Matrix product formula for Macdonald polynomials

Jan de Gier

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Collaborators:
Luigi Cantini
Michael Wheeler

Outline

1. Macdonald polynomials
2. Construction of Matrix Product form
3. General solution and combinatorics

I. What are Macdonald polynomials?

Symmetric group

Let s_i $(i = 1, \ldots, n-1)$ be generators of the symmetric group S_n:

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$
$$s_i^2 = 1;$$

There exist a natural t-deformation of S_n:

$$(T_i - t^{\frac{1}{2}})(T_i + t^{-\frac{1}{2}}) = 0, \qquad (i = 1, \ldots, n-1),$$
$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}.$$

This is the Hecke algebra (of type A_{n-1}) and S_n is recovered when $t \to 1$.

Polynomial action

The generators s_i act naturally on polynomials:

$$s_i f(\dots, x_i, x_{i+1}, \dots) = f(\dots, x_{i+1}, x_i, \dots) \qquad i = 1, \dots n-1$$

and the t-deformation also has an action:

$$T_i^{\pm 1} = t^{\pm \frac{1}{2}} - t^{-\frac{1}{2}} \frac{t x_i - x_{i+1}}{x_i - x_{i+1}} (1 - s_i).$$

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The shifted operator,

$$T_i(u) = T_i + \frac{t^{-\frac{1}{2}}}{[u]}, \qquad [u] = \frac{1 - t^u}{1 - t}.$$

satisfies the Yang–Baxter equation,

$$T_i(u) T_{i+1}(u+v) T_i(v) = T_{i+1}(v) T_i(u+v) T_{i+1}(u).$$

Nonsymmetric Macdonald polynomials

We can extend to the *affine* Hecke algebra by adding a cyclic shift operator:

$$(\omega f)(x_1, \ldots, x_n) = f(qx_n, x_1, \ldots, x_{n-1}),$$
$$\omega T_i = T_{i+1}\omega.$$

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This algebra has a family of commuting operators (Abelian subalgebra) generated by the Murphy elements:

$$Y_i = T_i \cdots T_{n-1}\omega T_1^{-1} \cdots T_{i-1}^{-1}.$$

which commute:

$$[Y_i, Y_j] = 0.$$

Remark: Symmetric functions of $\{Y_i\}$ are central, i.e. commute with all elements in the Hecke algebra.

Nonsymmetric Macdonald polynomials

Since the Y_i commute, they can be diagonalised simultaneously:

Definition (Nonsymmetric Macdonald polynomial E_λ)

$$Y_i E_\lambda = y_i(\lambda) E_\lambda,$$

The index $\lambda = (\lambda_1, \ldots, \lambda_n)$ is a composition, $\lambda_i \in \mathbb{N}_0$, and

$$y_i(\lambda) = t^{\rho(\lambda)_i} q^{\lambda_i}$$

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Example:
If $\lambda = (3, 0, 4, 4, 2)$, then define $\rho = (2, 1, 0, -1, -2)$

Dominant weight $\lambda^+ = (4, 4, 3, 2, 0)$

Reorder ρ in the same way as reordering $\lambda^+ \to \lambda$
$\rho(\lambda) = (0, -2, 2, 1, -1)$.

Macdonald polynomials

E_λ forms a basis in the ring of polynomials with top-degree λ^+,

$$E_\lambda(x_1, \ldots, x_n) = x_1^{\lambda_1} \cdots x_n^{\lambda_n} + \sum_{\mu < \lambda} c_{\lambda\mu} x^\mu$$

(summation in dominance ordering)

Definition (Symmetric Macdonald polynomials)

$$P_{\lambda^+} = \sum_{\lambda \leq \lambda^+} E_\lambda$$

Macdonald polynomials are (q, t) generalisations of Schur polynomials.

II. Matrix Product Form

Exchange relations

Let δ be the *anti-dominant* weight ($\delta_1 \leq \delta_2 \leq \ldots \leq \delta_n$).

Definition (The exchange basis)

$$f_\delta := E_\delta$$

$$f_{\ldots,\lambda_i,\lambda_{i+1},\ldots} := t^{-\frac{1}{2}} T_i^{-1} f_{\ldots,\lambda_{i+1},\lambda_i,\ldots} \quad \lambda_i > \lambda_{i+1}.$$

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Then f solves the exchange equations

$$\begin{aligned} T_i f_{\ldots,\lambda_i,\lambda_{i+1},\ldots} &= t^{\frac{1}{2}} f_{\ldots,\lambda_i,\lambda_{i+1},\ldots} \quad \lambda_i = \lambda_{i+1}, \\ T_i f_{\ldots,\lambda_i,\lambda_{i+1},\ldots} &= t^{-\frac{1}{2}} f_{\ldots,\lambda_{i+1},\lambda_i,\ldots} \quad \lambda_i > \lambda_{i+1}, \\ \omega f_{\lambda_n,\lambda_1,\ldots,\lambda_{n-1}} &= q^{\lambda_n} f_{\lambda_1,\ldots,\lambda_n}. \end{aligned}$$

- Dynamics of the multispecies asymmetric exclusion process
- $t^{1/2}$-deformed Knizhnik-Zamolodchikov equations

Matrix product ansatz

Assume

$$f_\lambda(x_1,\ldots,x_n) = \mathrm{Tr}\left[A_{\lambda_1}(x_1)\cdots A_{\lambda_n}(x_n)S\right],$$

This implies a matrix product for Macdonald polynomials as also

$$P_{\lambda^+} = \sum_{\lambda\leq\lambda^+} f_\lambda$$

'Normalisation of ASEP'.

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'Normalisation of ASEP'.

The exchange relations imply the following algebra for the 'matrices' A:

$$\begin{aligned} A_i(x)A_i(y) &= A_i(y)A_i(x), \\ tA_j(x)A_i(y) - \frac{tx-y}{x-y}\Big(A_j(x)A_i(y) - A_j(y)A_i(x)\Big) &= A_i(x)A_j(y), \\ SA_i(qx) &= q^i A_i(x)S, \end{aligned}$$

Zamolodchikov-Faddeev algebra

For $\lambda \subset r^n$ the algebra relations can be rephrased by writing

$$\mathbb{A}^{(r)}(x) = (A_0(x), \ldots, A_r(x))^T,$$

as an $(r+1)$-dimensional *operator valued* column vector.

Lemma (ZF algebra)

The exchange relations are equivalent to

$$\check{R}(x,y) \cdot [\mathbb{A}(x) \otimes \mathbb{A}(y)] = [\mathbb{A}(y) \otimes \mathbb{A}(x)]$$

$\check{R}(x,y)$ is the $U_{t^{\frac{1}{2}}}(sl_{r+1})$ R-matrix of dimension $(r+1)^2$ ($r = 1$ is the 6-vertex model).

Yang-Baxter algebra and Nested Matrix Product Form

More familiar is rank r Yang-Baxter algebra:

$$\check{R}(x,y) \cdot [L(x) \otimes L(y)] = [L(y) \otimes L(x)] \cdot \check{R}(x,y)$$

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Assume a solution of the following modified RLL relation

$$\check{R}^{(r)}(x,y)\cdot\left[\tilde{L}(x)\otimes\tilde{L}(y)\right] = \left[\tilde{L}(y)\otimes\tilde{L}(x)\right]\cdot\check{R}^{(r-1)}(x,y)$$
$$s\tilde{L}_{ij}(qx) = q^{i-j}\tilde{L}_{ij}(x)s.$$

in terms of an $(r+1)\times r$ operator-valued matrix $\tilde{L}(x)=\tilde{L}^{(r)}(x)$.

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Then

$$\mathbb{A}^{(r)}(x) = \tilde{L}^{(r)}(x)\cdot\tilde{L}^{(r-1)}(x)\cdots\tilde{L}^{(1)}(x)$$
$$S^{(r)} = s^{(r)}\cdot s^{(r-1)}\cdots s^{(1)}$$

Solves the ZF algebra

$$\check{R}(x,y)\cdot[\mathbb{A}(x)\otimes\mathbb{A}(y)] = [\mathbb{A}(y)\otimes\mathbb{A}(x)]$$

Rank 1 solution

$$\left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & c^- & b^+ & 0 \\ \hline 0 & b^- & c^+ & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \cdot \left[\begin{pmatrix} 1 \\ x \end{pmatrix} \otimes \begin{pmatrix} 1 \\ y \end{pmatrix}\right] = \left[\begin{pmatrix} 1 \\ y \end{pmatrix} \otimes \begin{pmatrix} 1 \\ x \end{pmatrix}\right].$$

The corresponding solution to the Yang–Baxter algebra is equal to

$$L^{(1)}(x) = \begin{pmatrix} 1 & a^\dagger \\ xa & x \end{pmatrix},$$

where the operators a, $a^\dagger$ and k satisfy the t-oscillator relations

$$a^\dagger k = tka^\dagger, \qquad ak = t^{-1}ka,$$
$$taa^\dagger - a^\dagger a = t - 1.$$

Trivialising reduces the rank, and thus obtain the solution $\mathbb{A}^{(1)}(x) = \tilde{L}^{(1)}(x)$:

$$\begin{pmatrix} 1 & a^\dagger \\ xa & x \end{pmatrix} \mapsto \begin{pmatrix} 1 & 1 \\ x & x \end{pmatrix}.$$

ACEMS

Rank 2 solution

$$
\left(\begin{array}{ccc|ccc|ccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & c^- & 0 & b^+ & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c^- & 0 & 0 & 0 & b^+ & 0 & 0 \\
\hline
0 & b^- & 0 & c^+ & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & c^- & 0 & b^+ & 0 \\
\hline
0 & 0 & b^- & 0 & 0 & 0 & c^+ & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b^- & 0 & c^+ & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}\right) \cdot \left[\begin{pmatrix} 1 & a^\dagger \\ xk & 0 \\ xa & x \end{pmatrix} \otimes \begin{pmatrix} 1 & a^\dagger \\ yk & 0 \\ ya & y \end{pmatrix} \right] =
$$

$$
\left[\begin{pmatrix} 1 & a^\dagger \\ yk & 0 \\ ya & y \end{pmatrix} \otimes \begin{pmatrix} 1 & a^\dagger \\ xk & 0 \\ xa & x \end{pmatrix} \right] \cdot \left(\begin{array}{cc|cc}
1 & 0 & 0 & 0 \\
0 & c^- & b^+ & 0 \\
\hline
0 & b^- & c^+ & 0 \\
0 & 0 & 0 & 1
\end{array}\right),
$$

We construct a solution of the ZF algebra in the following way:

$$\mathbb{A}^{(2)}(x) = \tilde{L}^{(2)}(x) \cdot \tilde{L}^{(1)}(x) = \begin{pmatrix} 1 & a^\dagger \\ xk & 0 \\ xa & x \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = \begin{pmatrix} 1 + xa^\dagger \\ kx \\ xa + x^2 \end{pmatrix}.$$

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The associated rank 2 solution to the Yang–Baxter algebra is

$$L^{(2)}(x) = \begin{pmatrix} 1 & a_1^\dagger & a_2^\dagger \\ xa_1 k_2 & xk_2 & 0 \\ xa_2 & xa_1^\dagger a_2 & x \end{pmatrix},$$

where $\{a_1, a_1^\dagger, k_1\}$ and $\{a_2, a_2^\dagger, k_2\}$ are two commuting copies of the t-oscillator algebra.

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The map $a_1^\dagger, a_1 \mapsto 1$ and $k_1 \mapsto 0$ reduces the rank of $L^{(2)}(x)$ by one

$$L^{(2)}(x) \mapsto \begin{pmatrix} 1 & 1 & a_2^\dagger \\ xk_2 & xk_2 & 0 \\ xa_2 & xa_2 & x \end{pmatrix} \Rightarrow \tilde{L}^{(2)}(x) = \begin{pmatrix} 1 & a_2^\dagger \\ xk_2 & 0 \\ xa_2 & x \end{pmatrix},$$

Example

$$E_\delta(x_1,\ldots,x_6;q=t^u,t)=\mathrm{Tr}\left[A_0(x_1)A_0(x_2)A_1(x_3)A_1(x_4)A_2(x_5)A_2(x_6)S\right],$$

$$\begin{aligned}A_0(x)&=1+xa^\dagger,\\A_1(x)&=xk,\\A_2(x)&=xa+x^2,\end{aligned}$$

S has the form

$$S=k^u=\mathrm{diag}\{1,t^{-u},t^{-2u},\ldots\}=\mathrm{diag}\{1,q^{-1},q^{-2},\ldots\}.$$

$$\begin{aligned}&\mathrm{Tr}\left[\left(1+x_1a^\dagger\right)\left(1+x_2a^\dagger\right)x_3kx_4kx_5\,(a+x_5)\,x_6\,(a+x_6)\,S\right]\\&=x_3x_4x_5x_6\,\mathrm{Tr}\left[\left(x_5x_6k^2+(x_1+x_2)(x_5+x_6)a^\dagger k^2a+x_1x_2(a^\dagger)^2k^2a^2\right)S\right],\end{aligned}$$

where other terms involving unequal powers of $a^\dagger$ and a have zero trace.

Example

Normalising with $\mathrm{Tr}(k^2 S)$ we finally get

$$
\begin{aligned}
E_\delta(x_1,\ldots,x_6; q=t^u, t) &= x_3x_4x_5^2x_6^2 \\
&+ x_3x_4x_5x_6(x_1+x_2)(x_5+x_6)t^2\frac{\mathrm{Tr}\, a^\dagger a k^2 S}{\mathrm{Tr}\, k^2 S} + x_1x_2x_3x_4x_5x_6t^4\frac{\mathrm{Tr}(a^\dagger)^2 a^2 k^2 S}{\mathrm{Tr}\, k^2 S}
\end{aligned}
$$

Traces can be easily calculated using a Fock space representation.

$$
\begin{aligned}
a^\dagger|m\rangle &= (1-t^{-m-1})^{\frac{1}{2}}|m+1\rangle \qquad a|m\rangle = (1-t^{-m})^{\frac{1}{2}}|m-1\rangle \\
k &= \mathrm{diag}\{1, t^{-1}, t^{-2}, \ldots\}.
\end{aligned}
$$

III. General solution and combinatorics

General L-matrix

Related work ($x = 1$) by (Ferrari& Martin), (Evans, Ferrari & Mallick), (Prolhac, Evans & Mallick), (Arita, Ayyer, Mallick & Prolhac), (Linusson&Ayyer),. . .

Theorem

The matrix $L^{(r)}(x)$ is given by

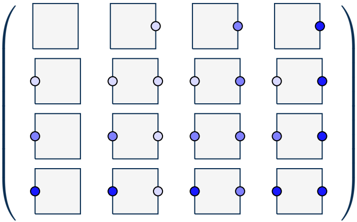
$$L^{(r)}_{ij}(x) = \begin{cases} x \prod_{m=i+1}^{r} k_m, & i = j \\ x a_i a_j^\dagger \prod_{m=i+1}^{r} k_m, & i > j \\ 0, & i < j \end{cases}$$

for all $1 \leq i, j \leq r$, and

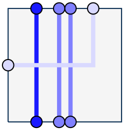
$$L^{(r)}_{0j} = a_j^\dagger,\ 1 \leq j \leq r, \qquad L^{(r)}_{i0}(x) = x a_i \prod_{m=i+1}^{r} k_m,\ 1 \leq i \leq r, \qquad L^{(r)}_{00} = 1,$$

where $\{a_i, a_i^\dagger, k_i\}$, $1 \leq i \leq r$ are r commuting copies of the t-oscillator algebra.

$L^{(3)}(x) =$



$$= \begin{pmatrix} 1 & a_1^\dagger & a_2^\dagger & a_3^\dagger \\ xk_3k_2a_1 & xk_3k_2 & 0 & 0 \\ xk_3a_2 & xk_3a_2a_1^\dagger & xk_3 & 0 \\ xa_3 & xa_3a_1^\dagger & xa_3a_2^\dagger & x \end{pmatrix}$$

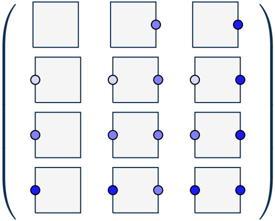


corresponds with $L^{(3)}_{1,0} = k_3k_2a_1$,

Trivialising a_1

$$a_1 = a_1^\dagger = 1, \qquad k_1 = 0.$$

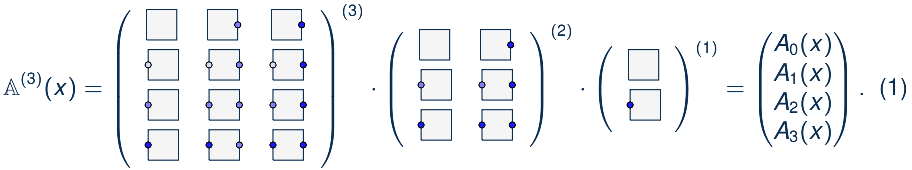
$\tilde{L}^{(3)}(x) =$



$$= \begin{pmatrix} 1 & a_2^\dagger & a_3^\dagger \\ xk_3k_2 & 0 & 0 \\ xk_3a_2 & xk_3 & 0 \\ xa_3 & xa_3a_2^\dagger & x \end{pmatrix}.$$

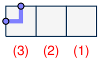
Solution of ZF algebra

$$\mathbb{A}^{(3)}(x) = \begin{pmatrix} 1 & a_2^\dagger & a_3^\dagger \\ xk_3k_2 & 0 & 0 \\ xk_3a_2 & xk_3 & 0 \\ xa_3 & xa_3a_2^\dagger & x \end{pmatrix}^{(3)} \cdot \begin{pmatrix} 1 & a_2^\dagger \\ xk_2 & 0 \\ xa_2 & x \end{pmatrix}^{(2)} \cdot \begin{pmatrix} 1 \\ x \end{pmatrix}^{(1)} = \begin{pmatrix} A_0(x) \\ A_1(x) \\ A_2(x) \\ A_3(x) \end{pmatrix}.$$



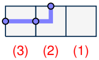
From this it is easy to extract individual components, for example:

$A_2(x) =$



$xk_3^{(3)} a_2^{(3)}$

$+$

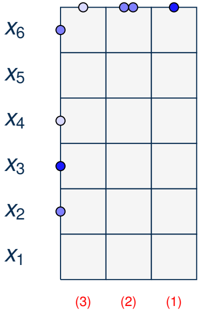


$x^2 k_3^{(3)} k_2^{(2)}$

Combinatorial rule

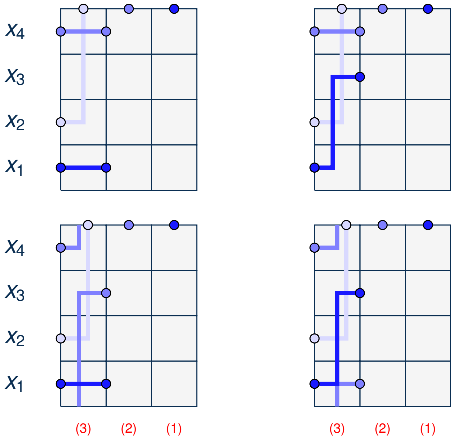
For $r = 3$ and $\lambda = (0, 2, 3, 1, 0, 2)$, the matrix product can be represented in the following way:

$$\mathrm{Tr}(A_0(x_1)A_2(x_2)A_3(x_3)A_1(x_4)A_0(x_5)A_2(x_6)S) =$$



Column by column transition

With $\lambda = (3, 1, 0, 2)$. We obtain the following four terms:



Macdonald polynomials

Recall symmetric Macdonald polynomials:

$$P_{\lambda^+} = \sum_{\lambda \leq \lambda^+} f_\lambda$$

Theorem (Cantini, dG, Wheeler)

For $\lambda \subset r^n$

$$P_\lambda(x_1, \ldots, x_n; q, t) = \sum_{\mu | \mu^+ = \lambda} \mathrm{Tr}\left[S \prod_{i=1}^{n} A_{\mu_i}(x_i) \right],$$

where the sum is over all permutations μ of λ.