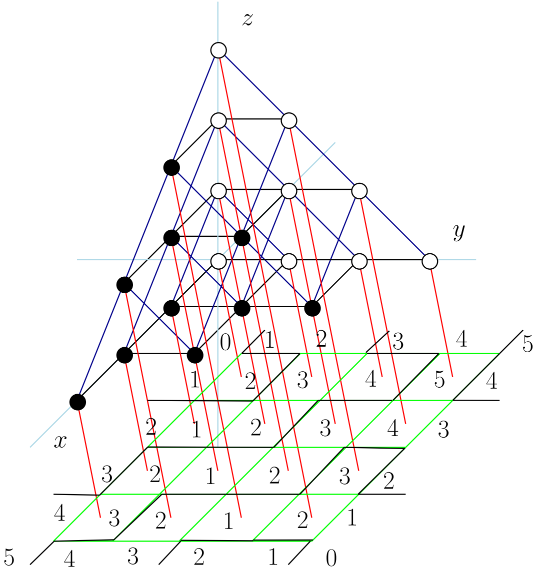

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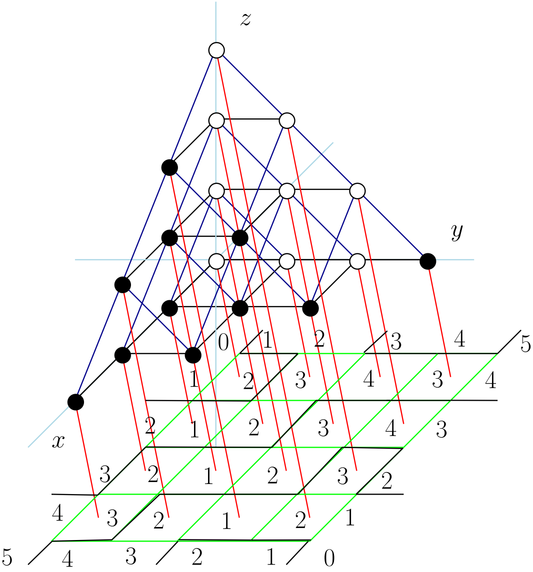
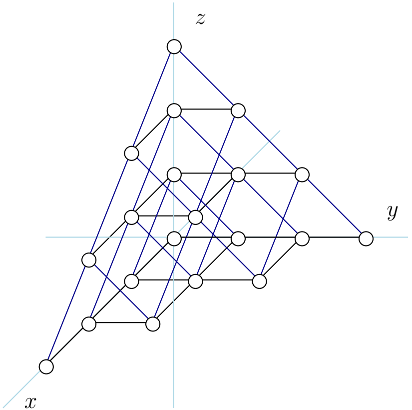


Gyration as a toggle group action

Theorem (N. Williams and S. 2012)

Gyration on fully-packed loops is equivalent to toggling even then odd ranks in the ASM poset.



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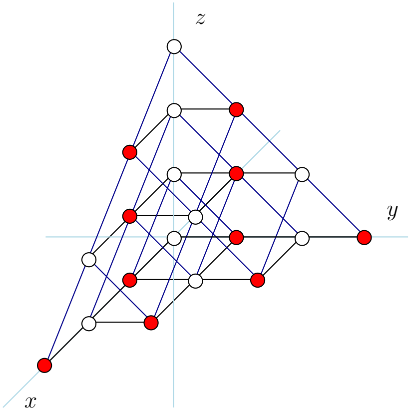

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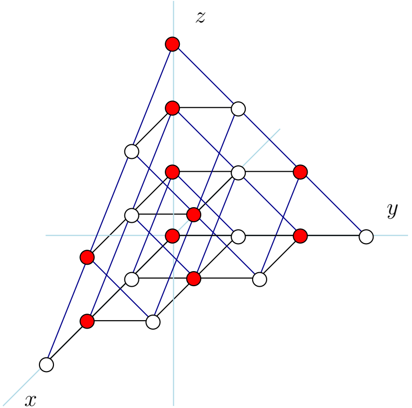
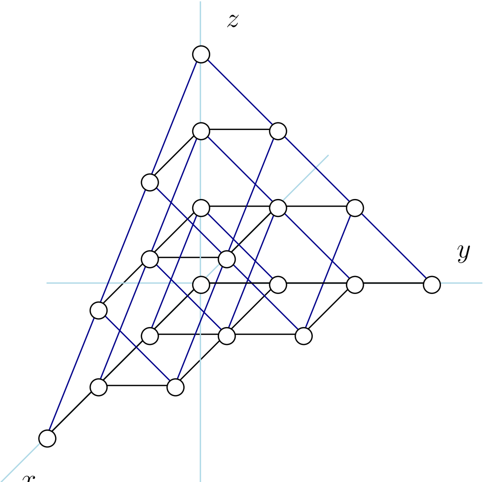


The "$3n-2$" problem

With N. Williams, we found another toggle group action on this poset, called *superpromotion*, that exhibits resonance with pseudo-period $3n-2$.



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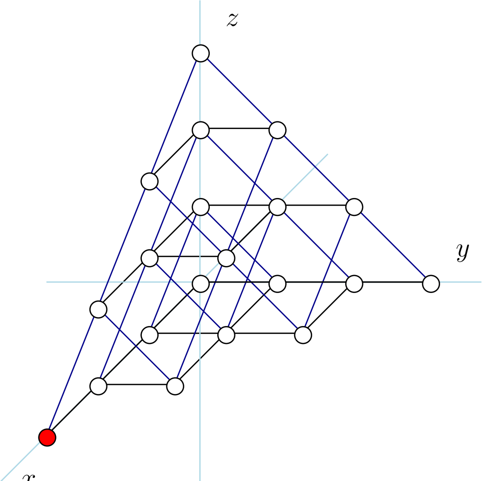

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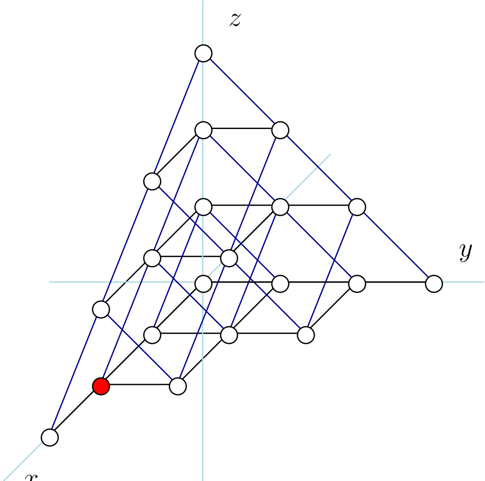

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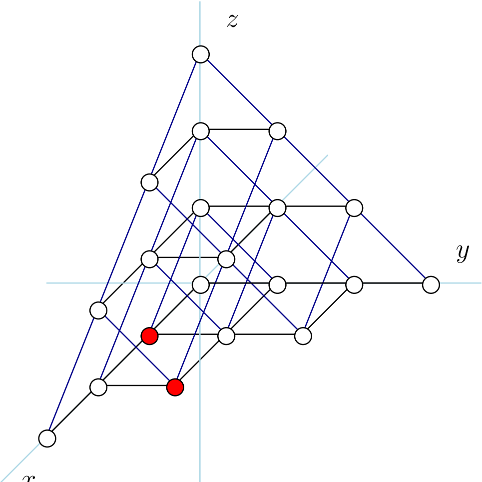

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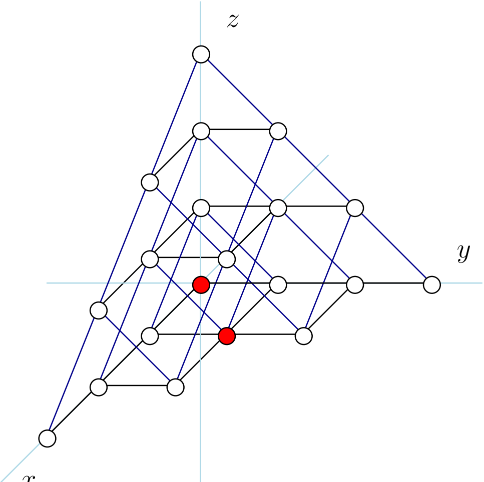

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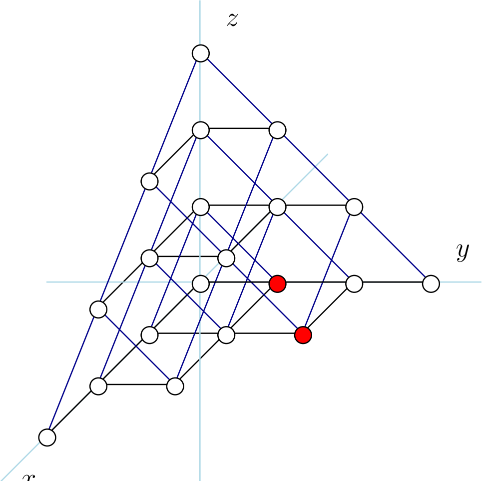

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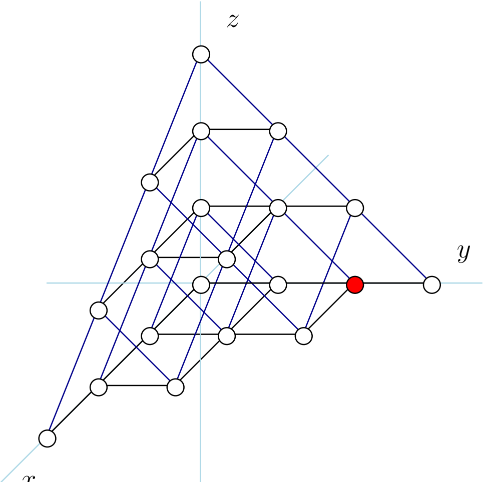

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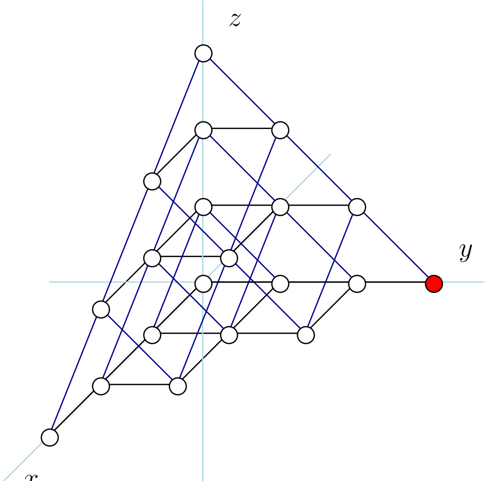

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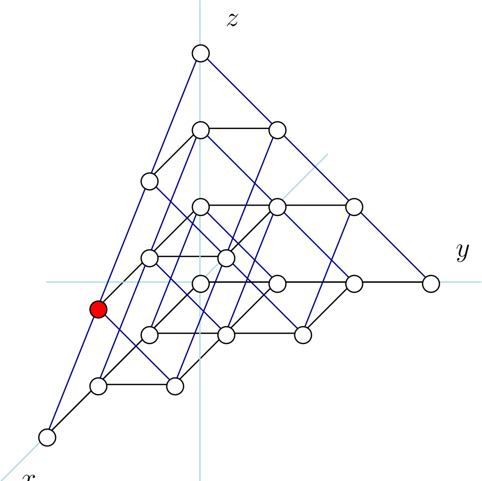

The "$3n - 2$" problem

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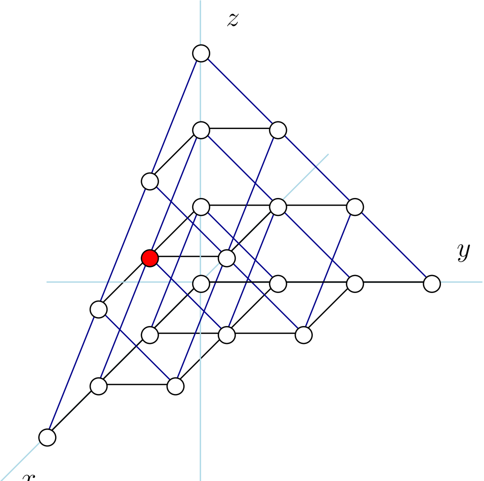

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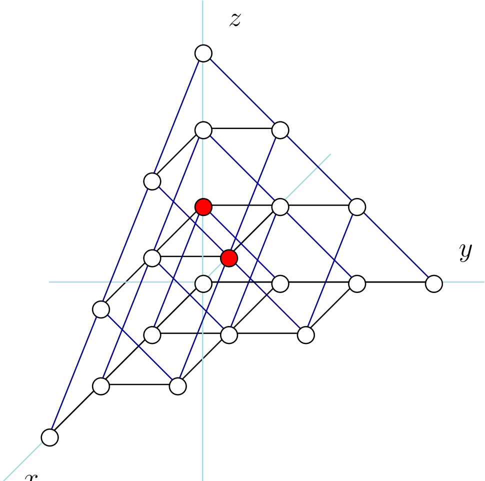

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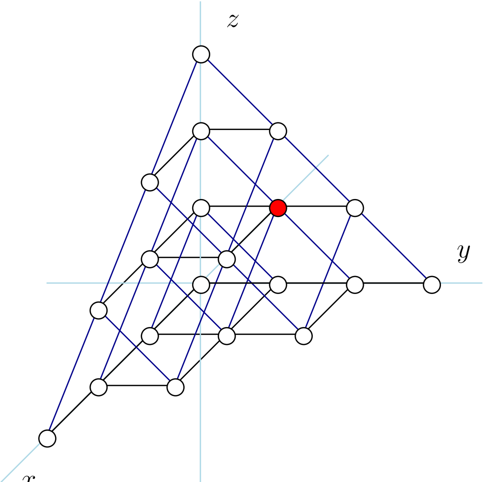

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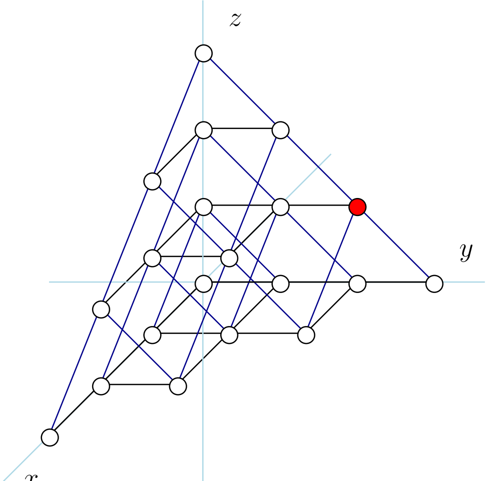

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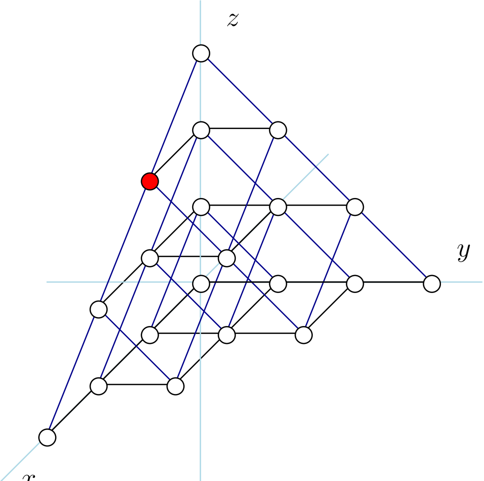

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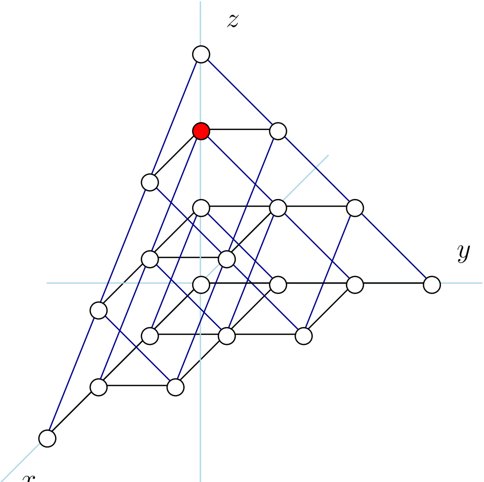

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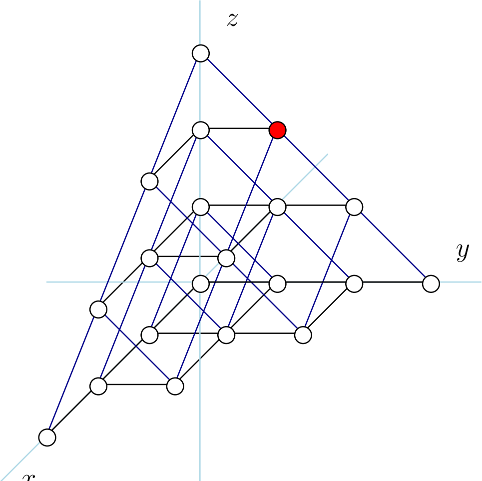

The "$3n-2$" problem

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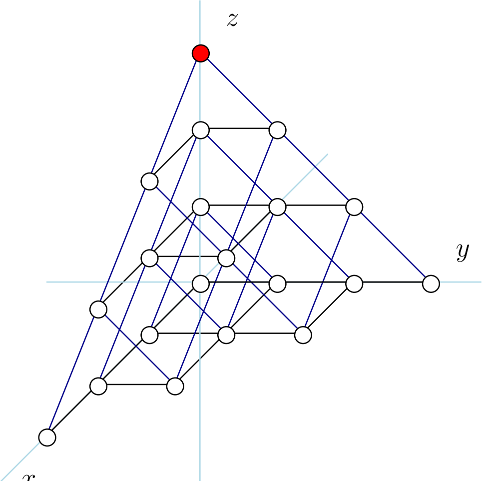


The "$3n-2$" problem

With N. Williams, we found another toggle group action on this poset, called *superpromotion*, that exhibits resonance with pseudo-period $3n-2$.

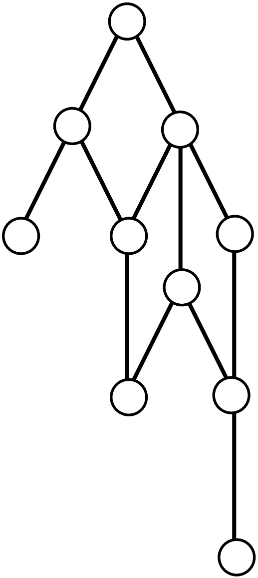
Problem

What is the underlying combinatorial structure that superpromotion is rotating with period $3n-2$?

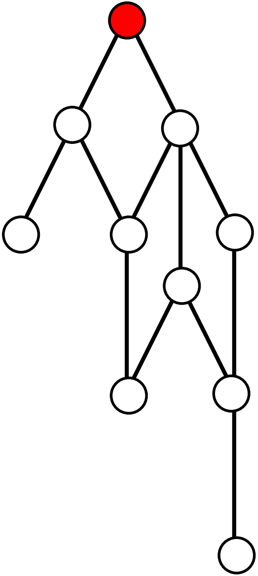
Motivation for the "$3n - 2$" problem

With N. Williams, we studied another toggle group action, called *rowmotion*, which on the TSSCPP poset has a very similar orbit structure to superpromotion on ASMs.

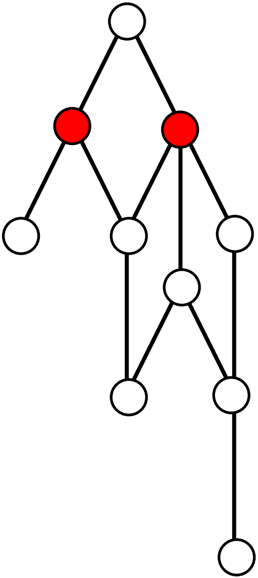
Rowmotion on TSSCPPs



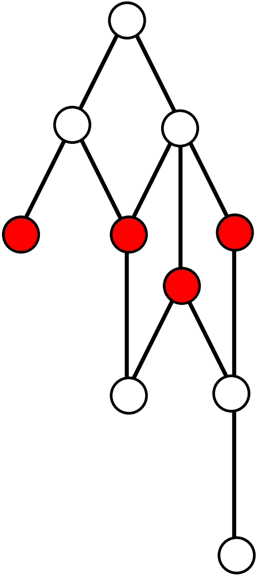
Rowmotion on TSSCPPs



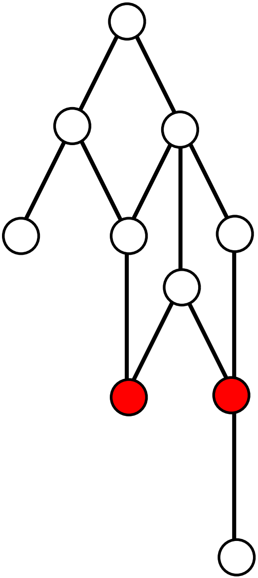
Rowmotion on TSSCPPs



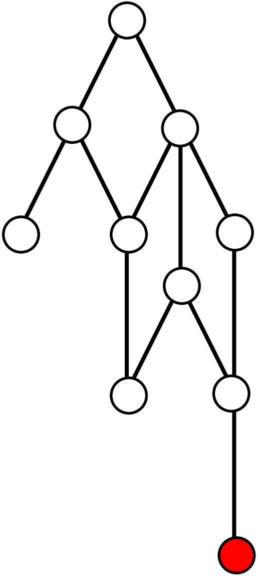
Rowmotion on TSSCPPs



Rowmotion on TSSCPPs



Rowmotion on TSSCPPs



Motivation for the "$3n-2$" problem

With N. Williams, we studied another toggle group action, called *rowmotion*, which on the TSSCPP poset has a very similar orbit structure to superpromotion on ASMs.

Idea: Find a bijection on orbits of ASM superpromotion and TSSCPP rowmotion.

Orbit size data for these actions

	ASM under SPro		TSSCPP under Row	
	Orbit Size	Number of Orbits	Orbit Size	Number of Orbits
$n = 1$	1	1	1	1
$n = 2$	2	1	2	1
$n = 3$	7	1	7	1
$n = 4$	10	3	10	3
	5	2	5	2
	2	1	2	1
$n = 5$			39	1
			26	1
	13	33	13	28
$n = 6$			$8k, k > 2$	65
	16	456	16	277
	8	16	8	13
	4	2		
	2	2	2	2
$n = 7$	57	55		
	19	11327	*	*

Motivation for the "$3n-2$" problem

With N. Williams, we studied another toggle group action, called *rowmotion*, which on the TSSCPP poset has a very similar orbit structure to superpromotion on ASMs.

Idea: Find a bijection on orbits of ASM superpromotion and TSSCPP rowmotion

Problem: Orbit sizes don't match

What came of it: Inspiration for studying the 'resonance' phenomenon

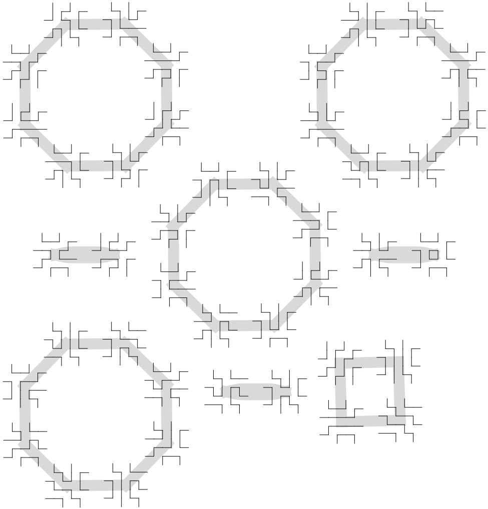
The conjugacy of rowmotion, promotion, and gyration

Theorem (N. Williams and S. 2012)

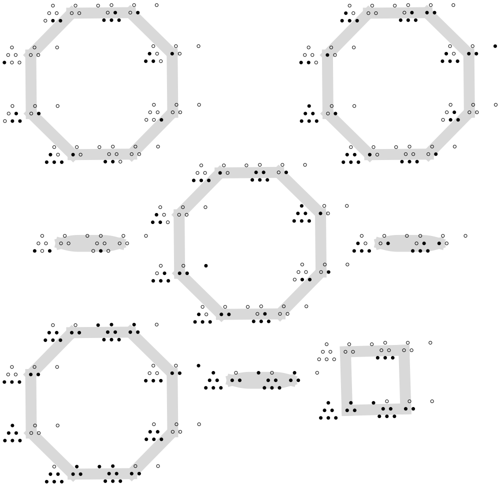
In any ranked poset, there are equivariant bijections between the order ideals under under rowmotion (toggle top to bottom), promotion (toggle left to right), and gyration (toggle even then odd ranks).

In an *equivariant* bijection, the orbit structure is preserved.

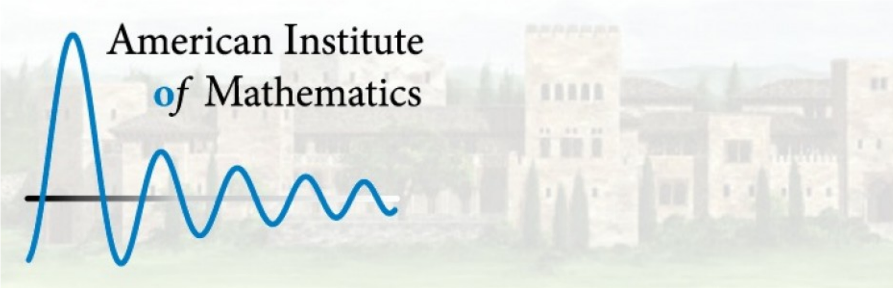
Fully-packed loop orbits under gyration



Order ideals in the ASM poset under rowmotion



Dynamical algebraic combinatorics



Dynamical algebraic combinatorics

March 23 to March 27, 2015

at the

American Institute of Mathematics, Palo Alto, California

organized by

James Propp, Tom Roby, Jessica Striker, and Nathan Williams

Plane partitions and increasing tableaux

Theorem (K. Dilks, O. Pechenik, S. 2015)

There is an equivariant bijection between plane partitions in $[a] \times [b] \times [c]$ *under rowmotion (toggle from top to bottom) and increasing tableaux of rectangular shape* $a \times b$ *and entries at most* $a + b + c - 1$ *under* K*-promotion.*

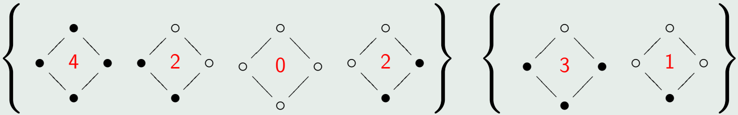
This correspondence explains observed resonance phenomena on both sides of this bijection.

Theorem (J. Propp and T. Roby 2013)

The order ideal size statistic in $J([n] \times [k])$ is homomesic (orbit-average $=$ global-average) with respect to rowmotion or promotion.

Example

The promotion orbits of $J([2] \times [2])$



$$\frac{4+2+0+2}{4} = 2 \qquad \frac{3+1}{2} = 2$$

Toggleability homomesy

Definition

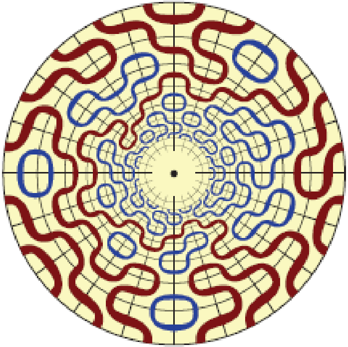
Fix a poset P. For each $e \in P$, define the *toggleability* statistic $\mathfrak{T}_e : J(P) \to \{0, 1, -1\}$ as:

$$\mathfrak{T}_e(X) = \begin{cases} 1 & \text{if } e \text{ can be toggled } \textit{out} \text{ of } X, \\ -1 & \text{if } e \text{ can be toggled } \textit{in} \text{ to } X, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (S. 2015)

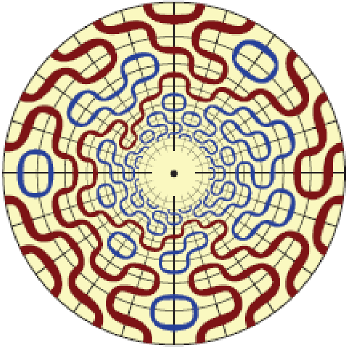
Given any ranked poset P and $e \in P$, $\mathfrak{T}_e$ on $J(P)$ is homomesic with average value 0 *with respect to gyration (toggle even then odd ranks).*

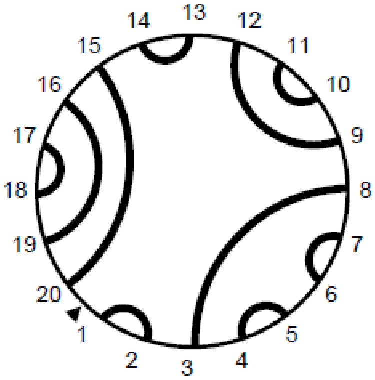
$O(1)$ dense loop model on a semi-infinite cylinder



http://old-lipn.univ-paris13.fr/journee_calin/Slides/sportiello.pdf

$O(1)$ dense loop model on a semi-infinite cylinder





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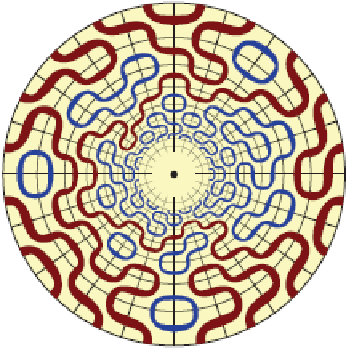
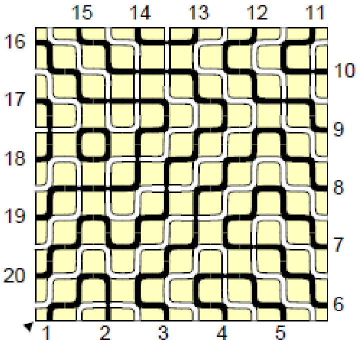
Razumov-Stroganov correspondence

$O(1)$ dense loop model   Fully-packed loop model





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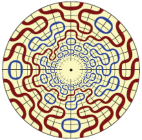
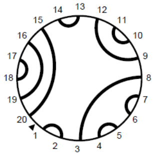
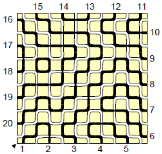
Gyration was used to prove...

Conjecture (A. Razumov and Y. Stroganov 2004)

The probability that a configuration of the $O(1)$ dense loop model on a semi-infinite cylinder of perimeter $2n$ has link pattern π equals the probability that a fully-packed loop of order n has link pattern π.

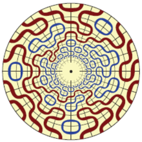


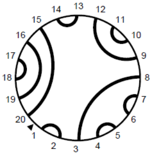


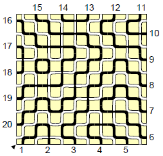


Theorem (L. Cantini and A. Sportiello 2011)

The probability that a configuration of the $O(1)$ dense loop model on a semi-infinite cylinder of perimeter $2n$ has link pattern π equals the probability that a fully-packed loop of order n has link pattern π.







Homomesy applied to the Razumov-Stroganov

Theorem (S. 2015)

Given any ranked poset P and $e \in P$, $\mathfrak{T}_e$ on $J(P)$ is homomesic with average value 0 with respect to gyration (toggle even then odd ranks).

When applied to the ASM poset, we recover the following lemma from Cantini and Sportiello's first proof of the Razumov-Stroganov conjecture.

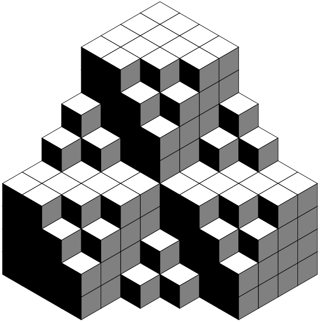
Lemma (Cantini and Sportiello 2011)

Fix any square α. Then the number of FPLs in an orbit of gyration with edge configuration $|\alpha|$ equals the number with configuration $\underline{\overline{\ \alpha\ }}$.

Outline

A missing bijection

Totally symmetric self-complementary plane partitions inside a $2n \times 2n \times 2n$ box are also counted by $\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$ (Andrews 1994), but **no explicit bijection is known**.

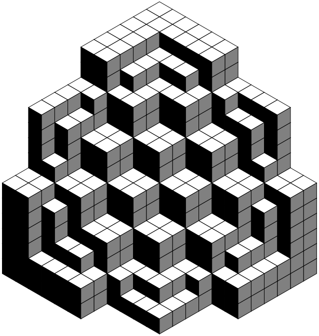


$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Permutation case progress (S. 2013)

Progress: I found nice, statistic-preserving bijection in the special case of *permutations*.

TSSCPP



$$\Leftrightarrow \quad \text{Permutation matrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Permutation case progress (S. 2013)

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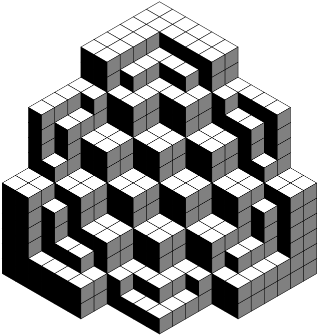
TSSCPP



$\Leftrightarrow$

Permutation matrix

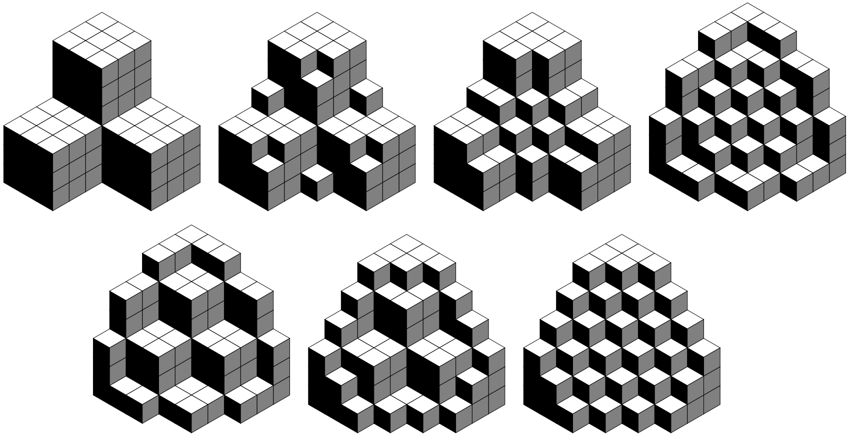
$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Which ones are permutations?

Monotone triangle ‘−1’s

ASM		Column partial sums		Monotone triangle
$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$	$\Leftrightarrow$	$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$	$\Leftrightarrow$	$\begin{array}{ccccccc} & & & {\color{red}2} & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$

Permutation TSSCPPs?



Q: Which one has a '-1' in it?

ASM inversions

Definition

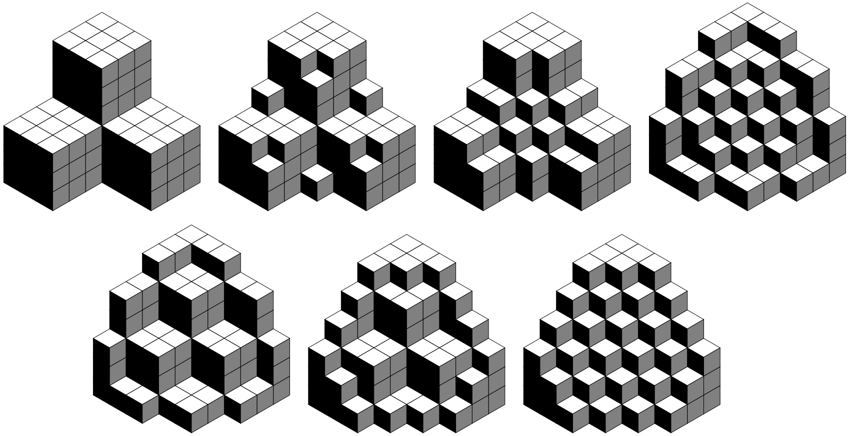
The *inversion number* of an ASM A is defined as

$$I(A) = \sum A_{ij}A_{k\ell}$$

where the sum is over all i, j, k, ℓ such that $i > k$ and $j < \ell$.

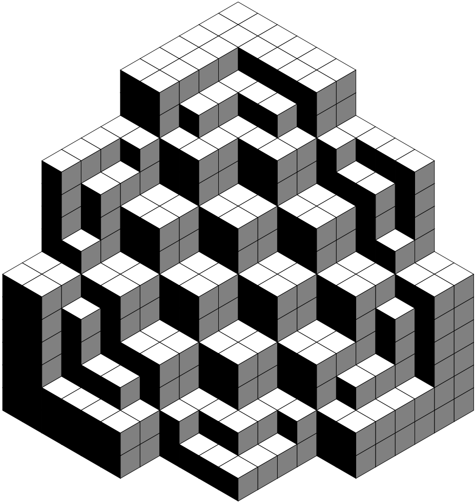
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Leftrightarrow \begin{array}{ccccccc} & & & \color{blue}{2} & & & \\ & & 1 & & \color{blue}{4} & & \\ & 1 & & \color{blue}{3} & & \color{blue}{4} & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

TSSCPP inversions?

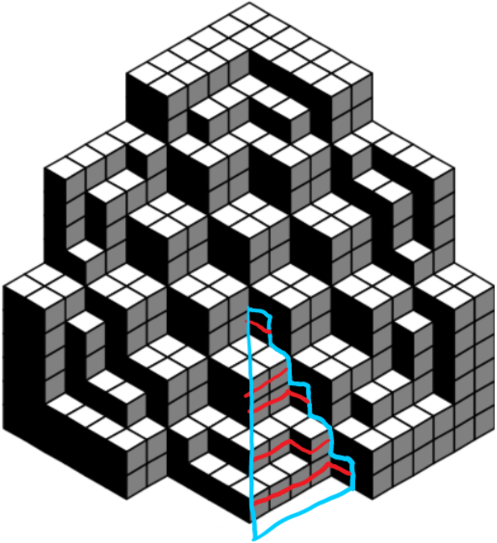


Q: What are TSSCPP 'inversions'?

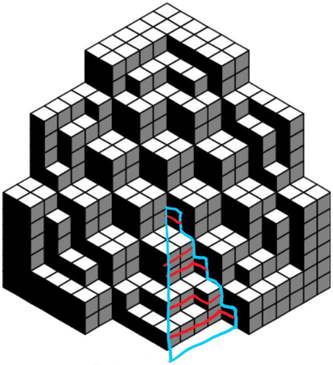
TSSCPP to non-intersecting lattice paths

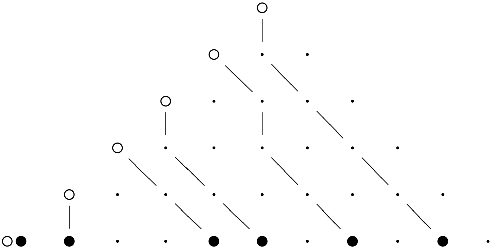


TSSCPP to non-intersecting lattice paths



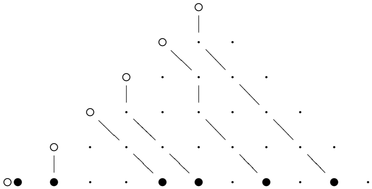
TSSCPP to non-intersecting lattice paths





Paths to boolean triangle

TSSCPP NILP



$\Leftrightarrow$

Boolean triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 0 & & 0 & & 0 & & 0
\end{array}$$

Boolean triangle definition

Definition

A *boolean triangle* of order n is a triangular integer array $\{b_{i,j}\}$ for $1 \le i \le n-1$, $n-i \le j \le n-1$ with entries in $\{0,1\}$ such that the diagonal partial sums satisfy $1 + \sum_{i=j+1}^{i'} b_{i,n-j-1} \ge \sum_{i=j}^{i'} b_{i,n-j}$.

$$
\begin{array}{ccccccccc}
 & & & & b_{1,n-1} & & & & \\
 & & & b_{2,n-2} & & b_{2,n-1} & & & \\
 & & b_{3,n-3} & & b_{3,n-2} & & b_{3,n-1} & & \\
 & & & & \vdots & & & & \\
b_{n-1,1} & & b_{n-1,2} & & \cdots & & b_{n-1,n-2} & & b_{n-1,n-1}
\end{array}
$$

Permutation TSSCPPs

Definition (S.)

Let *permutation TSSCPPs* be all TSSCPPs whose corresponding boolean triangles have weakly decreasing rows.

Not a permutation TSSCPP	A permutation TSSCPP
$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&0&&1&&1&& \\ &0&&0&&0&&0& \\ 0&&0&&1&&0&&0 \end{array}$	$\begin{array}{ccccccccc} &&&&1&&&& \\ &&&0&&0&&& \\ &&1&&1&&0&& \\ &0&&0&&0&&0& \\ 1&&0&&0&&0&&0 \end{array}$

Permutation TSSCPP inversions

The ‘inversions’ of permutation TSSCPPs are the zeros.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 1 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 1 & & 0 & & 0 & & 0 & & 0 \end{array}$$

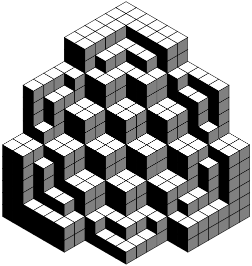
ASM–TSSCPP bijection in the permutation case

Theorem (S.)

There is a natural, statistic-preserving bijection between permutation matrices and permutation TSSCPPs which maps the number of inversions of the permutation to the number of zeros in the boolean triangle.

ASM–TSSCPP bijection in the permutation case

TSSCPP



$\Leftrightarrow$

Boolean triangle

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & & 0 & & & \\
 & & 1 & & 1 & & 0 & & \\
 & 0 & & 0 & & 0 & & 0 & \\
1 & & 0 & & 0 & & 0 & & 0
\end{array}$$

Monotone triangle

$\Leftrightarrow$

$$\begin{array}{ccccccccccc}
 & & & & & 4 & & & & & \\
 & & & & 4 & & 6 & & & & \\
 & & & 3 & & 4 & & 6 & & & \\
 & & 3 & & 4 & & 5 & & 6 & & \\
 & 1 & & 3 & & 4 & & 5 & & 6 & \\
1 & & 2 & & 3 & & 4 & & 5 & & 6
\end{array}$$

Permutation matrix

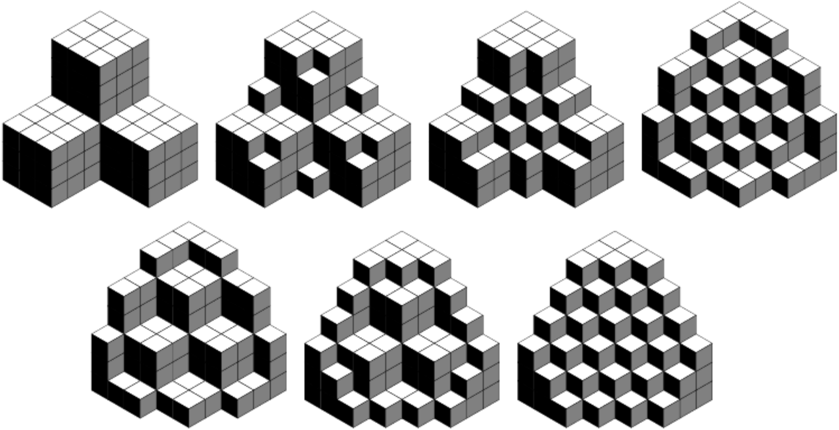
$\Leftrightarrow$

$$\begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Statistics

DPP	ASM	TSSCPP
no special parts*	no -1's	rows weakly decrease
number of parts*	number of inversions	number of zeros
number of n's*	position of 1 in last column	position of lowest 1 in last diagonal
largest part value that does not appear	position of 1 in last row	number of zeros in last row*

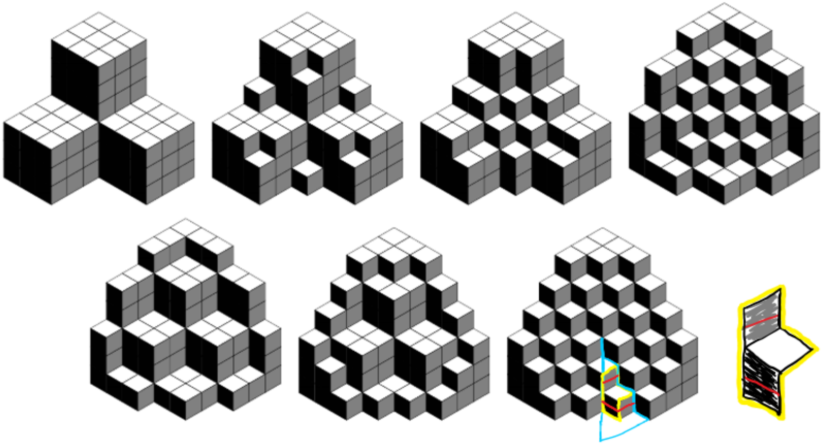
Permutation TSSCPPs



Q: Which one has a '-1' in it?

A: This one

Permutation TSSCPPs



Q: Which one has a '-1' in it?

A: This one

Questions

How does this permutation case bijection relate to the other subclass bijections?

- ASM $\cap$ TSSCPP / 132–avoiding ASMs (Ayyer, Cori, Goyou-Beauchamps 2011, S. 2008/2011)

- Two-diagonal case (Biane–Cheballah 2011)

Questions

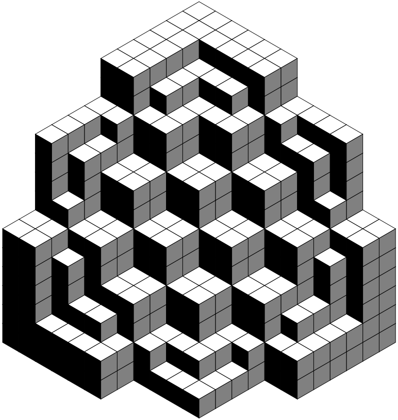
How does this permutation case bijection relate to the other subclass bijections?

- ASM $\cap$ TSSCPP / 132–avoiding ASMs (Ayyer, Cori, Goyou-Beauchamps 2011, S. 2008/2011)
  Does NOT correspond on the intersection
- Two-diagonal case (Biane–Cheballah 2011)
  Seems to correspond on the intersection

Outline

1. Alternating sign matrices and totally symmetric self-complementary plane partitions
2. Poset structures
3. Toggle group dynamics
4. A permutation case bijection

THANKS



A new poset structure on TSSCPPs

Definition

Define the *boolean partial order* T_n^{Bool} on TSSCPPs of order n by componentwise comparison of their boolean triangles.

Proposition

T_n^{Bool} is a lattice for $n \leq 3$, but for $n \geq 4$ it is not a lattice.

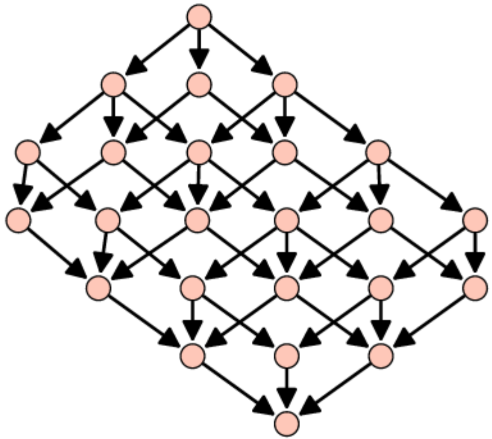
Theorem

The induced subposet of T_n^{Bool} consisting of all the permutation boolean triangles is $[2] \times [3] \times \cdots \times [n]$.

A new poset structure on TSSCPPs

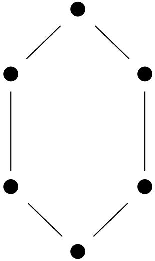
Theorem

The induced subposet of T_n^{Bool} consisting of all the permutation boolean triangles is $[2] \times [3] \times \cdots \times [n]$.

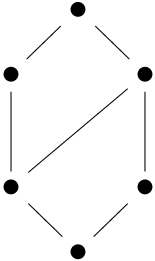


A new poset structure on TSSCPPs

Weak Bruhat



TSSCPP Boolean



Strong Bruhat