# Dimer Model: Full Asymptotic Expansion of the Partition Function

**Pavel Bleher**

Indiana University-Purdue University Indianapolis, USA

Joint work with **Brad Elwood** and **Dražen Petrović**

**GGI, Florence**

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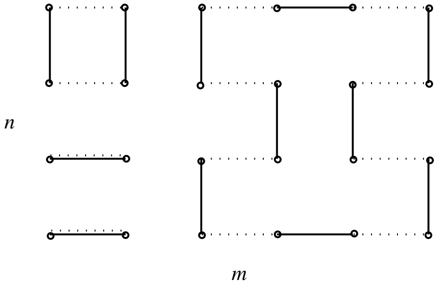
# Dimer Model

Consider a square $m \times n$ lattice $\Gamma_{mn}$ on the plane. We label the vertices of $\Gamma_{mn}$ as $x = (j, k)$, where $1 \leq j \leq m$ and $1 \leq k \leq n$. A *dimer* on $\Gamma_{mn}$ is a set of two neighboring vertices $\langle x, y \rangle$ connected by an edge. A *dimer configuration* $\sigma$ on $\Gamma_{mn}$ is a set of dimers $\{\langle x_k, y_k \rangle,\ k = 1, \dots, \frac{mn}{2}\}$ which cover $\Gamma_{mn}$ without overlapping. An obvious necessary condition for a dimer configuration to exist is that at least one of the numbers $m$ and $n$ be even, so we will assume that $m$ is even.

# An example of a dimer configuration




An example of a dimer configuration on a $6 \times 4$ lattice with *free boundary conditions*. The dot lines show the *standard* configuration. The superposition of the dimer configuration and the standard one produces a set of disjoint *contours*.

# Configuration weights and the partition function

To each *horizontal* dimer we assign a *weight* $z_h$, and to each *vertical* dimer, a *weight* $z_v$. If for a given dimer configuration, $\sigma$, we denote the total number of horizontal dimers by $N_h(\sigma)$ and the total number of vertical dimers by $N_v(\sigma)$, then the *dimer configuration weight* is

$$w(\sigma) = z_h^{N_h(\sigma)} z_v^{N_v(\sigma)}.$$

# Configuration weights and the partition function

The *partition function* of the dimer model is given by

$$Z=\sum_{\sigma} w(\sigma),$$

where the sum runs over all possible dimer configurations $\sigma$, and the *Gibbs probability measure* is given by

$$p(\sigma)=\frac{w(\sigma)}{Z}\,.$$

# Boundary Conditions

In this work we obtain the full asymptotic expansion of the dimer model partition function for the lattice with different boundary conditions:

1. *Free boundary conditions.*
2. *Cylindrical boundary conditions.*
3. *Periodic boundary conditions,*

by using the methods developed in the paper

E.V. Ivashkevich, N.Sh. Izmailian, and Chin-Kun Hu, *Kronecker's double series and exact asymptotic expansions for free models of statistical mechanics on torus,* J. Phys. A: Math. Gen. **35** (2002), 5543–5561

# Full Asymptotic Expansion: Notations

**Function $g(x)$.** Introduce the function

$$g(x) = \ln\left[\zeta \sin(\pi x) + \sqrt{1 + \zeta^2 \sin^2(\pi x)}\right].$$

Observe that $g(x)$ has the following properties:

1. $g(-x) = -g(x), \quad g(x+1) = -g(x),$
2. $g(x)$ is real analytic on $[0,1]$ and

$$g(x) = \sum_{p=0}^{\infty} g_{2p+1} x^{2p+1},$$

where

$$g_1 = \pi\zeta, \quad g_3 = -\frac{\pi^3 \zeta(\zeta^2+1)}{6},$$
$$g_5 = \frac{\pi^5 \zeta(\zeta^2+1)(9\zeta^2+1)}{120}, \;\ldots.$$

# Full Asymptotic Expansion: Notations

**Differential operator $\Delta_p$.** Let $\mathcal{S}_p$ be the set of collections of integers $(p_1, \ldots, p_r; q_1, \ldots, q_r)$, $1 \le r \le p$, such that

$$\mathcal{S}_p = \big\{ 0 < p_1 < \ldots < p_r;\; q_1 > 0, \ldots, q_r > 0; \\ 1 \le r \le p \;\big|\; p_1 q_1 + \ldots + p_r q_r = p \big\}.$$

Introduce the differential operator

$$\Delta_p = \sum_{\mathcal{S}_p} \frac{(g_{2p_1+1})^{q_1} \ldots (g_{2p_r+1})^{q_r}}{q_1! \ldots q_r!} \, \frac{d^q}{d\lambda^q}\,,$$

$$q = q_1 + \ldots + q_r - 1\,.$$

# Full Asymptotic Expansion: Notations

In particular,

$$\Delta_1 = g_3, \quad \Delta_2 = \frac{g_3^2}{2}\frac{d}{d\lambda} + g_5,$$
$$\Delta_3 = \frac{g_3^3}{3!}\frac{d^2}{d\lambda^2} + g_3 g_5 \frac{d}{d\lambda} + g_7, \quad \dots .$$

# Full Asymptotic Expansion: Notations

**Kronecker's double series.** The *Kronecker double series* of order $p$ with parameters $\alpha, \beta$ is defined as

$$K_p^{\alpha,\beta}(\tau) = -\frac{p!}{(-2\pi i)^p} \sum_{(j,k)\neq(0,0)} \frac{e(j\alpha + k\beta)}{(k+\tau j)^p},$$

where $e(x) = e^{-2\pi i x}$.

# Full Asymptotic Expansion: Notations

We will use the Kronecker double series with parameters $(\alpha,\beta)=(\frac12,\frac12),(0,\frac12),(\frac12,0)$:

$$K_p^{\frac12,\frac12}(\tau) = -\frac{p!}{(-2\pi i)^p}\sum_{(j,k)\neq(0,0)}\frac{(-1)^{j+k}}{(k+\tau j)^p},$$

$$K_p^{0,\frac12}(\tau) = -\frac{p!}{(-2\pi i)^p}\sum_{(j,k)\neq(0,0)}\frac{(-1)^{k}}{(k+\tau j)^p},$$

$$K_p^{\frac12,0}(\tau) = -\frac{p!}{(-2\pi i)^p}\sum_{(j,k)\neq(0,0)}\frac{(-1)^{j}}{(k+\tau j)^p}.$$

We will use it for $\tau$ pure imaginary and $p\geq 4$. Then the double series is absolutely convergent.

# Full Asymptotic Expansion: Notations

**Dedekind's eta function.** The *Dedekind eta function* is defined as

$$\eta = \eta(\tau) = e^{\frac{\pi i \tau}{12}} \prod_{k=1}^{\infty} \left(1 - e^{2\pi i \tau k}\right)$$
$$= q^{\frac{1}{12}} \prod_{k=1}^{\infty} \left(1 - q^{2k}\right),$$

where

$$q = e^{\pi i \tau}$$

is the *elliptic nome*.

# Full Asymptotic Expansion: Notations

**Jacobi's theta functions.** There are four *Jacobi theta functions*:

$$\theta_1(z,\tau) = 2\sum_{k=0}^{\infty}(-1)^k q^{\left(k+\frac{1}{2}\right)^2}\sin\big((2k+1)z\big),$$

$$\theta_2(z,\tau) = 2\sum_{k=0}^{\infty} q^{\left(k+\frac{1}{2}\right)^2}\cos\big((2k+1)z\big),$$

$$\theta_3(z,\tau) = 1+2\sum_{k=1}^{\infty} q^{k^2}\cos(2kz),$$

$$\theta_4(z,\tau) = 1+2\sum_{k=1}^{\infty}(-1)^k q^{k^2}\cos(2kz),$$

where $q = e^{\pi i\tau}$ is the elliptic nome.

**Reduction of the parameters**

To simplify the calculations, we write $Z_{mn}(z_h, z_v)$ as

$$Z_{mn}(z_h, z_v) = z_h^{\frac{mn}{2}} Z_{mn}(1, \zeta), \quad \zeta = \frac{z_v}{z_h} > 0,$$

and we will evaluate $Z_{mn}(1, \zeta)$ as $m, n \to \infty$. We will assume that the ratio $\frac{m}{n}$ is separated from $0$ and $\infty$, so that there are constants $C_2 > C_1 > 0$ such that

$$C_1 \leq \frac{m}{n} \leq C_2.$$

# Full Asymptotic Expansion: Notations

Denote

$$S = (m+1)(n+1), \qquad r = \frac{m+1}{n+1}.$$

We will set

$$\tau = i\zeta r,$$

so that the *elliptic nome* is equal to

$$q = e^{\pi i \tau} = e^{-\pi\zeta r}.$$

For brevity we will also denote

$$\eta = \eta(\tau), \qquad \theta_k = \theta_k(0,\tau),\ k = 2,3,4.$$

To indicate the free boundary conditions, we denote $Z_{mn}(1,\zeta)$ as $Z^{(f)}_{mn}(1,\zeta)$.

# Full Asymptotic Expansion for Free Boundary Conditions

Our main result for FBC is the following theorem:

Theorem 1. *As $m, n \to \infty$, we have that*

$$Z^{(f)}_{mn}(1,\zeta) = Ce^{SF-(m+1)J-(n+1)I+R},$$

*where* $S = (m+1)(n+1)$,

$$F = F(\zeta) = \frac{1}{\pi}\int_0^{\zeta} \frac{\arctan x}{x}\,dx,$$
$$J = \frac{1}{2}\ln\left(\zeta + \sqrt{1+\zeta^2}\right),$$
$$I = \frac{1}{2}\ln\left(1 + \sqrt{1+\zeta^2}\right),$$

# Full Asymptotic Expansion for Free Boundary Conditions

$$C = \begin{cases} (1+\zeta^2)^{1/4}\left(\dfrac{2\theta_3}{\eta}\right)^{1/2}, \text{if } n \text{ is even,} \\ (1+\zeta^2)^{1/4}\left(\dfrac{2\theta_2}{\eta}\right)^{1/2}, \text{if } n \text{ is odd,} \end{cases}$$

# Full Asymptotic Expansion for Free Boundary Conditions

*and* $R \sim \sum_{p=1}^{\infty} \frac{R_p}{S^p}$, *where*

$$R_p = \begin{cases} -\dfrac{r^{p+1}}{2p+2}\,\Delta_p\left[K_{2p+2}^{\frac{1}{2},\frac{1}{2}}\left(\dfrac{ir\lambda}{\pi}\right)\right]\Big|_{\lambda=\pi\zeta}, \\ \text{if } n \text{ is even}, \\ -\dfrac{r^{p+1}}{2p+2}\,\Delta_p\left[K_{2p+2}^{0,\frac{1}{2}}\left(\dfrac{ir\lambda}{\pi}\right)\right]\Big|_{\lambda=\pi\zeta}, \\ \text{if } n \text{ is odd}. \end{cases}$$

*In particular,*

$$R_1 = \begin{cases} -\dfrac{r^2 g_3}{120}\left(\dfrac{7}{8}\theta_3^8 + \theta_2^4\theta_4^4\right), & \text{if } n \text{ is even}, \\ -\dfrac{r^2 g_3}{120}\left(\dfrac{7}{8}\theta_2^8 - \theta_3^4\theta_4^4\right), & \text{if } n \text{ is odd}. \end{cases}$$

# Full Asymptotic Expansion for Free Boundary Conditions

*Remark.* The relation

$$R \sim \sum_{p=1}^{\infty} \frac{R_p}{S^p}$$

means that $R$ admits an asymptotic expansion in powers of $S^{-1}$, so that for all $\ell = 1, 2, \ldots,$

$$R = \sum_{p=1}^{\ell} \frac{R_p}{S^p} + \mathcal{O}(S^{-\ell-1}),$$

uniformly with respect to $m, n \to \infty$ satisfying $C_1 \leq \frac{m}{n} \leq C_2$, where $S = (m+1)(n+1)$.

# Full Asymptotic Expansion for Cylindrical Boundary Conditions

In the case of *cylindrical boundary conditions*, we impose periodic boundary conditions (PBC) along one direction, horizontal or vertical, and free boundary conditions (FBC) along the other direction. We assume that $m$ is even, and therefore we have the following three distinct cases to consider:

1. $n$ is even, horizontal PBC.
2. $n$ is odd, horizontal PBC.
3. $n$ is odd, vertical PBC.

For simplicity, we will consider Cases 1 and 2.

# Full Asymptotic Expansion for Cylindrical Boundary Conditions

**Notations.** In the case of cylindrical boundary conditions we set

$$S = m(n+1), \qquad r = \frac{m}{n+1},$$

and

$$\tau = \frac{i\zeta r}{2}.$$

Respectively, the elliptic nome is equal in this case to

$$q = e^{\pi i \tau} = e^{-\frac{\pi \zeta r}{2}}.$$

For brevity we also denote $\eta = \eta(\tau)$, and $\theta_k = \theta_k(0, \tau)$. To indicate the cylindrical boundary conditions, we denote $Z_{mn}(1, \zeta)$ as $Z^{(c)}_{mn}(1, \zeta)$.

# Full Asymptotic Expansion for Cylindrical Boundary Conditions

Our main result for CBC is the following theorem:

Theorem 2. *As $m, n \to \infty$,*

$$Z_{mn}^{(c)}(1, \zeta) = Ce^{SF - mJ + R},$$

*where $F$ and $J$ are the same as in Theorem 1,*

$$C = \begin{cases} \dfrac{\theta_3}{\eta}, & \text{if } n \text{ is even}, \\[2ex] \dfrac{\theta_2}{\eta}, & \text{if } n \text{ is odd}, \end{cases}$$

# Full Asymptotic Expansion for Cylindrical Boundary Conditions

*and*

$$R \sim \sum_{p=1}^{\infty} \frac{R_p}{S^p}\,,$$

*where*

$$R_p = \begin{cases} -\dfrac{r^{p+1}}{2p+2}\,\Delta_p\left[K_{2p+2}^{\frac{1}{2},\frac{1}{2}}\left(\dfrac{ir\lambda}{2\pi}\right)\right]\Big|_{\lambda=\pi\zeta}, \\ \text{if } n \text{ is even,} \\ -\dfrac{r^{p+1}}{2p+2}\,\Delta_p\left[K_{2p+2}^{0,\frac{1}{2}}\left(\dfrac{ir\lambda}{2\pi}\right)\right]\Big|_{\lambda=\pi\zeta}, \\ \text{if } n \text{ is odd.} \end{cases}$$

# Full Asymptotic Expansion for Cylindrical Boundary Conditions

*In particular,*

$$R_1 = \begin{cases} -\dfrac{r^2 g_3}{120}\left(\dfrac{7}{8}\theta_3^8 + \theta_2^4\theta_4^4\right), & \text{if } n \text{ is even,} \\ -\dfrac{r^2 g_3}{120}\left(\dfrac{7}{8}\theta_2^8 - \theta_3^4\theta_4^4\right), & \text{if } n \text{ is odd.} \end{cases}$$

# Full Asymptotic Expansion for Toroidal (Periodic) Boundary Conditions

We would like to evaluate the asymptotic behavior of the dimer model partition function $Z_{mn}^{(t)}(1,\zeta)$ on the toroidal quadratic lattice $\Gamma_{mn}^{(t)}$ of dimensions $m \times n$, when $m, n \to \infty$. As before, we assume that $m$ is even. Denote

$$S = mn, \qquad r = \frac{m}{n}\,.$$

# Full Asymptotic Expansion for Toroidal (Periodic) Boundary Conditions

We set

$$\tau = \begin{cases} i\zeta r, & n \text{ even}, \\ \dfrac{i\zeta r}{2}, & n \text{ odd}, \end{cases}$$

so that the elliptic nome is equal to

$$q = e^{\pi i \tau} = \begin{cases} e^{-\pi\zeta r}, & n \text{ even}, \\ e^{-\frac{\pi\zeta r}{2}}, & n \text{ odd}. \end{cases}$$

For brevity, we denote

$$\eta = \eta(\tau), \qquad \theta_k = \theta_k(0,\tau), \quad k = 2, 3, 4.$$

# Full Asymptotic Expansion for Toroidal (Periodic) Boundary Conditions

Observe that for the toroidal boundary conditions, the partition function is a sum of 4 Pfaffians, one of which is equal to $0$. Therefore, the asymptotic expansion of $Z_{mn}^{(t)}(1,\zeta)$ is given by a sum of three terms. The main result is the following theorem, proven by *Ivashkevich, Izmailian and Hu* in the case $\zeta = 1$ and $n$ even:

# Full Asymptotic Expansion for Toroidal (Periodic) Boundary Conditions

Observe that for the toroidal boundary conditions, the partition function is a sum of *four Pfaffians*, one of which is equal to $0$. Therefore, the asymptotic expansion of $Z_{mn}^{(t)}(1,\zeta)$ is given by a *sum of three terms*. The main result is the following theorem, proven by *Ivashkevich, Izmailian and Hu* in the case $\zeta = 1$ and $n$ even:

# Full Asymptotic Expansion for Toroidal Boundary Conditions

Theorem 3. *As $m, n \to \infty$,*

$$Z_{mn}^{(t)}(\zeta) = e^{SF}\left[C^{(2)}e^{R^{(2)}} + C^{(3)}e^{R^{(3)}} + C^{(4)}e^{R^{(4)}}\right],$$

*where $F$ is the same as in Theorem 1, and*

$$C^{(2)} = \frac{\theta_2^2}{2\eta^2}, \quad C^{(3)} = \frac{\theta_4^2}{2\eta^2}, \quad C^{(4)} = \frac{\theta_3^2}{2\eta^2},$$

if $n$ is even,

$$C^{(2)} = C^{(4)} = \frac{\theta_2}{2\eta}, \quad C^{(3)} = 0, \quad \text{if } n \text{ is odd.}$$

# Full Asymptotic Expansion for Toroidal Boundary Conditions

*and* $R^{(\ell)} \sim \sum_{p=1}^{\infty} \frac{R_p^{(\ell)}}{S^p}, \quad \ell = 2, 3, 4,$ *where*

$$\left.\begin{aligned} R_p^{(2)} &= -\frac{2^{2p+1} r^{p+1} \Delta_p K_{2p+2}^{0,\frac{1}{2}}(\tau)}{p+1}, \\ R_p^{(3)} &= -\frac{2^{2p+1} r^{p+1} \Delta_p K_{2p+2}^{\frac{1}{2},0}(\tau)}{p+1}, \\ R_p^{(4)} &= -\frac{2^{2p+1} r^{p+1} \Delta_p K_{2p+2}^{\frac{1}{2},\frac{1}{2}}(\tau)}{p+1}, \end{aligned}\right\} \quad \text{if } n \text{ is even,}$$

*and*

$$R_p^{(2)} = R_p^{(4)} = -\frac{r^{p+1} \Delta_p K_{2p+2}^{0,\frac{1}{2}}(\tau)}{p+1}, \quad \text{if } n \text{ is odd.}$$

# Full Asymptotic Expansion for Toroidal Boundary Conditions

*In particular,*

$$\left.\begin{aligned} R_1^{(2)} &= -\frac{2r^2 g_3}{15}\left(\frac{7}{8}\theta_2^8 - \theta_3^4\theta_4^4\right), \\ R_1^{(3)} &= -\frac{2r^2 g_3}{15}\left(\frac{7}{8}\theta_4^8 - \theta_2^4\theta_3^4\right), \\ R_1^{(4)} &= -\frac{2r^2 g_3}{15}\left(\frac{7}{8}\theta_3^8 + \theta_2^4\theta_4^4\right), \end{aligned}\right\} \quad \text{if } n \text{ is even,}$$

*and*

$$R_p^{(2)} = R_1^{(4)} = -\frac{r^2 g_3}{60}\left(\frac{7}{8}\theta_2^8 - \theta_3^4\theta_4^4\right), \quad \text{if } n \text{ is odd.}$$

# Sketch of the Proof

**The partition function as a Pfaffian**

The fundamental discovery of *Kasteleyn* is that the partition function $Z$ is equal to the *Pfaffian* of some antisymmetric matrix $A^{\mathrm{K}}$, which is called the *Kasteleyn matrix* (the superscript K in $A^{\mathrm{K}}$ stands for Kasteleyn), so that

$$Z = \mathrm{Pf}\, A^{\mathrm{K}}.$$

# The Kasteleyn Matrix

The *size* of the Kasteleyn matrix $A^{\mathrm{K}}$ is $(mn) \times (mn)$ and its matrix elements $a(x, y)$ are labeled by points $x, y \in V_{mn}$. The *matrix elements* $a(x, y)$ are defined as

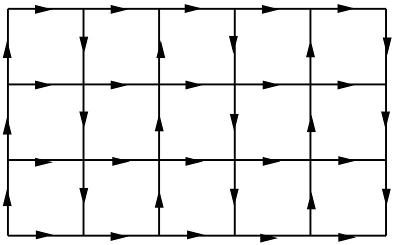
$$\begin{aligned} a((j,k),(j+1,k)) &= -a((j+1,k),(j,k)) \\ &= 1, \quad 1 \le j \le m-1, \quad 1 \le k \le n, \end{aligned}$$

$$\begin{aligned} a((j,k),(j,k+1)) &= -a((j,k+1),(j,k)) \\ &= (-1)^{j+1}\zeta, \quad 1 \le j \le m, \quad 1 \le k \le n-1, \end{aligned}$$

$$a(x, y) = 0, \quad |x - y| \ne 1.$$

Observe the factor $(-1)^{j+1}$ in the second line. It can be interpreted as the sign of the *Kasteleyn orientation* of the edges of the graph $\Gamma_{mn}$.

# The Kasteleyn orientation of the graph



The Kasteleyn orientation of the graph. The fundamental property of the Kasteleyn orientation is that the number of positively oriented edges along the boundary of every plaquette is *odd*.
The crucial fact, which allows us to evaluate the Pfaffian, is the classical formula

$$(\mathrm{Pf}\, A)^2 = \det A.$$

# The Kasteleyn double product formula

Our goal is to evaluate the asymptotic behavior of the dimer model partition function $Z_{mn}(1,\zeta)$ on the $m \times n$ lattice on the plane as $m, n \to \infty$. The first step in this study is to evaluate the determinant of the Kasteleyn matrix $A^{\mathrm{K}}$ by a *diagonalization process*. This leads to the *Kasteleyn double product* formula for $Z_{mn}(1,\zeta)$.

# The Kasteleyn double product formula

The *Kasteleyn double product* formula for $Z_{mn}(1,\zeta)$ is

$$Z_{mn}(1,\zeta) = \begin{cases} \displaystyle\prod_{j=0}^{\frac{m}{2}-1}\prod_{k=0}^{\frac{n}{2}-1}\left[4\left(\cos^2\frac{(j+1)\pi}{m+1} + \zeta^2\cos^2\frac{(k+1)\pi}{n+1}\right)\right], & (n \text{ even,}) \\ \displaystyle\prod_{j=0}^{\frac{m}{2}-1}\prod_{k=0}^{\frac{n-1}{2}-1}\left[4\left(\cos^2\frac{(j+1)\pi}{m+1} + \zeta^2\cos^2\frac{(k+1)\pi}{n+1}\right)\right], & (n \text{ odd.}) \end{cases}$$

# An identity

We will assume that $n$ is even. We have the classical *identity*:

$$\prod_{j=0}^{\frac{m}{2}-1}\left[4\left(u^2+\cos^2\frac{(j+1)\pi}{m+1}\right)\right]$$

$$=\frac{\left(u+\sqrt{1+u^2}\right)^{m+1}-\left(u-\sqrt{1+u^2}\right)^{m+1}}{2\sqrt{1+u^2}},$$

hence we obtain that

$$Z_{mn}(1,\zeta)$$

$$=\prod_{k=0}^{\frac{n}{2}-1}\frac{\left(u_k+\sqrt{1+u_k^2}\right)^{m+1}-\left(u_k-\sqrt{1+u_k^2}\right)^{m+1}}{2\sqrt{1+u_k^2}},$$

where

$$u_k=\zeta\sin\frac{\left(k+\frac{1}{2}\right)\pi}{n+1}.$$

# Factorization of the partition function

We write now

$$Z_{mn}(1,\zeta) = \frac{B_{mn} C_{mn}}{D_n} \, ,$$

where

$$B_{mn} = \prod_{k=0}^{\frac{n}{2}-1} \left( u_k + \sqrt{1+u_k^2} \right)^{m+1} ,$$

$$C_{mn} = \prod_{k=0}^{\frac{n}{2}-1} \left[ 1 + \frac{1}{\left( u_k + \sqrt{1+u_k^2} \right)^{2(m+1)}} \right] ,$$

$$D_n = \prod_{k=0}^{\frac{n}{2}-1} \left( 2\sqrt{1+u_k^2} \right) .$$

# Decomposition of the logarithm of the partition function

Respectively,

$$\ln Z_{mn}(1,\zeta) = G_{mn} + H_{mn} - I_n\,,$$

where

$$G_{mn} = (m+1)\sum_{k=0}^{\frac{n}{2}-1} \ln\left(u_k + \sqrt{1+u_k^2}\right),$$

$$H_{mn} = \sum_{k=0}^{\frac{n}{2}-1} \ln\left[1 + \frac{1}{\left(u_k + \sqrt{1+u_k^2}\right)^{2(m+1)}}\right],$$

$$I_n = \sum_{k=0}^{\frac{n}{2}-1} \ln\left(2\sqrt{1+u_k^2}\right).$$

# Evaluation of $G_{mn}$

Using the *Euler–MacLaurin formula*, we prove the following lemma:

Lemma 4. *As $n, m \to \infty$, we have that $G_{mn}$ admits the following asymptotic expansion:*

$$G_{mn} \sim SF + \frac{\pi\zeta r}{24} - (m+1)J - \frac{1}{2}(m+1)\sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right) g_{2p+1}}{(p+1)(n+1)^{2p+1}},$$

*where $B_{2p+2}\left(\frac{1}{2}\right)$ are the Bernoulli polynomials evaluated at $\frac{1}{2}$.*

# Evaluation of $H_{mn}$

The evaluation of $H_{mn}$ is the most difficult, technical part of the work. Remind that

$$H_{mn} = \sum_{k=0}^{\frac{n}{2}-1} \ln \left[ 1 + \frac{1}{\left( u_k + \sqrt{1+u_k^2} \right)^{2(m+1)}} \right],$$

where

$$u_k = \zeta \sin \frac{\left(k+\frac{1}{2}\right)\pi}{n+1}.$$

# Evaluation of $H_{mn}$

Lemma 5. *As $n, m \to \infty$, we have that*

$$H_{mn} = A + B\,,$$

*where*

$$A = \sum_{k=0}^{\infty} \ln\left(1 + e^{-2r\lambda(k+\frac{1}{2})}\right)\,, \quad \lambda = \pi\zeta\,,$$

*and $B$ admits the following asymptotic expansion:*

$$B = \frac{1}{2}(m+1)\sum_{p=1}^{\infty} \frac{B_{2p+2}\left(\frac{1}{2}\right) g_{2p+1}}{(p+1)(n+1)^{2p+1}} - \sum_{p=1}^{\infty} \frac{r^{p+1}}{S^p(2p+2)} \Delta_p \left[K_{2p+2}^{\frac{1}{2},\frac{1}{2}}\left(\frac{ir\lambda}{\pi}\right)\right]\Big|_{\lambda=\pi\zeta}\,.$$

# Evaluation of $H_{mn}$

*Remark.* Observe that in $G_{mn} + H_{mn}$ the terms, involving the Bernoulli functions, cancel out!

# Evaluation of $I_n$

Remind that

$$I_n=\sum_{k=0}^{\frac{n}{2}-1}\ln\left(2\sqrt{1+u_k^2}\right),\quad u_k=\zeta\sin\frac{\left(k+\frac{1}{2}\right)\pi}{n+1}.$$

Using the Euler–MacLaurin formula, we obtain the following result:

Lemma 6. *As $m,n\to\infty$,*

$$I_n=(n+1)I-\frac{\ln 2}{2}-\frac{1}{4}\ln\left(1+\zeta^2\right)$$
$$+\mathcal{O}(n^{-M}),\quad \forall M>0,$$

*where*

$$I=\frac{1}{2}\ln\left(1+\sqrt{1+\zeta^2}\right).$$

# Evaluation of $\ln Z_{mn}(1,\zeta)$

In summary, we obtain that

$$\begin{aligned}
&\ln Z_{mn}(1,\zeta) = G_{mn} + H_{mn} - I_n \\
&\sim SF + \frac{\pi\zeta r}{24} + \frac{\ln 2}{2} + \frac{1}{4}\ln\left(1+\zeta^2\right) \\
&+ \sum_{k=0}^{\infty} \ln\left(1 + e^{-2r\pi\zeta(k+\frac{1}{2})}\right) \\
&- (m+1)J - (n+1)I \\
&- \sum_{p=1}^{\infty} \frac{r^{p+1}}{S^p} \frac{\Delta_p K_{2p+2}^{\frac{1}{2},\frac{1}{2}}\left(\frac{ir\lambda}{\pi}\right)}{2p+2}\Bigg|_{\lambda=\pi\zeta}.
\end{aligned}$$

# Evaluation of $Z_{mn}(1,\zeta)$

If we denote

$$q = e^{-\pi\zeta r} = e^{\pi i\tau}, \quad \tau = i\zeta r\,,$$

then after exponentiating the previous formula, we obtain that

$$Z_{mn}(1,\zeta) = \sqrt{2}\left(1+\zeta^2\right)^{\frac{1}{4}} q^{-\frac{1}{24}} \prod_{k=0}^{\infty}\left(1+q^{2k+1}\right) \times e^{SF-(m+1)J-(n+1)I+R},$$

where

$$R \sim \sum_{p=1}^{\infty} \frac{R_p}{S^p},$$

$$R_p = -\frac{r^{p+1}}{2p+2}\,\Delta_p\left[K_{2p+2}^{\frac{1}{2},\frac{1}{2}}\left(\frac{ir\lambda}{\pi}\right)\right]\Big|_{\lambda=\pi\zeta}.$$

# Evaluation of the constant factor

Furthermore, we can express the constant factor

$$q^{-\frac{1}{24}} \prod_{k=0}^{\infty} \left(1 + q^{2k+1}\right) = q^{-\frac{1}{24}} \prod_{k=1}^{\infty} \left(1 + q^{2k-1}\right)$$

in terms of the Dedekind eta function as

$$\frac{[\eta(\tau)]^2}{\eta(2\tau)\eta(\frac{\tau}{2})}\,.$$

# Evaluation of the constant factor

Thus, the constant factor is:

$$C = \sqrt{2}\,(1+\zeta^2)^{1/4}\, q^{-\frac{1}{24}} \prod_{k=0}^{\infty} \left(1+q^{2k+1}\right)$$
$$= \sqrt{2}\,(1+\zeta^2)^{1/4}\, \frac{[\eta(\tau)]^2}{\eta(2\tau)\eta(\frac{\tau}{2})}\,.$$

This can be further simplified as

$$C = (1+\zeta^2)^{1/4} \left(\frac{2\theta_3}{\eta}\right)^{1/2},$$

and this finishes the proof of the exact asymptotic expansion of the partition function $Z_{mn}(1,\zeta)$.

# Numerical Check

We checked the asymptotic formula

$$Z_{mn}(1,\zeta) = Ce^{SF-(m+1)J-(n+1)I+R},$$

numerically for various values of $\zeta$ and values of $m, n$ of the order of $10^3$, and we obtained an agreement with a relative error of the order of $10^{-12}$.

**The End**

☺

**Thank you!**