Lozenge tilings and other lattice models – via symmetric functions

Greta Panova (University of Pennsylvania)

based on:

V.Gorin, G.Panova, Asymptotics of symmetric polynomials with applications to statistical mechanics and representation theory, *Annals of Probability* arXiv:1301.0634

G. Panova, Lozenge tilings with free boundaries, arXiv:1408.0417.

Firenze, Maggio 2015

Overview

Characters of $U(\infty)$, boundary of the Gelfand-Tsetlin graph

1	1	1	2	2	...
2	2	3	...		
...					

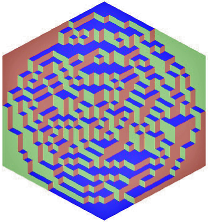
Alternating Sign Matrices (ASM)/ 6-Vertex model:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

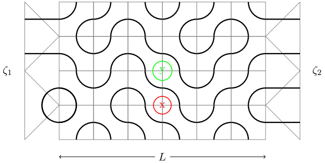
Normalized Schur functions:

$$S_\lambda(x_1, \dots, x_k; N) = \frac{s_\lambda(x_1, \dots, x_k, 1^{N-k})}{s_\lambda(1^N)}$$

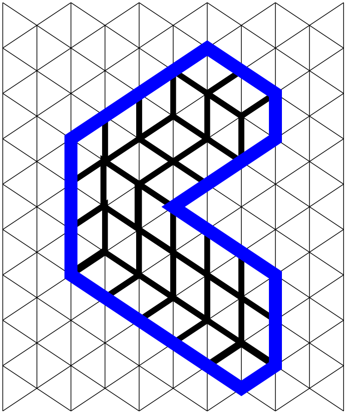
Lozenge tilings:



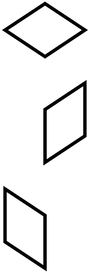
Dense loop model:



Lozenge tilings

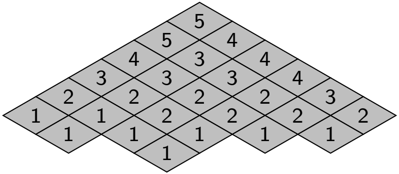


Tilings of a domain Ω (on a triangular lattice) with elementary rhombi of 3 types ("lozenges").



The many faces of lozenge tilings

5	4	4	4	3	2
5	3	3	2	2	1
4	3	2	2	1	
3	2	2	1		
2	1	1	1		
1	1				



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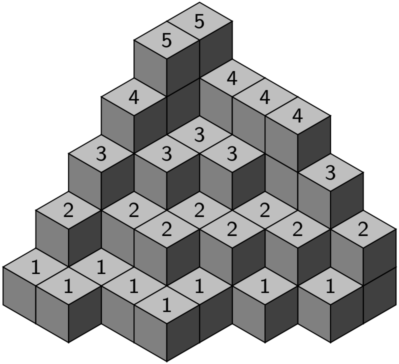

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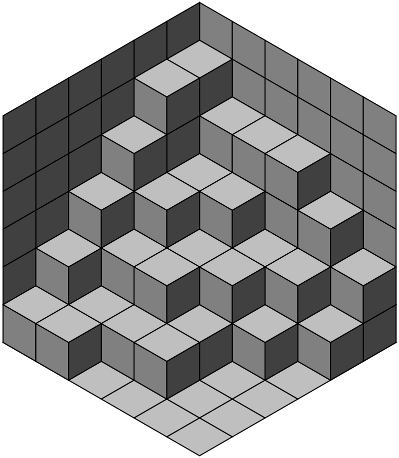

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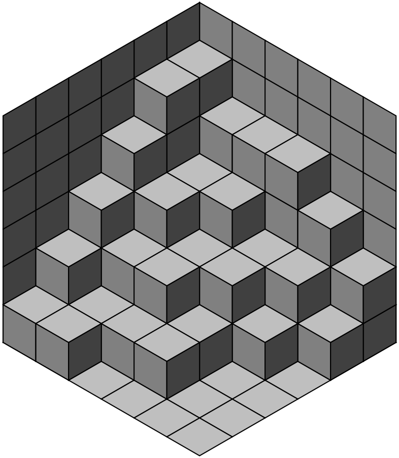

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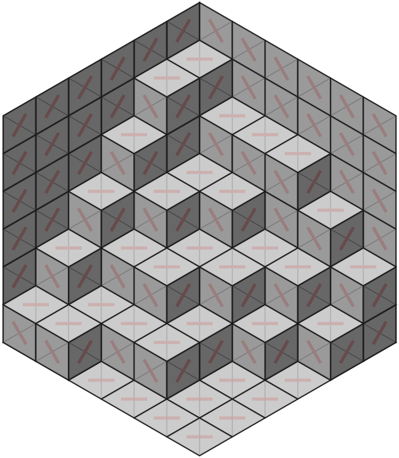

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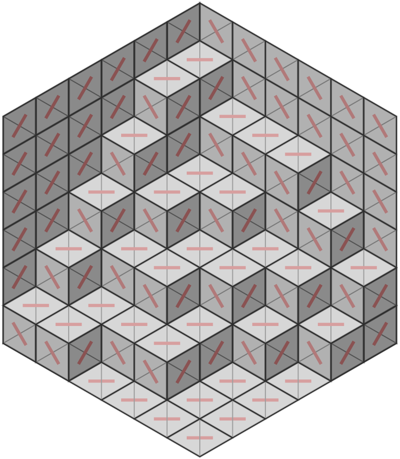

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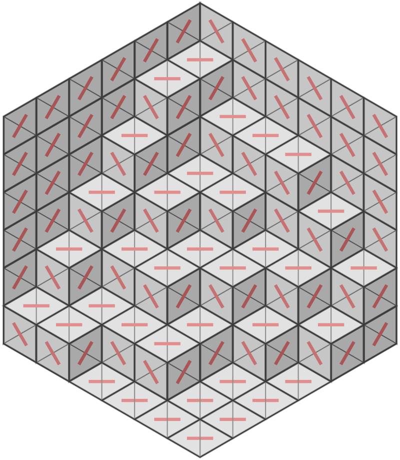

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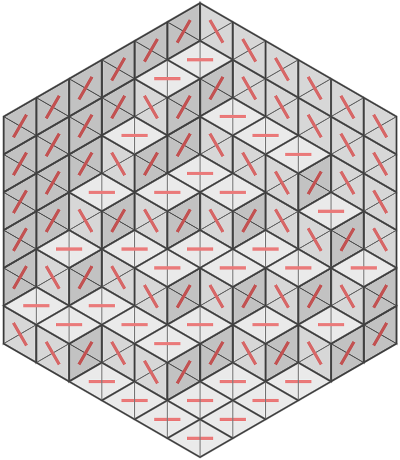

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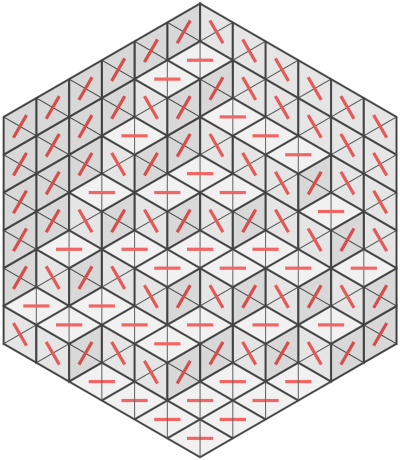

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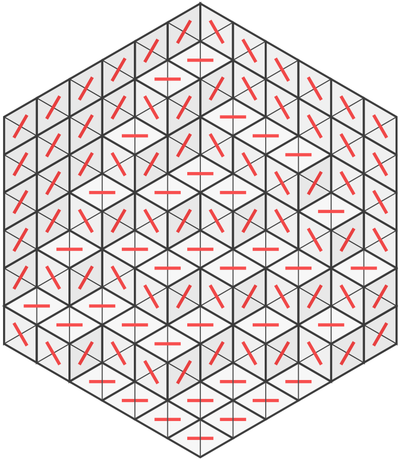

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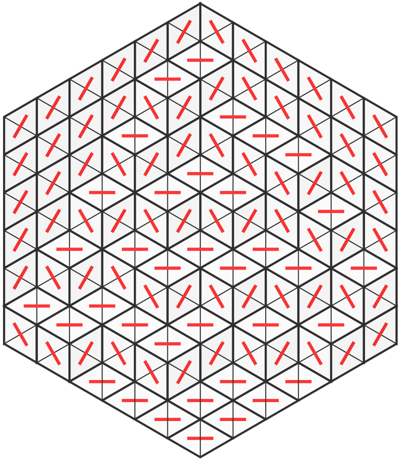

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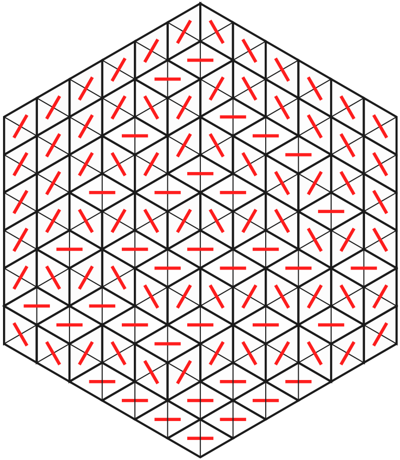

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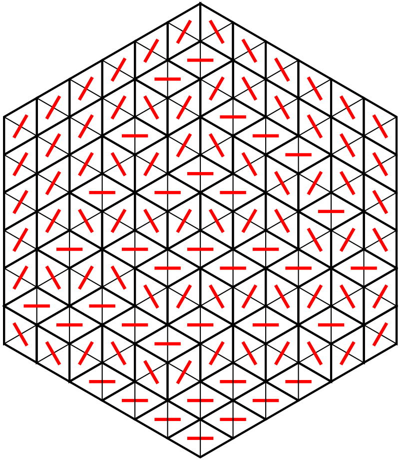
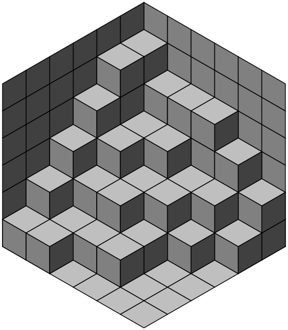


Combinatorics: how many?

5	4	4	4	3	2
5	3	3	2	2	1
4	3	2	2	1	
3	2	2	1		
2	1	1	1		
1	1				



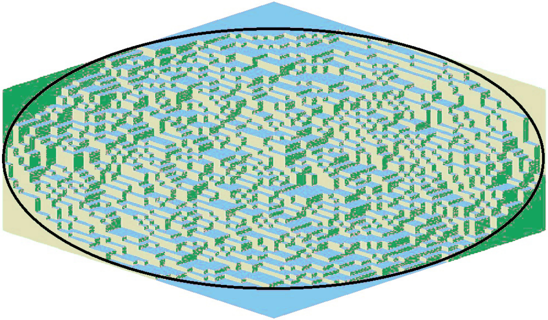
[MacMahon]: Boxed plane partitions (tilings of $a \times b \times c \times a \times b \times c$ hexagon)

$$= \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{i+j+k-1}{i+j+k-2}$$

General: Lindström-Gessel-Viennot determinants; hook-content formula.

Probability: limit behavior

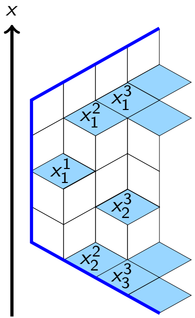
Question: Fix Ω in the plane and let *grid size* $\to 0$, what are the properties of *uniformly random* tilings of Ω?

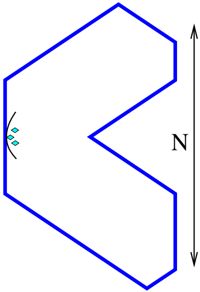


2.jpg

Frozen regions (polygonal domains), "limit shapes" of the surface of the height function (plane partitions).
([Cohn–Larsen–Propp, 1998], [Kenyon–Okounkov, 2005], [Cohn–Kenyon–Propp, 2001; Kenyon-Okounkov-Sheffield, 2006])

Behavior near the boundary, interlacing particles



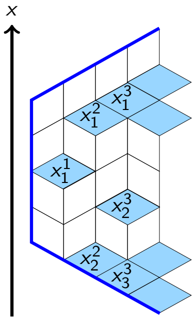


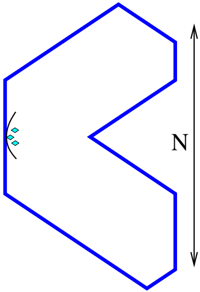
Horizontal lozenges near a flat boundary:

$$\begin{array}{ccccccccc} & & & & x_1^1 & & & & \\ & & & \leq & & \leq & & & \\ & & x_2^2 & & & & x_1^2 & & \\ & \leq & & \leq & & \leq & & \leq & \\ x_3^3 & & & & x_2^3 & & & & x_1^3 \end{array}$$

Question: Joint distribution of $\{x_j^i\}_{i=1}^k$ as $N \to \infty$ (rescaled)?

Behavior near the boundary, interlacing particles





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Question: Joint distribution of $\{x_j^i\}_{i=1}^k$ as $N \to \infty$ (rescaled)?

Conjecture [Okounkov–Reshetikhin, 2006]:
The joint distribution converges to a *GUE*-corners (aka *GUE*-minors) process: eigenvalues of GUE matrices.

Proofs: hexagonal domain [Johansson-Nordenstam, 2006], more general domains [Gorin-P,2012], [Novak, 2014], unbounded [Mkrtchyan, 2013]

The Gaussian Unitary Ensemble (GUE)

GUE: matrices $[X_{ij}]_{i,j}$: $X = \overline{X^T}$
$\mathrm{Re}X_{ij}, \mathrm{Im}X_{ij}$ – i.i.d. $\sim \mathcal{N}(0, 1/2)$, $i \neq j$
X_{ii} – i.i.d. $\sim \mathcal{N}(0, 1)$

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

$(x_1^k \geq x_2^k \geq \cdots \geq x_k^k)$ – eigenvalues of $[X_{i,j}]_{i,j=1}^k$

Interlacing condition: $\quad x_{i-1}^j \leq x_{i-1}^{j-1} \leq x_i^j$

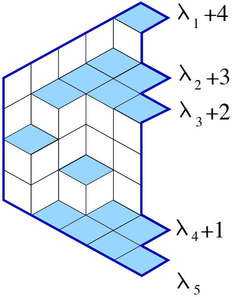
$$\begin{array}{ccccccc} x_1^4 & & x_2^4 & & x_3^4 & & x_4^4 \\ & x_1^3 & & x_2^3 & & x_3^3 & \\ \geq & & x_1^2 & & x_2^2 & & \leq \\ & & & x_1^1 & & & \end{array}$$

The *joint distribution* of $\{x_i^j\}_{1 \leq i \leq j \leq k}$ is the
GUE–corners (also, GUE-minors) process, $=: \mathbb{GUE}_k$.

GUE in tilings: our setup

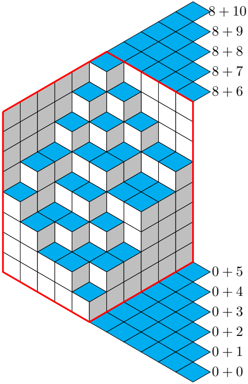
Domain $\Omega_{\lambda(N)}$:
positions of the N horizontal lozenges on right boundary are:

$$\lambda(N)_1 + N - 1 > \lambda(N)_2 + N - 2 > \cdots > \lambda(N)_N$$



$\lambda(5) = (4, 3, 3, 0, 0)$

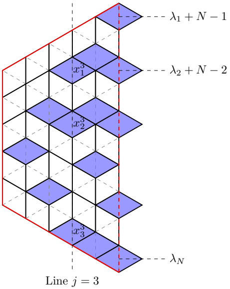
($\frac{1}{N}\Omega_{\lambda(N)}$ is *not necessarily a finite polygon as* $N \to \infty$, e.g. $\lambda(N) = (N, N-1, \ldots, 2, 1)$)

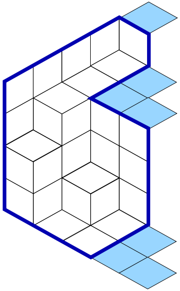


$\lambda = (\underbrace{a, \ldots, a}_{c}, \underbrace{0, \ldots, 0}_{b})$

$\leftrightarrow a \times b \times c$... hexagon.

Plane partitions/Gelfand-Tsetlin patterns





N

5				
4	4			
4	3	3		
3	3	2	1	
2	1	0	0	0

$\lambda = (5, 4, 3, 1, 0)$ $\quad x^3 = (4, 3, 0)$

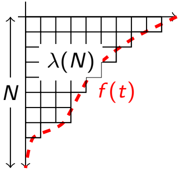
4				
3	3			
3	3	3		
3	1	0	0	
2	0	0	0	0

$\lambda = (4, 3, 3, 0, 0)$

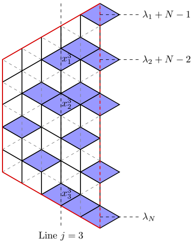
Question: Joint distribution of the (rescaled) positions $\left\{x_j^i\right\}_{i=1}^{k}$ as $N \to \infty$?

GUE in tilings: our results

Limit profile $f(t)$ of $\lambda(N)$ as $N \to \infty$: $\frac{\lambda(N)_i}{N} \to f\left(\frac{i}{N}\right)$



$\Omega_{\lambda(N)}$ domain:



Theorem (Gorin-P (2012), Novak (2014))

Let $\lambda(N) = (\lambda_1(N) \geq \ldots \geq \lambda_N(N))$, $N = 1, 2, \ldots$. If $\exists$ a piecewise-differentiable weakly decreasing function $f(t)$ (limit profile of $\lambda(N)$) s.t. $\sum_{i=1}^{N} \left| \frac{\lambda_i(N)}{N} - f\left(\frac{i}{N}\right) \right| = o(\sqrt{N})$ *as $N \to \infty$ and* $\sup_{i,N} |\lambda_i(N)/N| < \infty$.

Let $\Upsilon^k_{\lambda(N)} = \{x_i^j\}_{j=1}^k$. Then $\forall$ fixed k, as $N \to \infty$

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k \text{ (GUE-corners proc. rank } k\text{)}$$

in the sense of weak convergence, where

$$E(f) = \int_0^1 f(t)dt,$$

$$S(f) = \int_0^1 \left(f(t) - t + \frac{1}{2}\right)^2 dt - \frac{1}{6} - E(f)^2$$

Towards the proof: Schur functions

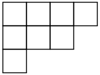
(symmetric functions, Lie group characters)

Irreducible (rational) representations V_λ of $GL(N)$:

dominant weights (signatures/Young diagrams/integer partitions) λ:

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N,$$

where $\lambda_i \in \mathbb{Z}$, e.g. $\lambda = (4,3,1)$,



Schur functions: $s_\lambda(x_1, \ldots, x_N)$ – characters of V_λ.

Weyl's determinantal formula:

$$s_\lambda(x_1, \ldots, x_N) = \frac{\det \left[x_i^{\lambda_j + N - j} \right]_{ij=1}^{N}}{\prod_{i<j}(x_i - x_j)}$$

Semi-Standard Young tableaux($\Leftrightarrow$ Gelfand-Tsetlin patterns) of shape λ :

$$s_{(2,2)}(x_1,x_2,x_3) = s_{\begin{array}{|c|c|}\hline \ & \ \\ \hline \ & \ \\ \hline\end{array}}(x_1,x_2,x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\ \hline 2&2\\ \hline\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\ \hline 3&3\\ \hline\end{array}}{x_1^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\ \hline 3&3\\ \hline\end{array}}{x_2^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\ \hline 2&3\\ \hline\end{array}}{x_1^2x_2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\ \hline 2&3\\ \hline\end{array}}{x_1x_2^2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\ \hline 3&3\\ \hline\end{array}}{x_1x_2x_3^2}.$$

Main object: Normalized Schur functions

$$S_{\lambda(N)}(x_1,\dots,x_k) := \frac{s_{\lambda(N)}(x_1,\dots,x_k,\overbrace{1,\dots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\dots,1}_{N})}$$

or other normalized Lie group characters:

$$X_{\gamma(N)}(x_1,\dots,x_k) := \frac{\chi_{\gamma(N)}(x_1,\dots,x_k,1^{N-k})}{\chi_{\gamma(N)}(1^N)}$$

Harish-Chandra/Itzykson–Zuber integral:

$$\left.\frac{s_\lambda(e^{a_1},\dots,e^{a_n})}{s_\lambda(1,\dots,1)}\right|_{b_j=\lambda_j+N-j} = \prod_{i<j}\frac{a_i-a_j}{e^{a_i}-e^{a_j}}\int_{U(N)} \exp(\mathrm{Trace}(AUBU^{-1}))dU$$

Integral formula, $k=1$ asymptotics

Theorem (Gorin-P)

For any partition λ and any $x \in \mathbb{C} \setminus \{0,1\}$ we have

$$S_\lambda(x;N,1) = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi\mathbf{i}} \oint_C \frac{x^z}{\prod_{i=1}^N (z-(\lambda_i+N-i))} dz,$$

where the contour C includes all the poles of the integrand. Similar formulas hold for the other normalized Lie group characters.

Theorem (Gorin-P)

If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ [under certain convergence conditions], for all fixed $y \neq 0$:

$$\lim_{N\to\infty} \frac{1}{N} \ln S_{\lambda(N)}(e^y;N,1) = yw_0 - \mathcal{F}(w_0) - 1 - \ln(e^y-1),$$

where $\mathcal{F}(w;f) = \int_0^1 \ln(w - f(t) - 1 + t)dt$, w_0 – root of $\frac{\partial}{\partial w}\mathcal{F}(w;f) = y$. If $\frac{\lambda(N)}{N} \to f\left(\frac{i}{N}\right)$ ["other" conv. cond.], for any fixed $h \in \mathbb{R}$:

$$S_{\lambda(N)}(e^{h/\sqrt{N}};N,1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right),$$

$$\text{where} \quad E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 (f(t)-t+1/2)^2 dt - 1/6 - E(f)^2.$$

Remark 1. Similarly for others: symplectic characters, Jacobi...+ q–analogues.
Remark 2. Integral formula also in [Colomo,Pronko,Zinn-Justin], [Guionnet–Maida] (rand. matr), new analysis in [Novak]...

From $k = 1$ asymptotics to general k, multiplicativity

Theorem (Gorin-P[1])

Let $D_{i,1} = x_i \frac{\partial}{\partial x_i}$, $\Delta-$ Vandermonde det. Then $\forall\ \lambda$, $k \le N$, we have

$$S_\lambda(x_1, \dots, x_k; N) = \frac{s_\lambda(x_1, \dots, x_k, \overbrace{1, \dots, 1}^{N-k})}{s_\lambda(\underbrace{1, \dots, 1}_{N})}$$

$$= \prod_{i=1}^{k} \frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}} \times \frac{\det\left[D_{i,1}^{j-1}\right]_{i,j=1}^{k}}{\Delta(x_1, \dots, x_k)} \prod_{j=1}^{k} S_\lambda(x_j; N, 1)(x_j-1)^{N-1}.$$

Corollary (Gorin-P)

Suppose that the sequence $\lambda(N)$ is such that, as $N \to \infty$,

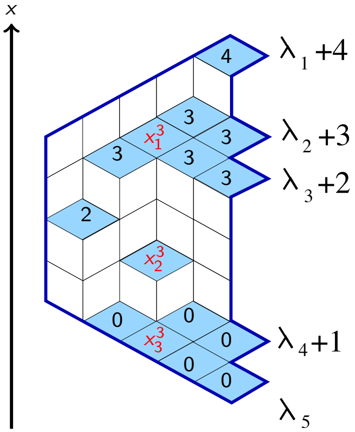
$$\frac{\ln\left(S_{\lambda(N)}(x; N, 1)\right)}{N} \to \Psi(x) \quad \text{uniformly on a compact } M \subset \mathbb{C}. \text{ Then for any } k$$

$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x_1, \dots, x_k; N, 1)\right)}{N} = \Psi(x_1) + \dots + \Psi(x_k)$$

uniformly on M^k. More informally, under various regimes of convergence for $\lambda(N)$ we have $S_{\lambda(N)}(x_1, \dots, x_k) \sim S_{\lambda(N)}(x_1) \cdots S_{\lambda(N)}(x_k)$.

[1] Similar for symplectic characters, Jacobi; and q-analogues. Sympl. chars in *[de Gier, Nienhuis, Ponsaing]*

GUE in tilings I: combinatorics



Tilings of $\Omega_{\lambda(N)}$

$\Leftrightarrow$ Gelfand-Tsetlin schemes, bottom row $\lambda(N)$

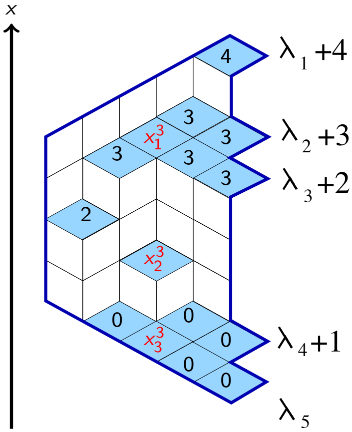
$$\begin{array}{ccccccccc} & & & & 2 & & & & \\ & & & 0 & & 3 & & & \\ & & 0 & & 1 & & 3 & & \\ & 0 & & 0 & & 3 & & 3 & \\ 0 & & 0 & & 3 & & 3 & & 4 \end{array} \quad \Leftrightarrow$$

SSYT of shape $\lambda(N)$

$$T = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 2 & 5 \\ \hline 3 & 4 & 4 & \\ \cline{1-3} 5 & 5 & 5 & \\ \cline{1-3} \end{array}$$

Line j:
(x^j) = shape of the subtableaux of T of the entries $1, \dots, j$.

GUE in tilings I: combinatorics



Tilings of $\Omega_{\lambda(N)}$

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$$
\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & 0 & & 3 & & & \\
 & & 0 & & 1 & & 3 & & \\
 & 0 & & 0 & & 3 & & 3 & \\
0 & & 0 & & 3 & & 3 & & 4
\end{array}
\quad \Leftrightarrow
$$

SSYT of shape $\lambda(N)$

$$
T = \begin{array}{|c|c|c|c|}
\hline
1 & 1 & 2 & 5 \\
\hline
3 & 4 & 4 \\
\cline{1-3}
5 & 5 & 5 \\
\cline{1-3}
\end{array}
\qquad x^3 = (3, 1, 0).
$$

Line j:
(x^j) = shape of the subtableaux of T of the entries $1, \ldots, j$.

GUE in tilings II: moment generating functions

Proposition

In a uniformly random tiling of Ω_λ

$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

where $s_{\lambda/\eta}$ *is the skew Schur polynomial.*

Proof: combinatorial definition of Schur functions as sums over SSYTs.

Proposition

For any variables $y_1, \ldots, y_k$*, the following m.g.f. of* x^k *(as above) is*

$$\mathbb{E}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right) = \frac{s_\lambda(y_1,\ldots,y_k,\overbrace{1,\ldots,1}^{N-k})}{s_\lambda(\underbrace{1,\ldots,1}_{N})} = S_\lambda(y_1,\ldots,y_k).$$

GUE in tilings III: MGF asymptotics

Proposition

$$\mathbb{E}\left[\frac{s_{\nu-\delta_k}(y_1,\dots,y_k)}{s_{\nu-\delta_k}(\underbrace{1,\dots,1}_{k})} \quad \nu \sim \mathbb{GUE}_k\right] = \exp\left(\frac{1}{2}(y_1^2+\cdots+y_k^2)\right),$$

GUE in tilings III: MGF asymptotics

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Compare:

$$S_\lambda(y_1,\dots,y_k) = \mathbb{E}_{tiling}\left(\frac{s_{x^k}(y_1,\dots,y_k)}{s_{x^k}(\underbrace{1,\dots,1}_{k})}\right)$$

Proposition (Gorin-P)

For any k real numbers $h_1,\dots,h_k$ and $\lambda(N)/N \to f$ (as earlier) we have:

$$\lim_{N\to\infty} S_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}},\dots,e^{\frac{h_k}{\sqrt{NS(f)}}}\right) e^{\left(-\frac{E(f)}{\sqrt{NS(f)}}\sum_{i=1}^k h_i\right)} \quad = \quad \exp\left(\frac{1}{2}\sum_{i=1}^k h_i^2\right).$$

GUE in tilings III: MGF asymptotics

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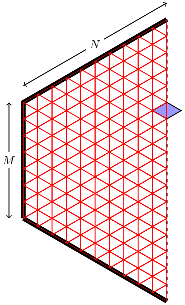
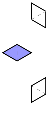
Theorem. Let $\Upsilon^k_{\lambda(N)} = \{x^k, x^{k-1}, \dots\}$ –collection of positions of the horizontal lozenges on lines $k, k-1, \dots, 1$ of tiling from $\Omega_{\lambda(N)}$, then

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k$$ (GUE-corners process of rank k).

Free boundary domains



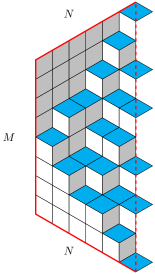
$$T_f(N,M) := \bigcup_{\substack{\lambda:\\ \ell(\lambda)=N,\lambda_1\le M}} \text{tilings of } \Omega_\lambda,$$

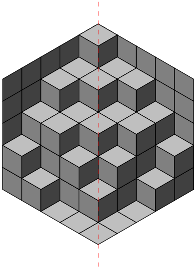


– N free (unrestricted) horizontal rhombi on the right.

$\Leftrightarrow$

symmetric boxed Plane Partitions.

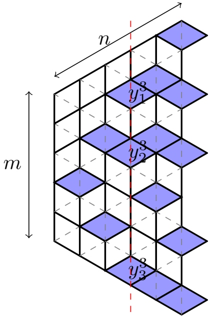




Questions:

1. Existence of a "limit shape" (surface)? Equation: [Di Francesco – Reshetikhin, 2009].
2. Behavior near boundary (GUE?).

Lozenge tilings with free boundaries: GUE



Line $k = 3$

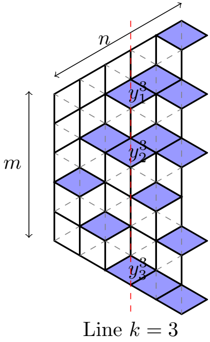
Theorem (P)

Let $Y_{n,m}^k = (y_1^k, \ldots, y_k^k)$ – horizontal lozenges on kth line. As $n, m \to \infty$ with $m/n \to a$ for $0 < a < \infty$ the collection

$$\left\{ \frac{Y_{n,m}^j - m/2}{\sqrt{n(a^2 + 2a)/8}} \right\}_{j=1}^k \to \mathbb{GUE}_k$$

weakly as RVs.

Lozenge tilings with free boundaries: GUE



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Let $Y^k_{n,m} = (y^k_1, \dots, y^k_k)$ – horizontal lozenges on kth line. As $n, m \to \infty$ with $m/n \to a$ for $0 < a < \infty$ the collection

$$\left\{ \frac{Y^j_{n,m} - m/2}{\sqrt{n(a^2 + 2a)/8}} \right\}_{j=1}^{k} \to \mathbb{GUE}_k$$

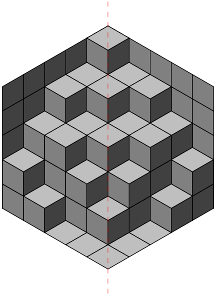
weakly as RVs.

Proof method: Generating function for "free boundary tilings" is an SO_{2n+1} character:

$$\sum_{\lambda \in (m^n)} s_\lambda(x_1, \dots, x_n) = \frac{\det[x_j^{m+2n-i} - x_j^{i-1}]_{1 \le i,j \le n}}{\det[x_j^{2n-i} - x_j^{i-1}]_{1 \le i,j \le n}} = \gamma_{(m^n)}(x),$$

Apply the same asymptotic techniques to this MGF as for Schur functions $\to$ GUE eigenvalues MGF as $N \to \infty$.

Lozenge tilings with free boundaries: limit shape (limit surface)



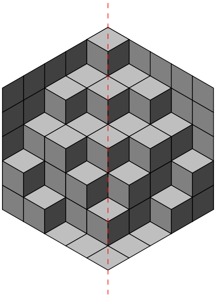
Theorem (P)

Let $n, m \in \mathbb{Z}$, such that $m/n \to a$ as $n \to \infty$, where $a \in (0, +\infty)$. Let $H_n(u, v)$ – height function of tiling from $T_f(n, m)$, i.e.

$$H_n(u, v) = \frac{1}{n} y_{\lfloor nv \rfloor}^{\lfloor nu \rfloor} - v.$$

For all $1 \geq u \geq v \geq 0$, as $n \to \infty$ we have that $H_n(u, v)$ converges uniformly in probability to a deterministic function $L(u, v)$ ("the limit shape").

Lozenge tilings with free boundaries: limit shape (limit surface)



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For any fixed $u \in (0, 1)$, $L(u, v)$ is the distribution function of the limit measure $\mathbf{m}$, given by its moments:

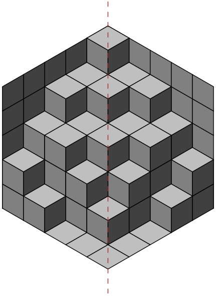
$$\int_{\mathbb{R}} x^r \mathbf{m}(dx) = \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{(\ell+1)!} \frac{\partial^\ell}{\partial t^\ell} \Phi_a(t) \Big|_{t=1},$$

where $\Phi_a(e^y) = y\frac{a}{2} + 2\phi(y; a) - 2$ and...

$$h(y) = \frac{1}{4}\left((e^y + 1) + \sqrt{(e^y + 1)^2 + 4(a^2 + a)(e^y - 1)^2} \right)$$

$$\phi(y; a) = (\frac{a}{2} + 1) \ln \left(h(y) - (\frac{a}{2} + 1)(e^y - 1) \right) - (\frac{a}{2} + \frac{1}{2}) \ln \left(h(y) - (\frac{a}{2} + \frac{1}{2})(e^y - 1) \right)$$
$$+ \frac{a}{2} \ln \left(h(y) + \frac{a}{2}(e^y - 1) \right) - (\frac{a}{2} + \frac{1}{2}) \ln \left(h(y) + (\frac{a}{2} + \frac{1}{2})(e^y - 1) \right)$$

Lozenge tilings with free boundaries: limit shape (limit surface)



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Proof, idea:[2]

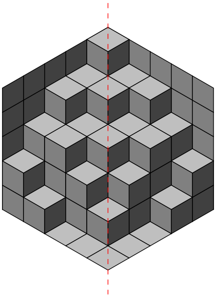
Horizontal lozenges at line $x = un$ at positions $\mu = y^{un} \sim \rho$ (unif. on all tilings), giving a sequence of *random measures* $m[\mu] = \frac{1}{n} \sum_i \delta\left(\frac{\mu_i}{n}\right)$.

$$S_\rho(u_1, \ldots, u_N) := \sum_{\mu:\ell(\mu)=N} \rho(\mu) \frac{s_\mu(u_1, \ldots, u_N)}{s_\mu(1^N)} = \left(\frac{\gamma_{m^n}(x_1, \ldots, x_N)}{\gamma_{(m^n)}(1^N)} \text{ – our MGF} \right)$$

Asymptotics of this MGF + ∇ operators $\Rightarrow$ *concentration phenomenon*: $m[\mu] \to \mathbf{m}$ – a deterministic measure, *the limit shape L*.

[2] after [Borodin-Bufetov-Olshanski, Bufetov-Gorin]

Lozenge tilings with free boundaries: limit shape (limit surface)



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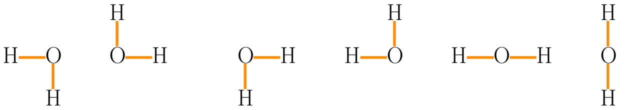
For all $1 \geq u \geq v \geq 0$, as $n \to \infty$ we have that $H_n(u, v)$ converges uniformly in probability to a deterministic function $L(u, v)$ ("the limit shape").

Corollary (P)

The height function of a half-hexagon with free right boundary converges (in probability) to a limit shape (surface) $H(x, y)$, which coincides with the limit shape for the tilings of the full hexagon (fixed boundary). After shifting by $m/2$ and rescaling by $\sqrt{n(a^2 + 2a)/8}$, as $n, m \to \infty, m/n \to a$, the positions of the horizontal lozenges on the k-th vertical line have the same joint distributions as in the full hexagon ($\mathbb{GUE}_k$).

6 Vertex model / ASM

Six vertex types:

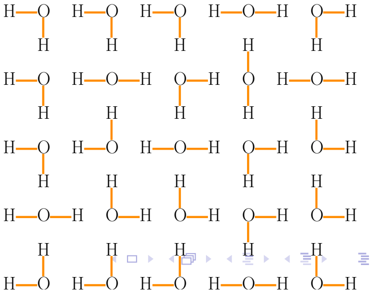


a	a	b	b	c	c
0	0	0	0	1	-1

Alternating Sign Matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

A 6 vertex model configuration:



Definitions and background on ASMs

Definition: A is an *Alternating Sign Matrix* **ASM** of size n if:

$$A \in \{0,+1,-1\}^{n\times n}, \qquad \sum_{i=1}^{n} A_{i,j} = 1, \qquad \sum_{j=1}^{n} A_{i,j} = 1$$

and $(A_{i,k}, i = 1 \dots n \text{ s.t. } A_{i,k} \neq 0) = (1,-1,1,-1,\dots,-1,1)$
A *monotone triangle* is a Gelfand-Tsetlin pattern with *strictly* increasing rows.

6 Vertex model (domain-wall bdry.cond) $\leftrightarrow$ ASM $\leftrightarrow$ monotone triangles.

ASM: $\begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$

Definitions and background on ASMs

Definition: A is an *Alternating Sign Matrix* **ASM** of size n if:

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Question: As $n \to \infty$:
Uniformly random ASM.
What is the *distribution* of *the positions of the 1s and -1s near the boundary* ?

Known:

- Limit behavior(conj): Behrend, Colomo, Pronko, Zinn-Justin, Di Francesco.
- Free fermions point (weight(1/-1)=2) $\leftrightarrow$ domino tilings.
- Exact gen. functions for special statistics.

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positions of 1s $\Updownarrow$ in sum of first k rows

Monotone triangle:

$$\begin{array}{ccccccccc} & & & & 4 & & & & \\ & & & 2 & & 5 & & & \\ & & 2 & & 3 & & 5 & & \\ & 1 & & 2 & & 3 & & 5 & \\ 1 & & 2 & & 3 & & 4 & & 5 \\ & & & & < & & & & \end{array}$$

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Uniformly random ASM.
What is the *distribution* of *the positions of the 1s and -1s near the boundary* ?

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- Limit behavior(conj): Behrend, Colomo, Pronko, Zinn-Justin, Di Francesco.
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- Exact gen. functions for special statistics.

ASMs/6Vertex: new results

$$\text{ASM } A: \qquad \Psi_k(A) := \sum_{j=1:n\ ,\ A_{kj}=1} j - \sum_{j=1:n\ ,\ A_{kj}=-1} j$$

$$\text{Monotone triangle } M = [m^i_j]_{j\le i}: \qquad \Psi_k(M) = \sum_{j=1}^{k} m^k_j - \sum_{j=1}^{k-1} m^{k-1}_j$$

$$k: \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ \color{red}0 & \color{red}1 & \color{red}0 & \color{red}-1 & \color{red}1 \\ \color{blue}0 & \color{blue}0 & \color{blue}1 & \color{blue}0 & \color{blue}0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{array}{ccccccccc} &&&& 4 &&&& \\ &&& \color{red}2 && \color{red}5 &&& \\ && \color{blue}2 && \color{blue}3 && \color{blue}5 && \\ & 1 && 2 && 3 && 5 & \\ 1 && 2 && 3 && 4 && 5 \end{array}$$

$\color{red}2$ $\quad \color{red}{\Psi_2 = 2+5-4 = 3}$ $\qquad \color{red}{\Psi_2 = (2+5)-(4) = 3}$

$\color{blue}3$ $\quad \color{blue}{\Psi_3 = 3}$ $\qquad \color{blue}{\Psi_3 = (2+3+5)-(2+5)}$

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$$\begin{array}{lcc} 2 & \Psi_2 = 2+5-4 = 3 & \Psi_2 = (2+5)-(4) = 3 \\ 3 & \Psi_3 = 3 & \Psi_3 = (2+3+5)-(2+5) \end{array}$$

Theorem (Gorin-P)

If $A \sim$ unif. rand. $n \times n$ ASM, then $\frac{\Psi_k(A)-n/2}{\sqrt{n}}$, $k = 1, 2, \ldots$ converge as $n \to \infty$ to the collection of i.i.d. Gaussian random variables, $N(0, \sqrt{3/8})$.

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Using this Theorem on $\Psi_k(n)$ and the Gibbs property:

Theorem (G, 2013; Conjecture in [Gorin-P])

Fix any k. The [centered,rescaled] positions of 1s on the first k rows (top k rows of the monotone triangle M) tend to the GUE-corners process:

$$\sqrt{\frac{8}{3n}}\left([M]_{i=1:k} - \frac{n}{2}\right) \to \mathbb{GUE}_k.$$

6Vertex/ASMs: proofs

Vertex weights at (i,j):

type:	a	b	c
wight:	$q^{-1}u_i^2 - qv_j^2$	$q^{-1}v_j^2 - qu_i^2$	$(q^{-1}-q)u_i v_j$

($v_1,\ldots,v_N$, $u_1,\ldots,u_N$ – parameters, $q = \exp(\pi\mathbf{i}/3)$)

Set $\lambda(N) := (N-1, N-1, N-2, N-2, \ldots, 1, 1, 0, 0) \in \mathbb{GT}_{2N}$.

Proposition (Okada;Stroganov)

Let $\beth_N$ be the set of all 6-Vertex configurations on an $N \times N$ grid.

$$\sum_{\vartheta \in \beth_N} \prod_{v \text{ vertex of} \vartheta} weight(v) = (-1)^{N(N-1)/2}(q^{-1}-q)^N \prod_{i=1}^{N}(v_i u_i)^{-1} s_{\lambda(N)}(u_1^2,\ldots,u_N^2,v_1^2,\ldots,v_N^2).$$

6Vertex/ASMs: proofs

Vertex weights at (i,j):

type:	a	b	c
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$(v_1,\dots,v_N,\ u_1,\dots,u_N$ – parameters, $q=\exp(\pi\mathbf{i}/3)$ $)$

Set $\lambda(N) := (N-1, N-1, N-2, N-2, \dots, 1, 1, 0, 0) \in \mathbb{GT}_{2N}$.

Proposition

Let $\widehat{x}_i$ =*number of vertices of type* x *on row* i, *then* $\forall$ *collection of rows* $i_1,\dots,i_m$

$$\mathbb{E}_N \prod_{\ell=1}^{m}\left[\left(\frac{q^{-1}-qv_\ell^2}{q^{-1}-q}\right)^{\widehat{a}_{i_\ell}}\left(\frac{q^{-1}v_\ell^2-q}{q^{-1}-q}\right)^{\widehat{b}_{i_\ell}}(v_\ell)^{\widehat{c}_{j_\ell}}\right]$$

$$=\left(\prod_{\ell=1}^{n} v_\ell^{-1}\right)\frac{s_{\lambda(N)}(v_1,\dots,v_m,1^{2N-m})}{s_{\lambda(N)}(1^{2N})}=\left(\prod_{\ell=1}^{n} v_\ell^{-1}\right)S_{\lambda(N)}(v_1,\dots,v_m)$$

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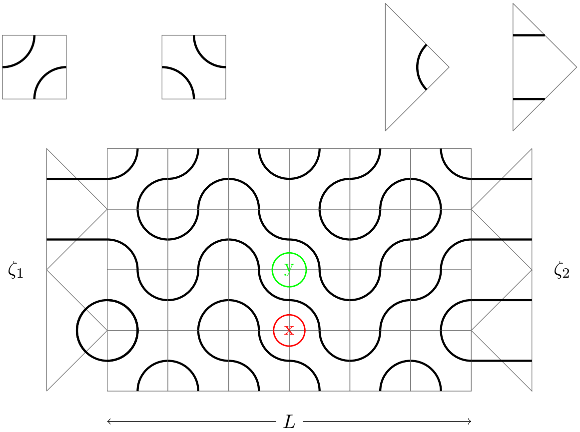
$$= \left(\prod_{\ell=1}^n v_\ell^{-1}\right) \frac{s_{\lambda(N)}(v_1,\dots,v_m,1^{2N-m})}{s_{\lambda(N)}(1^{2N})} = \left(\prod_{\ell=1}^n v_\ell^{-1}\right) S_{\lambda(N)}(v_1,\dots,v_m)$$

Proof of Theorem: Proposition $\to$ *MGF* for $\Psi_k = \widehat{a}_k$ – normalized Schur function. Approximations using $c_k \le 2k-1$ – get MGF for $\widehat{a}_k$. Asymptotics:

$$S_{\lambda(N)}(e^{y_1/\sqrt{n}},\dots,e^{y_k/\sqrt{n}}) = \prod_{i=1}^k \exp\left[\sqrt{n}y_i + \frac{5}{12}y_i^2 + o(1)\right]$$

The dense loop model

Vertical strip of width L and height $\to \infty$:
tiles – squares, boundary – triangales:



Mean total current between pts x and y:
$F^{x,y}$ = avg number of paths connecting the 2 boundaries, passing between x and y.
Similar observables in the critical percolation model [Smirnov, 2009].

Dense loop model: the mean current

$\lambda^L := (\lfloor \frac{L-1}{2} \rfloor, \lfloor \frac{L-2}{2} \rfloor, \ldots, 1, 0, 0)$

$$u_L(\zeta_1, \zeta_2; z_1, \ldots, z_L) := (-1)^L \imath \frac{\sqrt{3}}{2} \ln \left[\frac{\chi_{\lambda^{L+1}}(\zeta_1^2, z_1^2, \ldots, z_L^2)\chi_{\lambda^{L+1}}(\zeta_2^2, z_1^2, \ldots, z_L^2)}{\chi_{\lambda^L}(z_1^2, \ldots, z_L^2)\chi_{\lambda^{L+2}}(\zeta_1^2, \zeta_2^2, z_1^2, \ldots, z_L^2)} \right]$$

χ_ν – character for the $Sp(\mathbb{C})$-irrep of heighest weight ν.

$$X_L^{(j)} = z_j \frac{\partial}{\partial z_j} u_L(\zeta_1, \zeta_2; z_1, \ldots, z_L)$$

$$Y_L = w \frac{\partial}{\partial w} u_{L+2}(\zeta_1, \zeta_2; z_1, \ldots, z_L, vq^{-1}, w)|_{v=w},$$

Proposition (De Gier, Nienhuis, Ponsaing)

Under certain assumptions the mean total current between two horizontally adjacent points is

$$X_L^{(j)} = F^{(j,i),(j+1,i)},$$

and Y is the mean total current between two vertically adjacent points in the strip of width L:

$$Y_L^{(j)} = F^{(j,i),(j,i+1)}.$$

Dense loop model: asymptotics of the mean current

Theorem

As $L \to \infty$ we have

$$X_L^{(j)}\Big|_{z_j=z;\, z_i=1,\ i\neq j} = \frac{\mathbf{i}\sqrt{3}}{4L}(z^3 - z^{-3}) + o\left(\frac{1}{L}\right)$$

and

$$Y_L\Big|_{z_i=1,\ i=1,\dots,L} = \frac{\mathbf{i}\sqrt{3}}{4L}(w^3 - w^{-3}) + o\left(\frac{1}{L}\right)$$

Remark 1. When $z = 1$, $F^{(j,i),(j+1,i)}$ is (trivially) identically zero. ✓
Remark 2. The fully homogeneous case when $w = \exp^{-\mathbf{i}\pi/6}$, $q = e^{2\pi\mathbf{i}/3}$, then

$$Y_L = \frac{\sqrt{3}}{2L} + o\left(\frac{1}{L}\right).$$

Proof: same type of asymptotic methods and results for symplectic characters + some tricks with the multivariate formula.

Thank you