LOZENGE TILINGS WITH GAPS IN A 90° WEDGE DOMAIN WITH MIXED BOUNDARY CONDITIONS

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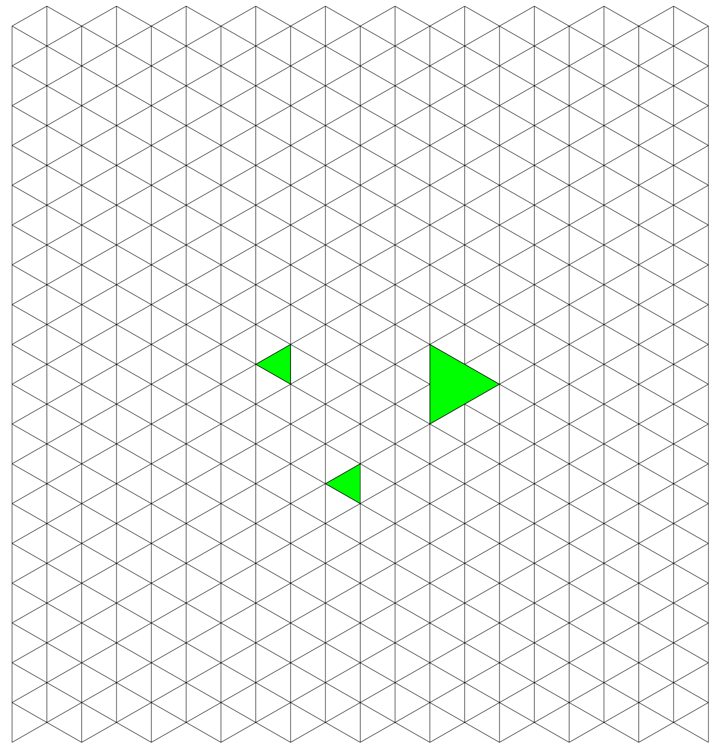


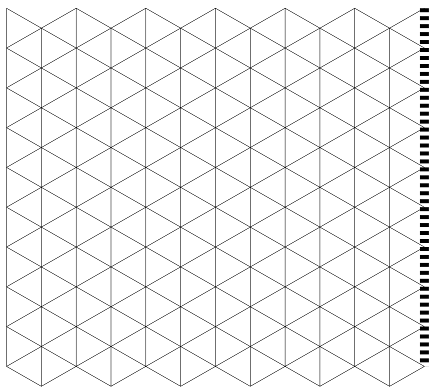
Correlation in a sea of dimers

[C, '05–'10]

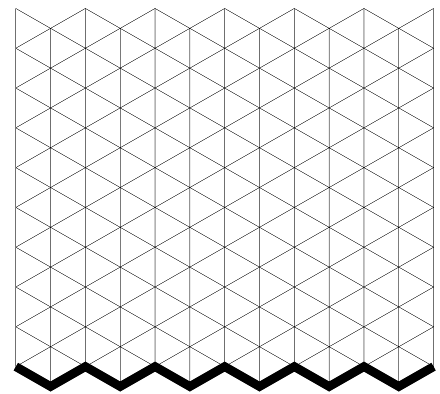
In bulk, for large separations, this is asymptotically 2D electrostatics

What about the interaction with boundary?

Two natural types:

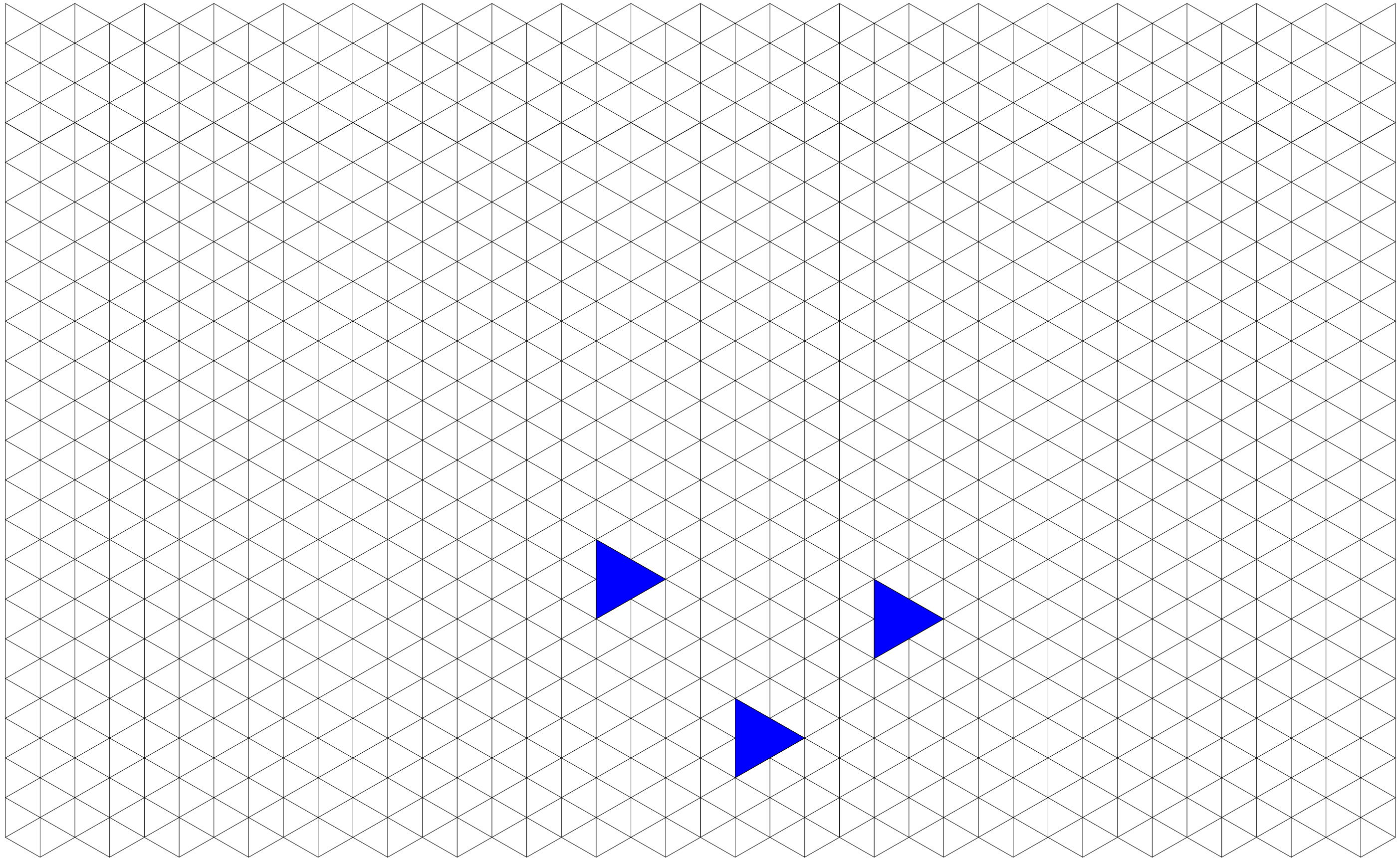


free boundary

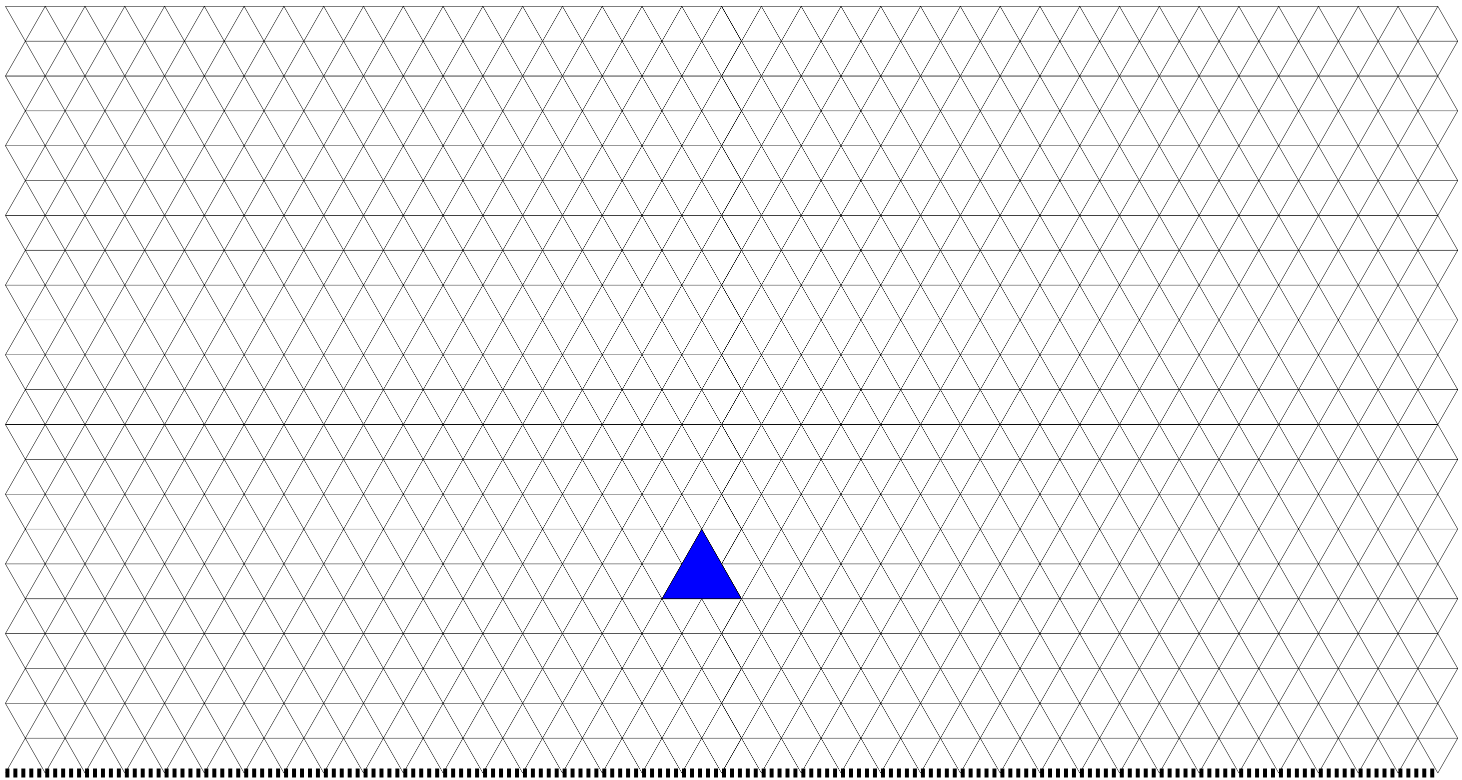


constrained boundary

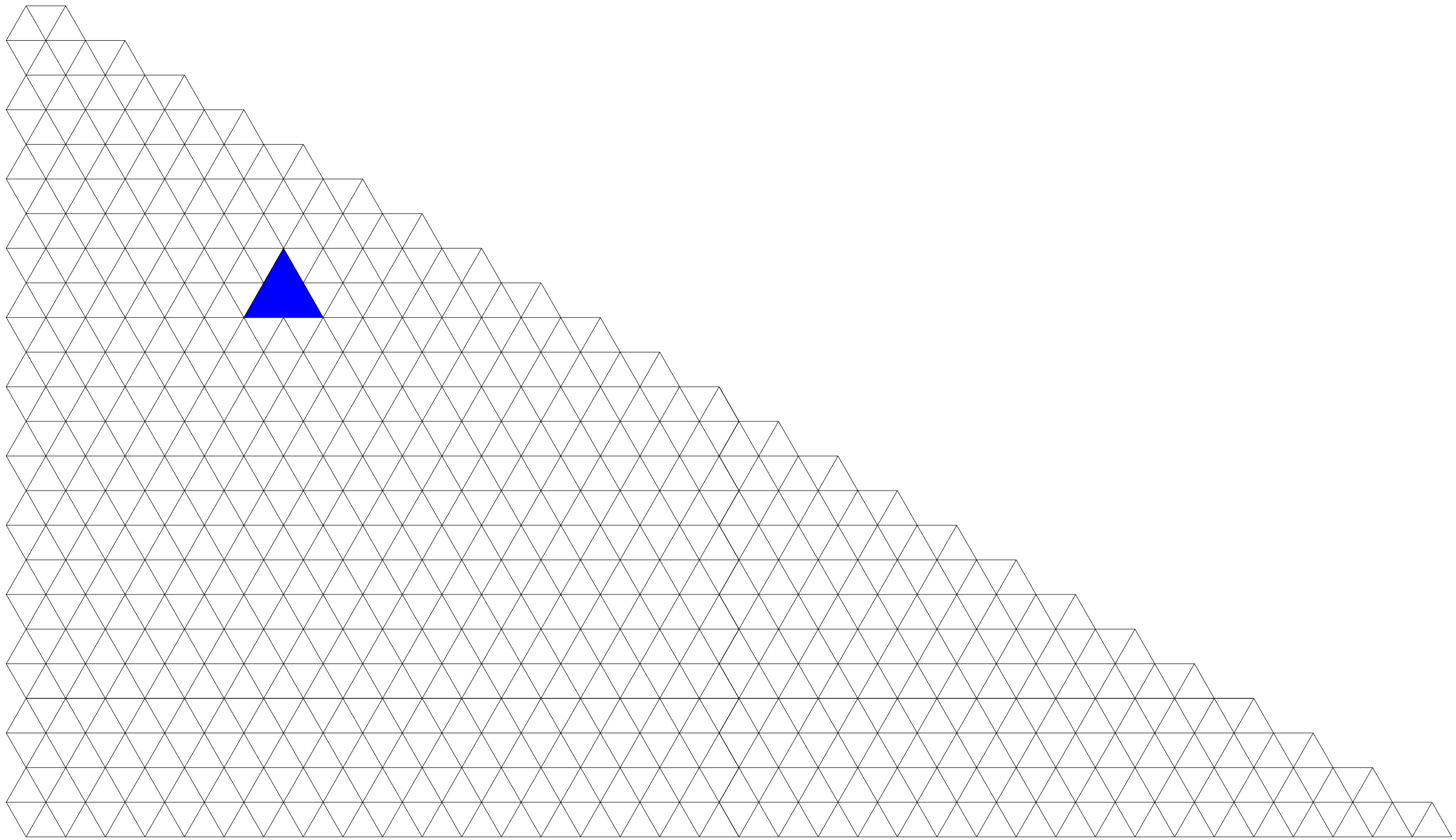
Previous examples



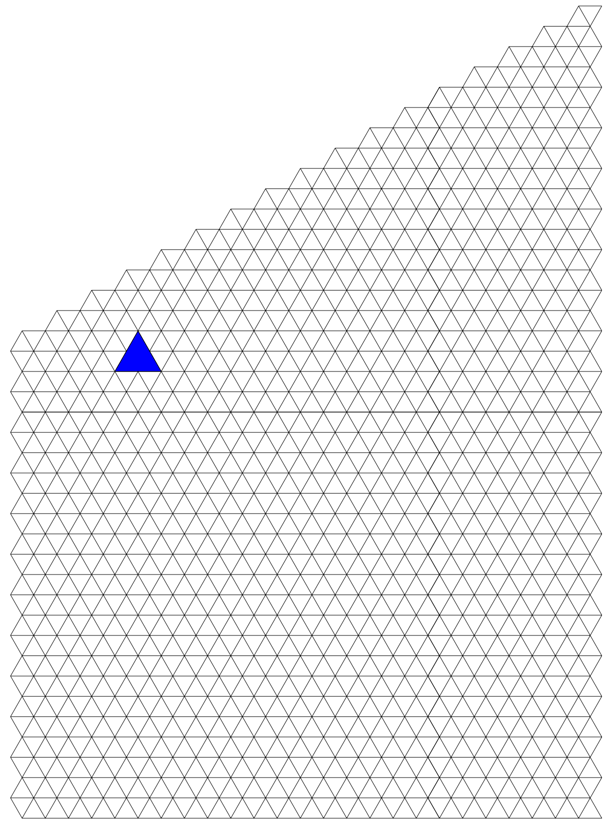
“straight line” constrained boundary



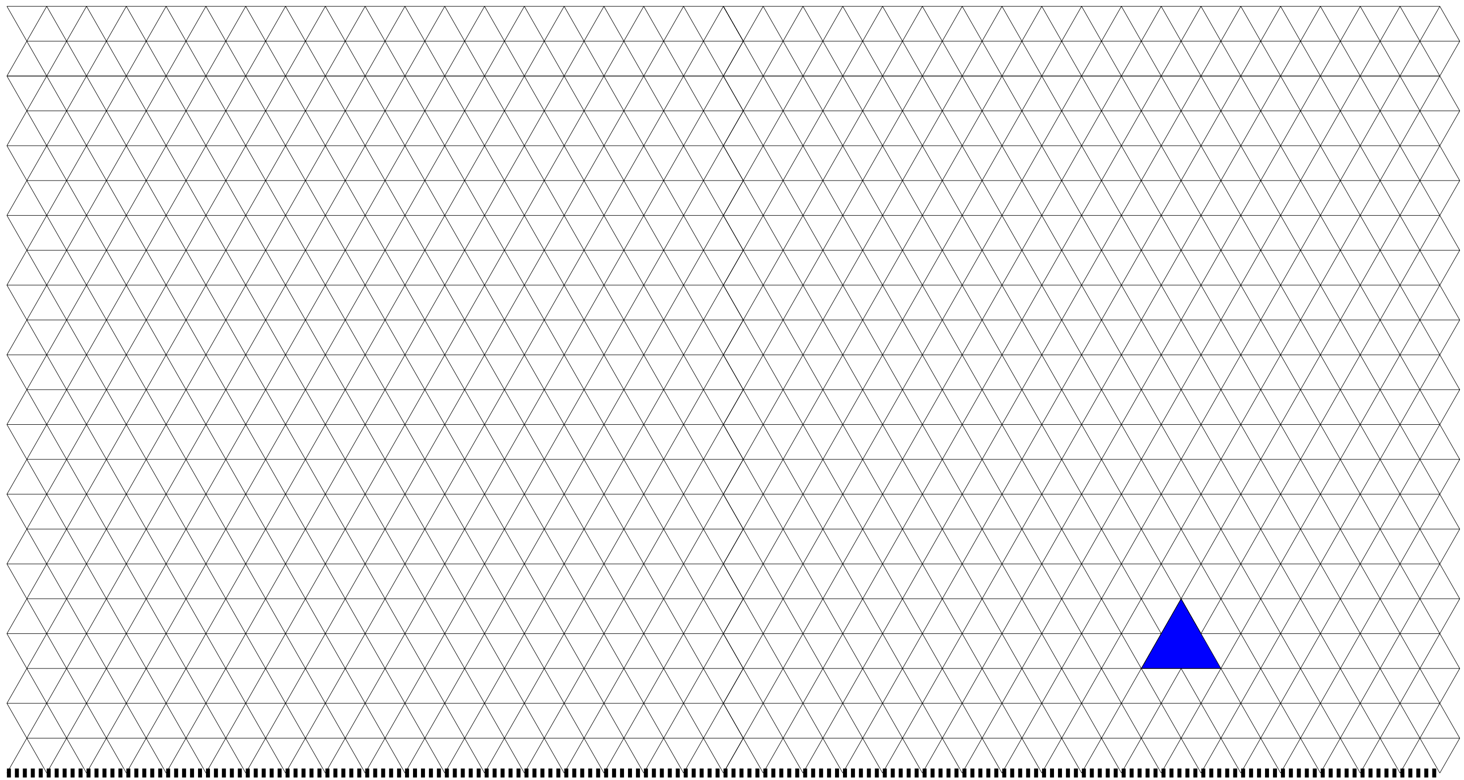
straight line free boundary



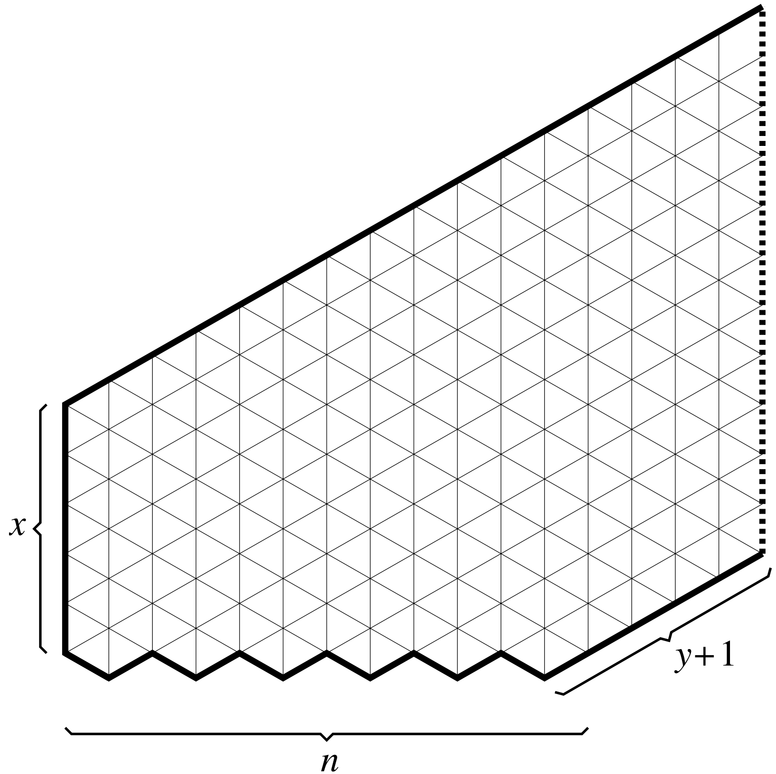
60° angle, constrained boundary



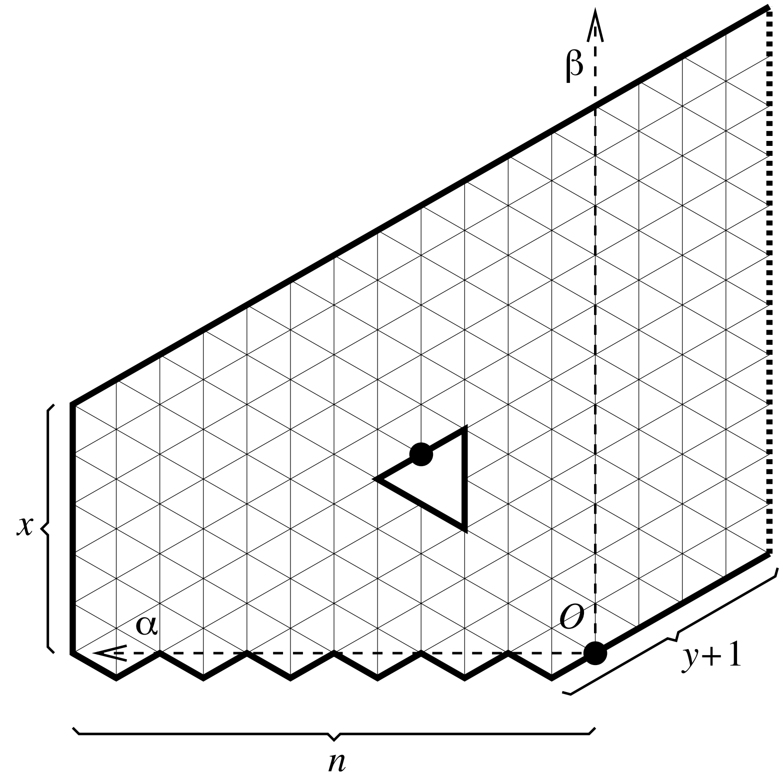
120° angle, constrained boundary



Current talk: 90° angle, mixed boundary



$D_{n,x,y}$: $n = 6, x = 5, y = 4$

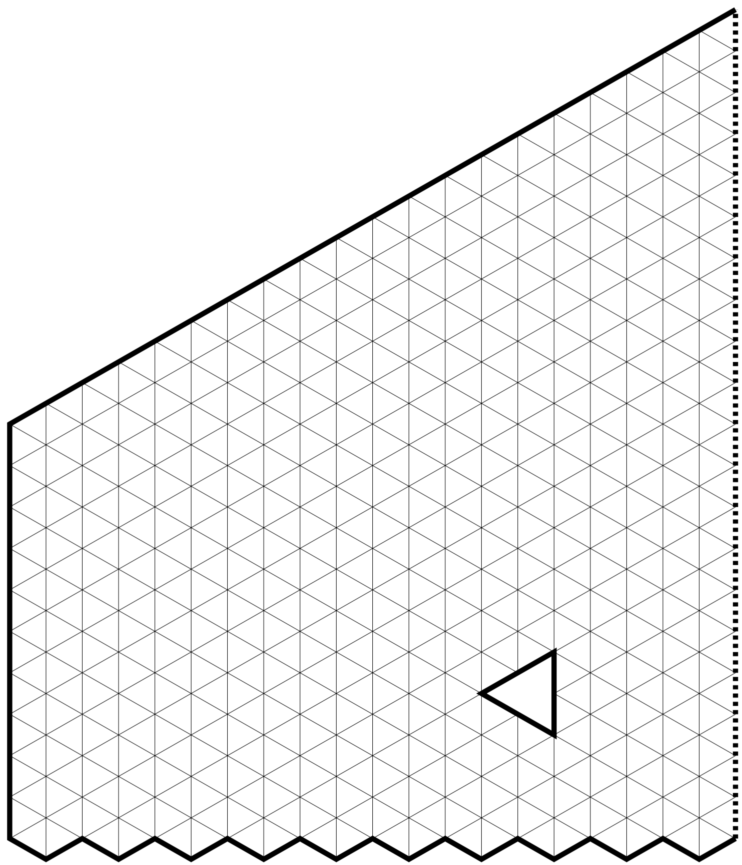


$D_{n,x,y}(\alpha, \beta)$: $n = 6, x = 5, y = 4, \alpha = 2, \beta = 4$

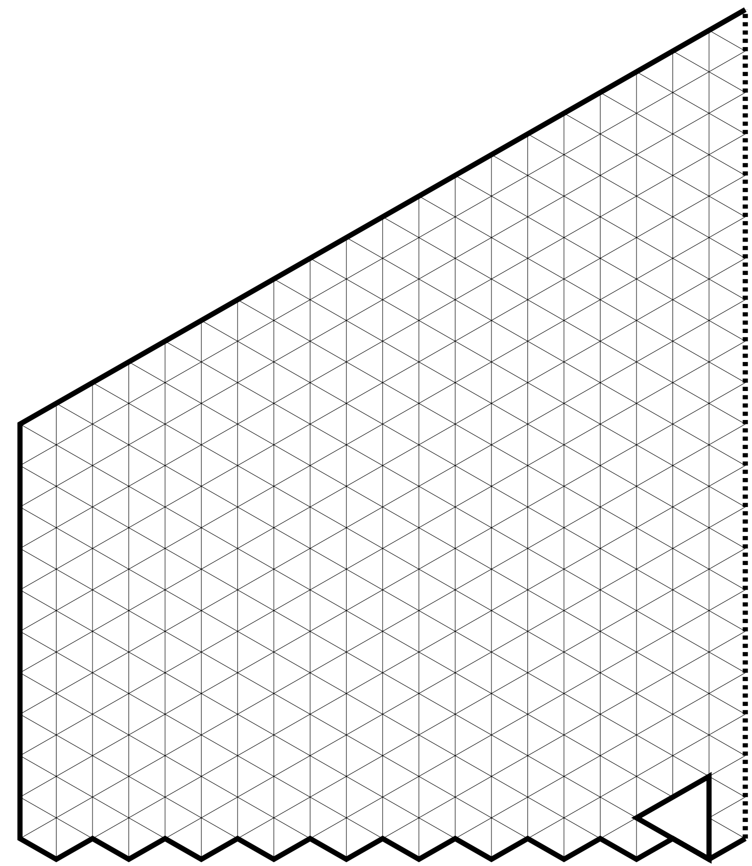
• $\mathrm{M}_f(D)$: # tilings of D with tiles allowed to protrude across free boundary portions

•: $\omega_c(\alpha,\beta)$ (correlation of the gap with the corner):

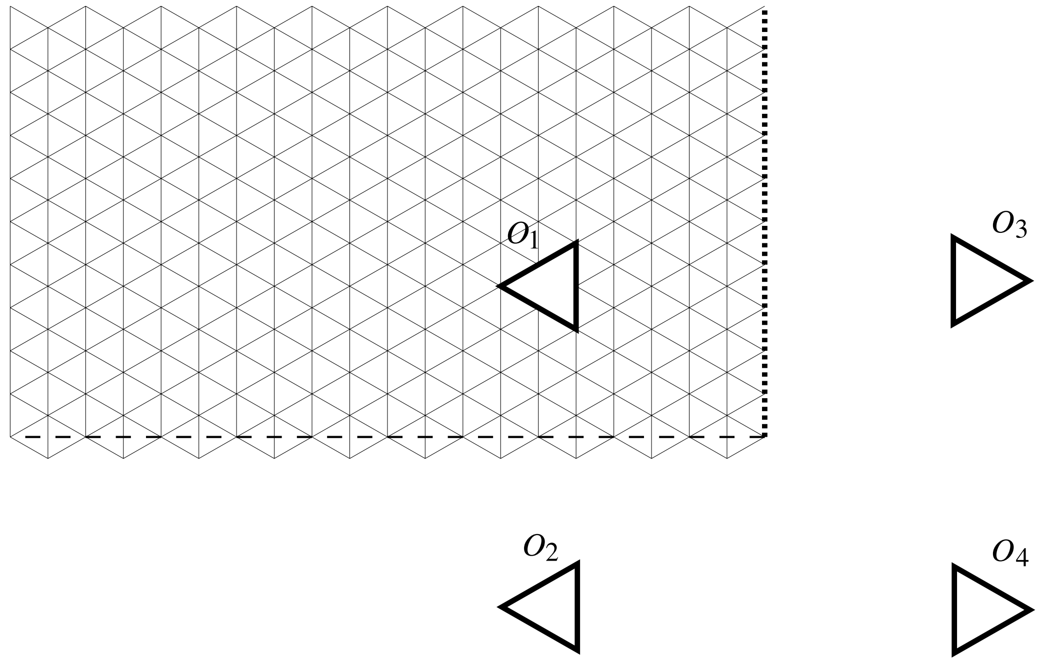
$$\omega_c(\alpha,\beta) := \lim_{n\to\infty} \frac{\mathrm{M}_f(D_{n,n,0}(\alpha,\beta))}{\mathrm{M}_f(D_{n,n,0}(1,1))}$$



$D_{10,10,0}(3,4)$.



$D_{10,10,0}(1,1)$.



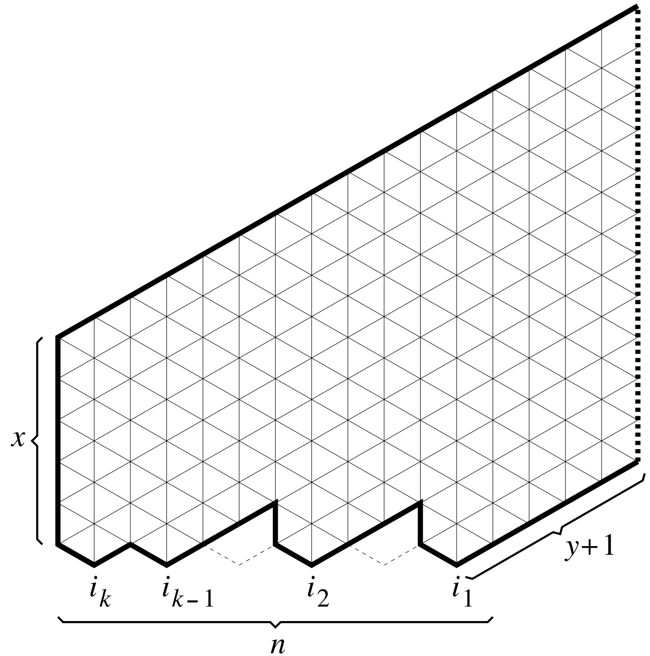
The gap and its three images for $\alpha = 3$, $\beta = 4$

The main result of this talk:

THEOREM. *Let q be a fixed positive rational number. As α and β approach infinity so that $\alpha = q\beta$, we have*

$$\omega_c(\alpha,\beta) \sim \frac{16}{3\pi R q\sqrt{q^2+\frac{1}{3}}} \sim \frac{32}{\pi}\sqrt{\frac{\mathrm{d}(O_1,O_2)\,\mathrm{d}(O_3,O_4)}{\mathrm{d}(O_1,O_3)\,\mathrm{d}(O_1,O_4)\,\mathrm{d}(O_2,O_3)\,\mathrm{d}(O_2,O_4)}},$$

where d *is the Euclidean distance.*



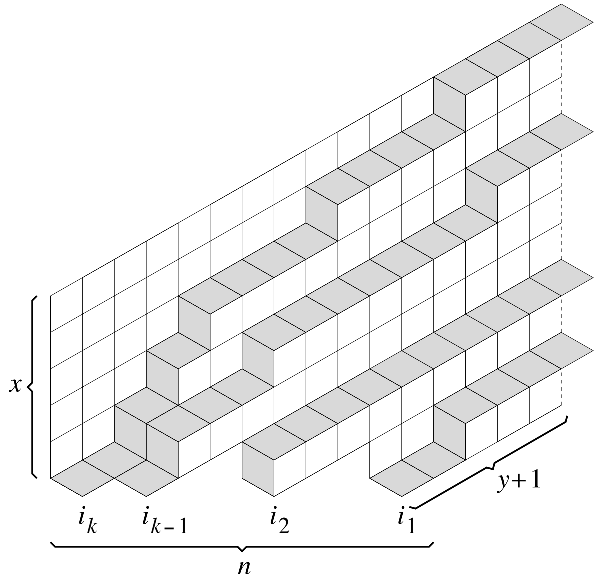
$D^{i_1,\ldots,i_k}_{n,x,y}$ for $n = 6$, $x = 5$, $y = 4$, $k = 4$, $i_1 = 1$, $i_2 = 3$, $i_3 = 5$, $i_4 = 6$.

It turns out we can reduce to enumerating tilings of such regions.

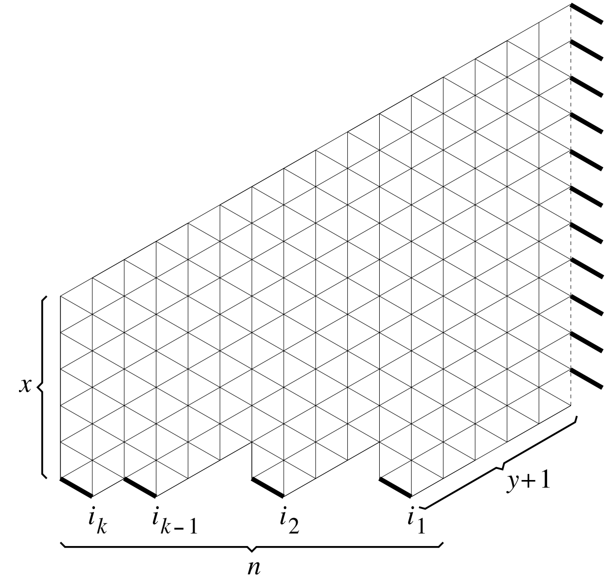
Great strike of luck: *They are given by "round" formulas!*

Proposition. *For any integers* $n, x \geq 0$ *and* $y \geq -1$, *and for any integers* $1 \leq i_1 < \cdots < i_k \leq n$, *we have*

$$\mathrm{M}_f(D_{n,x,y}^{i_1,\ldots,i_k}) = \prod_{a=1}^{k} \binom{x+y+n+i_a}{y+2i_a} \prod_{1\leq a<b\leq k} \frac{i_b - i_a}{y+i_b+i_a}.$$



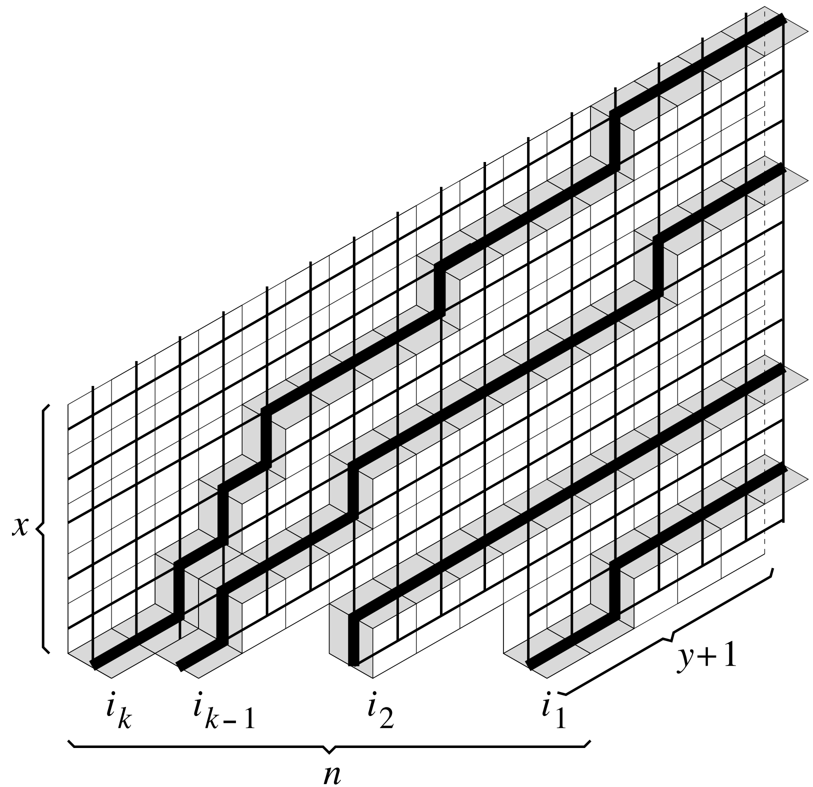
Tilings and paths



Starting and ending segments

The tilings are in bijection with non-intersecting families of paths of rhombi:

- starting points: fixed
- ending points: can vary among a specified set



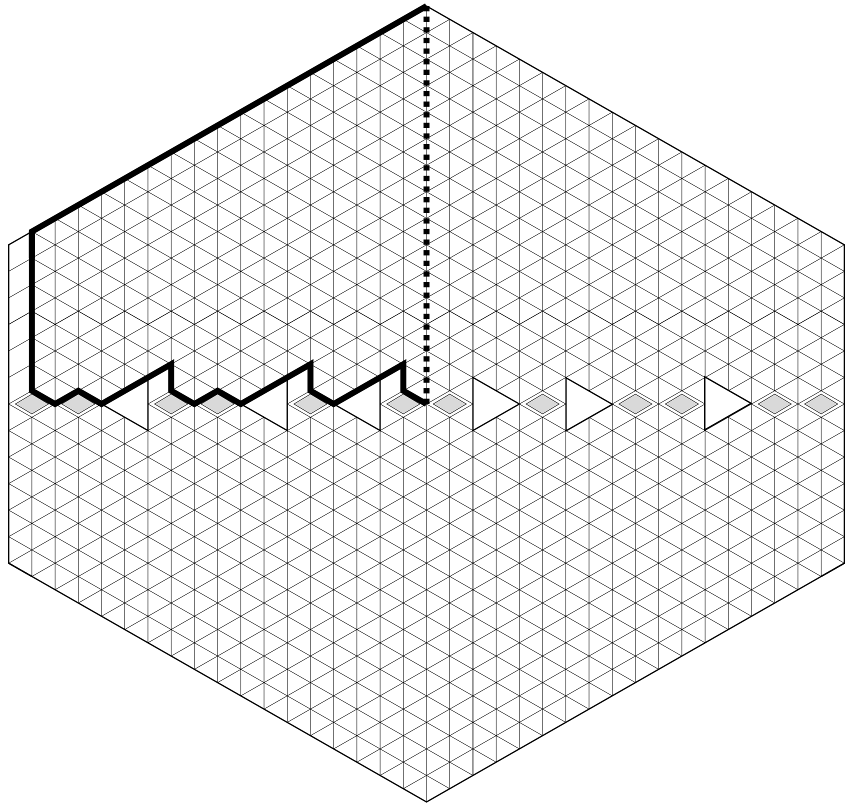
Regarding the paths of lozenges as lattice paths in $\mathbb{Z}^2$

A result of Stembridge expresses this as a Pfaffian.

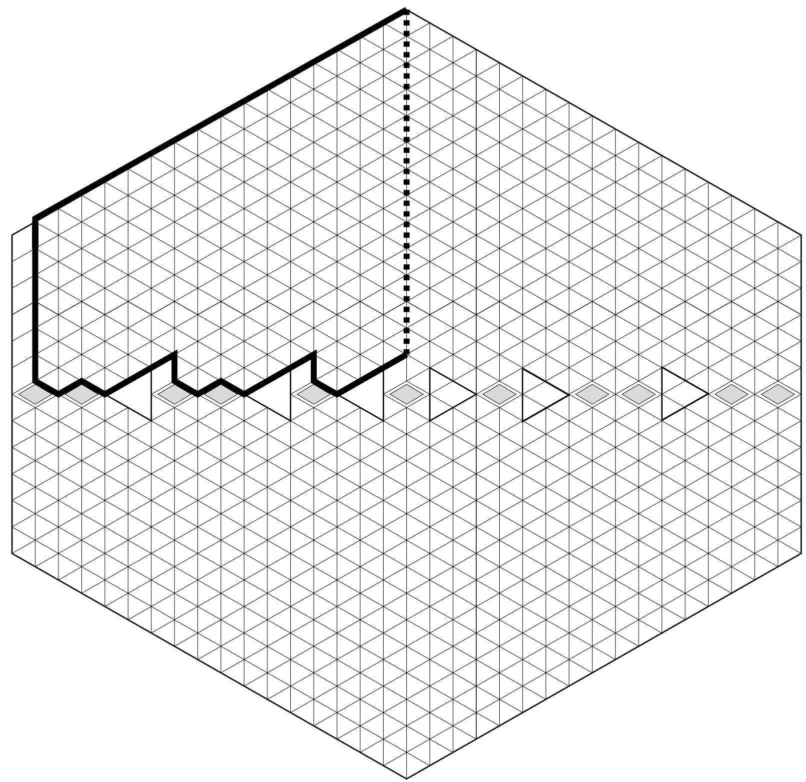
After using some combinatorial identities, this Pfaffian can be evaluated explicitly using Schur's Pfaffian Identity:

Theorem (Schur's Pfaffian Identity). *Let n be even, and let $x_1, \ldots, x_n$ be indeterminates. Then we have*

$$\mathrm{Pf}\left[\frac{x_j - x_i}{x_j + x_i}\right]_{i,j=1}^{n} = \prod_{1 \leq i < j \leq n} \frac{x_j - x_i}{x_j + x_i}.$$



Generalization of SSC plane partitions, even by even by even case.



Generalization of SSC plane partitions, even by odd by odd case.

Corollary (Generalization of SSC plane partitions). *Let $n, x \geq 0$ and $1 \leq k_1 < \cdots < k_s \leq n$ be integers. If $k_1 > 1$ set $t = 0$, otherwise define t by requiring $k_i - i = 0$, $i = 1, \ldots, t$, and $k_{t+1} - (t+1) > 0$. Let $\{1, \ldots, n\} \setminus \{k_1, \ldots, k_s\} = \{i_1, \ldots, i_{n-s}\}$.*
Then we have:
(a).

$$\mathrm{M}_{-,|}(H_{2n,2n,2x}(k_1, \ldots, k_s)) = \mathrm{M}_f(D^{i_1,\ldots,i_{n-s}}_{n,x,2t-1})$$

$$= \prod_{a=1}^{n-s} \binom{x + 2t + n + i_a - 1}{2t + 2i_a - 1} \prod_{1 \leq a < b \leq n-s} \frac{i_b - i_a}{2t + i_a + i_b - 1}.$$

(b).

$$\mathrm{M}_{-,|}(H_{2n+1,2n+1,2x}(k_1, \ldots, k_s)) = \mathrm{M}_f(D^{i_1,\ldots,i_{n-s}}_{n,x,2t})$$

$$= \prod_{a=1}^{n-s} \binom{x + 2t + n + i_a}{2t + 2i_a} \prod_{1 \leq a < b \leq n-s} \frac{i_b - i_a}{2t + i_a + i_b}.$$

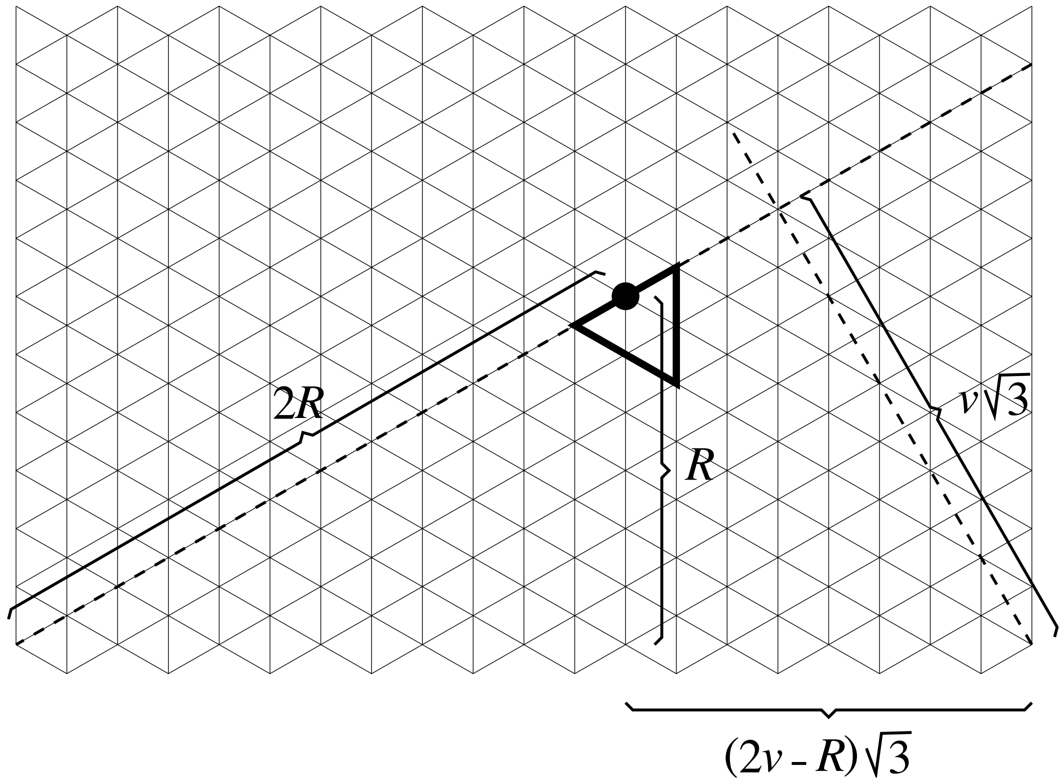
A limit formula for regions with two dents

Proposition. *For any fixed integers* $1 \le i < j$, *we have*

$$\lim_{n\to\infty} \frac{\mathrm{M}_f\left(D_{n,n,0}^{[n]\setminus\{i,j\}}\right)}{\mathrm{M}_f\left(D_{n,n,0}^{[n]\setminus\{1,2\}}\right)} = 4\,\frac{j-i}{j+i}\,\frac{1}{2^{2i-2}}\binom{2i-1}{i-1}\frac{1}{2^{2j-2}}\binom{2j-1}{j-1}.$$

To finish the proof:

- a double sum formula
- its asymptotic analysis



Changing from (α, β) to (R, v)-coordinates.

A double sum formula

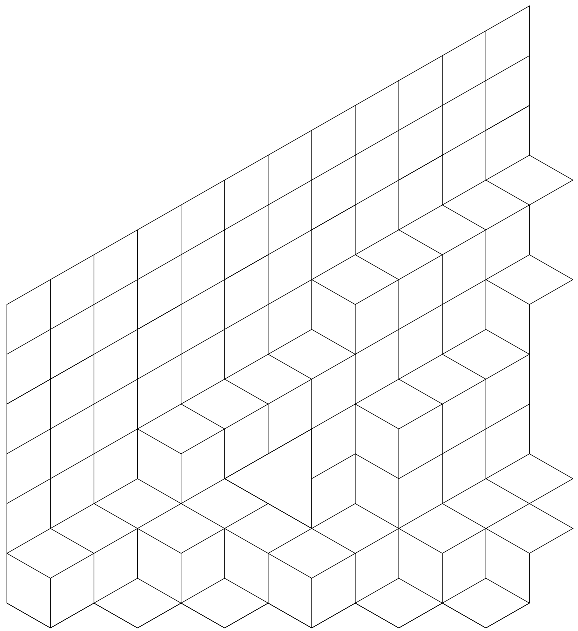
LEMMA. *Write $\alpha = 2v - R$, $\beta = R$, with R and v non-negative integers. Then we have*

$$\omega_c(\alpha,\beta) = \omega_c(2v-R,R)$$

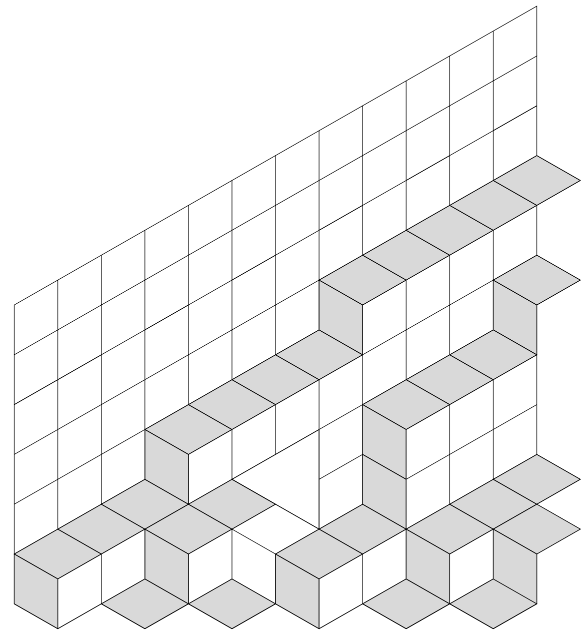
$$= 4R\left|\sum_{a=0}^{R}\sum_{b=0}^{R}(-1)^{a+b}\frac{(R+a-1)!\,(R+b-1)!}{(2a)!\,(R-a)!\,(2b)!\,(R-b)!}\right.$$

$$\left.\times\frac{(2v'+2a+1)!\,(2v'+2b+1)!}{2^{2(2v'+a+b)}(v'+a)!\,(v'+a+1)!\,(v'+b)!\,(v'+b+1)!}\quad\frac{(b-a)^2}{2v'+a+b+2}\right|,$$

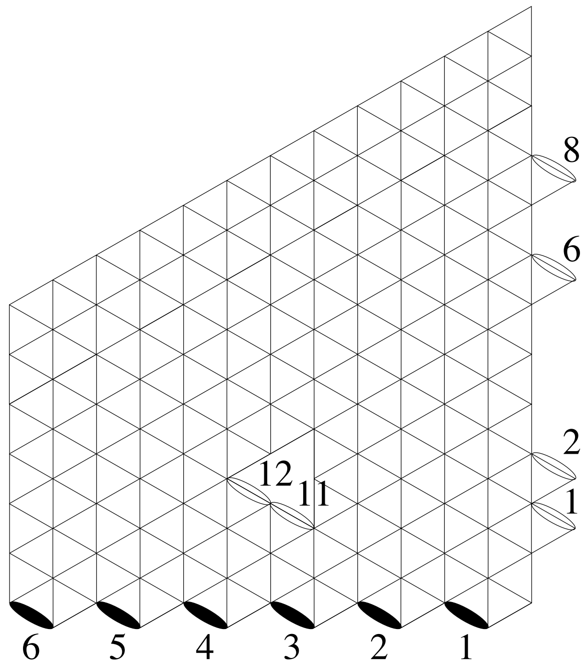
where $v' = 2v - R - 1$.



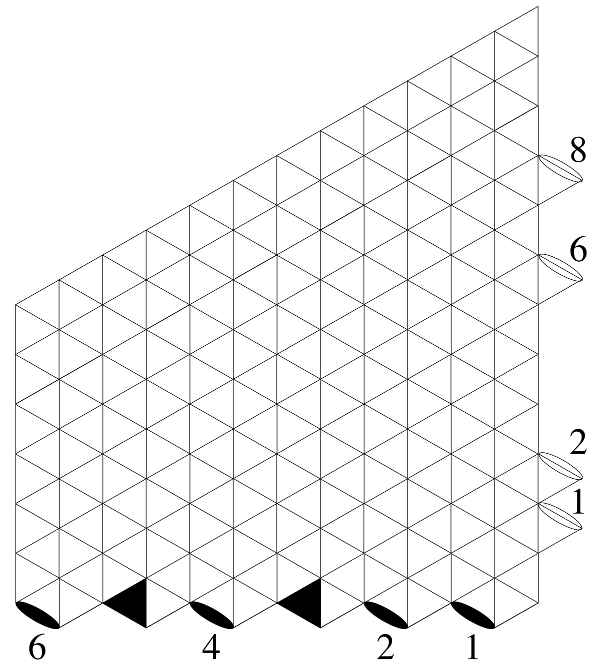
$D_{6,6,0}(3,3;\{1,2,6,8\})$



Paths of lozenges



Labeling starting and ending points



$D^{1,2,4,6}_{6,6,0}(\{1,2,6,8\})$

Outline of proof of double sum formula

- Free boundary is sum over constrained boundaries:

$$\mathrm{M}_f(D_{n,n,0}(\alpha,\beta)) = \sum_{\substack{S\subset T\\ |S|=n-2}} \mathrm{M}(D_{n,n,0}(\alpha,\beta;S))$$

- Use Pfaffian formula for lattice paths and Laplace expansion to get

$$\mathrm{M}(D_{n,n,0}(\alpha,\beta;S)) =$$
$$\left|\sum_{0\le a<b\le R} (-1)^{a+b} \frac{(b-a)(R+a-1)!\,(R+b-1)!}{(2a)!\,(R-a)!\,(2b)!\,(R-b)!}\,\mathrm{M}(D_{n,n,0}^{[n]\setminus\{2v-R+a,2v-R+b\}}(S))\right|$$

- Sum over boundaries to get

$$\mathrm{M}_f(D_{n,n,0}(\alpha,\beta)) =$$

$$2R\left|\sum_{0\le a<b\le R}(-1)^{a+b}\frac{(b-a)(R+a-1)!\,(R+b-1)!}{(2a)!\,(R-a)!\,(2b)!\,(R-b)!}\,\mathrm{M}_f(D_{n,n,0}^{[n]\setminus\{2v-R+a,2v-R+b\}})\right|$$

- divide by $\mathrm{M}_f(D_{n,n,0}(1,1))$, let $n\to\infty$, and use 2-dent limit formula

Reduction of the double sum to simple sums

- The double sum separates if we write

$$\frac{1}{2v' + a + b + 2} = \int_0^1 x^{2v'+a+b+1}\, dx$$

- Moment sums ($k \in \mathbb{Z}$, $x \in [0, 1]$):

$$T^{(k)}(R, v; x) := \frac{1}{R} \sum_{a=0}^{R} \frac{(-R)_a (R)_a (3/2)_{v+a}}{(1)_a (1/2)_a (2)_{v+a}} \left(\frac{x}{4}\right)^a a^k$$

Lemma. *We have that*

$$\omega_c(2v - R, R) =$$

$$8R \left| \int_0^1 T^{(2)}(R, v'; x) T^{(0)}(R, v'; x) x^{2v'+1} dx - \int_0^1 \left(T^{(1)}(R, v'; x) \right)^2 x^{2v'+1} dx \right|,$$

where $v' = 2v - R - 1$.

The asymptotics of the integrals in the lemma

It follows from results in [C, Mem. AMS, 2005] that:

$$\int_0^1 T^{(2)}(R,v';x)T^{(0)}(R,v';x)x^{2v'+1}dx$$

$$\sim \frac{2}{\pi R}\int_0^1 x^{2qR}\frac{1}{(4-x)\sqrt{q^2+\frac{x}{4-x}}}\cos\left[2R\arccos\left(1-\frac{x}{2}\right)-\arctan\frac{1}{q}\sqrt{\frac{x}{4-x}}+\pi\right]dx$$

and

$$\int_0^1\left(T^{(1)}(R,v';x)\right)^2 dx\sim$$

$$\frac{2}{\pi R}\int_0^1 x^{2qR}\frac{1}{(4-x)\sqrt{q^2+\frac{x}{4-x}}}\left\{1+\cos\left[2R\arccos\left(1-\frac{x}{2}\right)-\arctan\frac{1}{q}\sqrt{\frac{x}{4-x}}+\pi\right]\right\}dx$$

Lemma then implies

$$\omega_c(2v - R, R) \sim \frac{16}{\pi} \left| \int_0^1 x^{2qR} \frac{1}{(4-x)\sqrt{q^2 + \frac{x}{4-x}}} dx \right|$$

as R and v approach infinity so that $2v - R = qR$.

We have

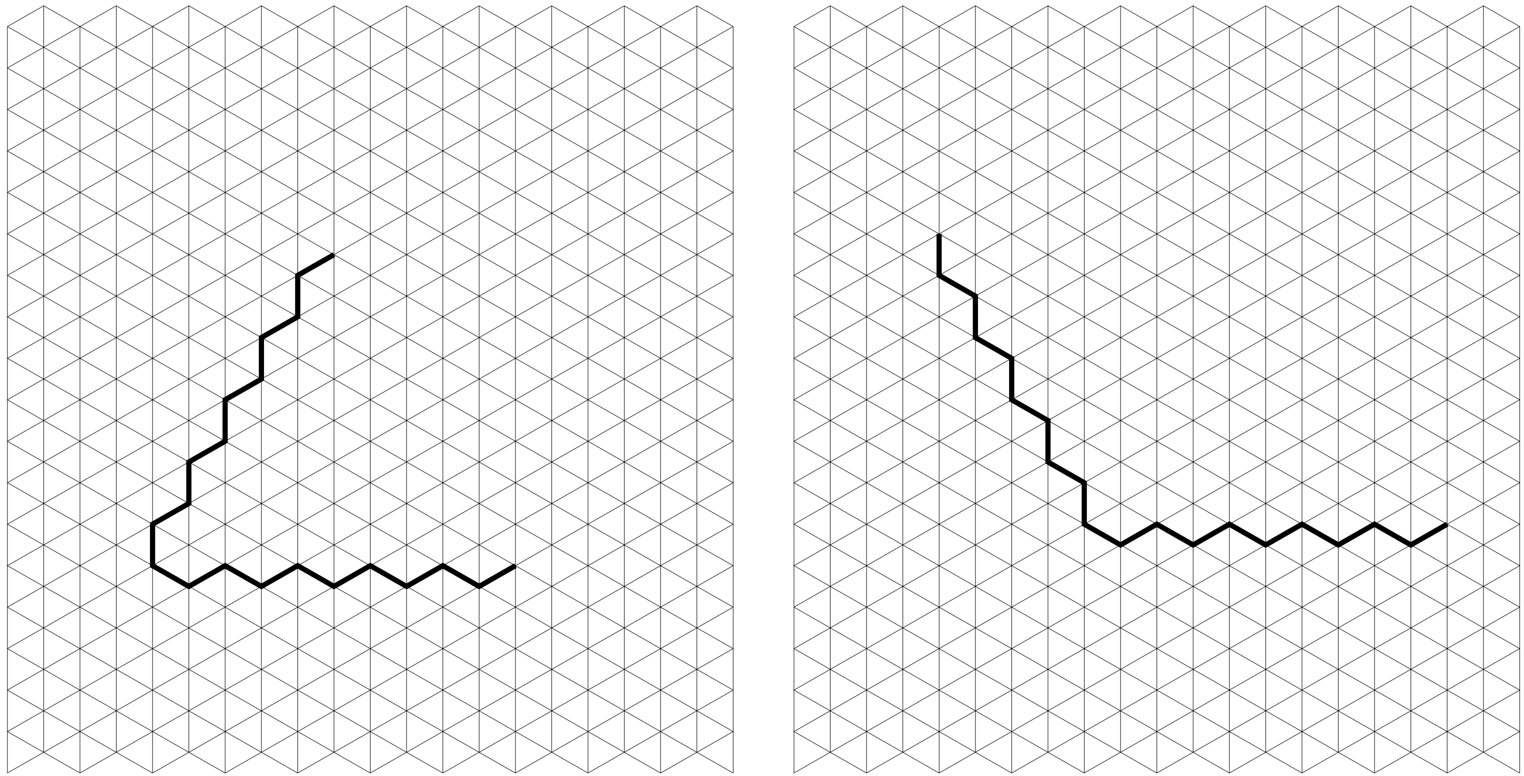
$$\int_0^1 x^{2qR} \frac{1}{(4-x)\sqrt{q^2 + \frac{x}{4-x}}} dx \sim \frac{1}{3q\sqrt{q^2 + \frac{1}{3}}} \frac{1}{R}, \qquad R \to \infty$$

Then we get

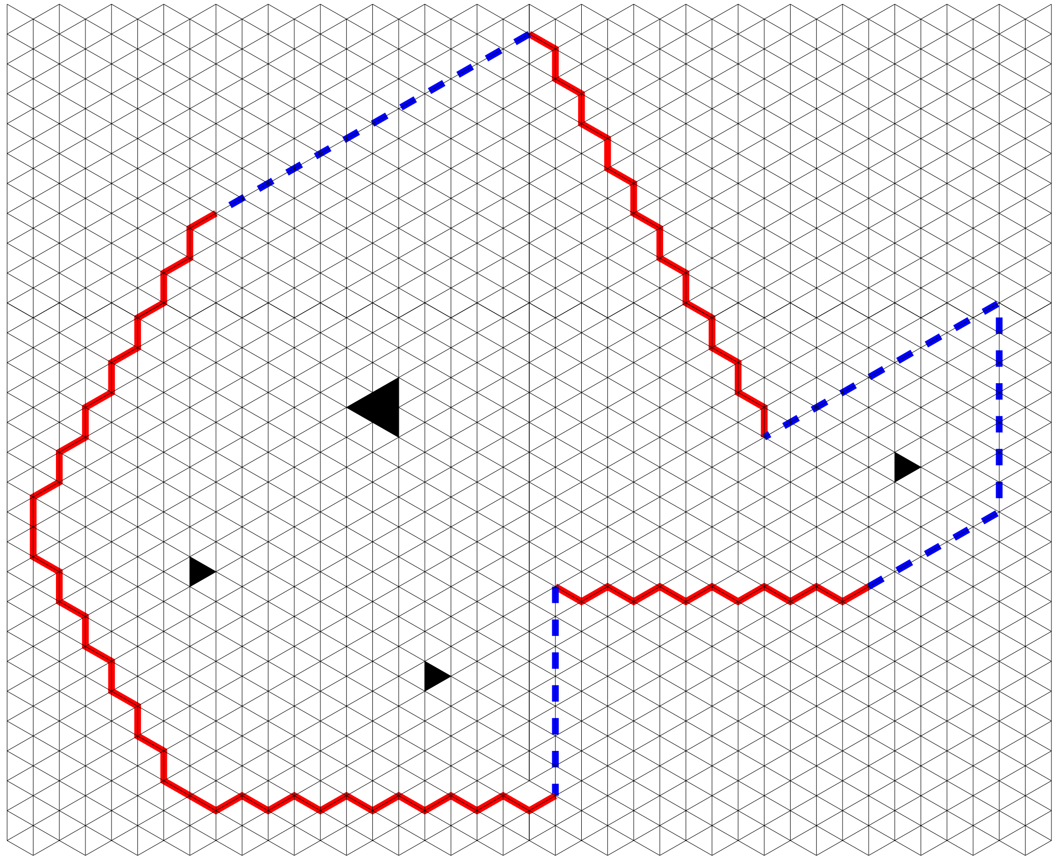
$$\omega_c(2v - R, R) \sim \frac{16}{3\pi q\sqrt{q^2 + \frac{1}{3}}} \frac{1}{R},$$

which proves the Theorem.

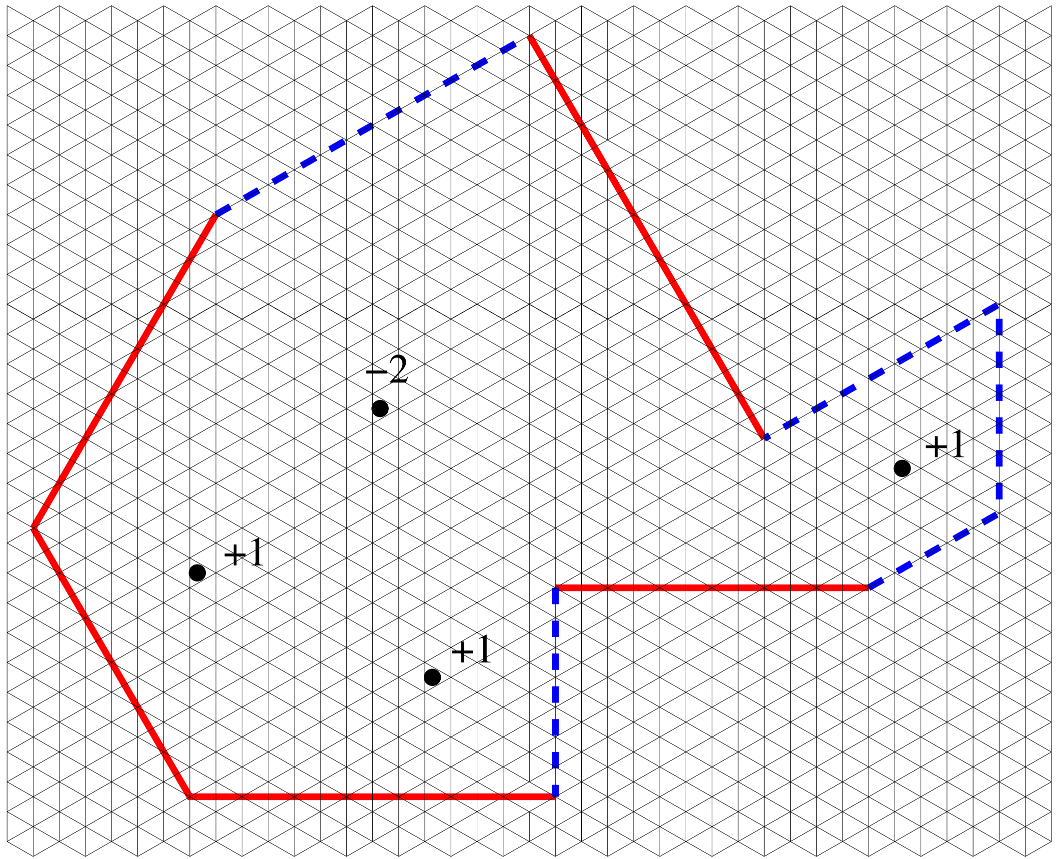
A general conjecture for regions Ω_n on the triangular lattice



The two types of zig-zag corners in Ω_n



An example of Ω_n



The corresponding steady state heat flow problem

- $O_1^{(n)}, \ldots, O_k^{(n)}$: finite unions of unit triangles from the interior of $\Omega\ _n$ (the gaps)
- for fixed i, $O_i^{(n)}$'s are translates of one another for all $n \geq 1$
- $O_i^{(n)}$ shrinks to point $a_i \in \Omega$ in scaling limit, $i = 1, \ldots, k$
- $\Omega_n \to \Omega$, $n \to \infty$
- E: heat energy when sources/sinks are at positions $a_1, \ldots, a_k$

Conjecture. *Let $O_i'^{(n)}$'s be translations of the $O_i^{(n)}$'s that shrink to distinct points $a_1', \ldots, a_k' \in \Omega$ in the scaling limit as $n \to \infty$. Then*

$$\frac{\mathrm{M}_f(\Omega_n \setminus O_1^{(n)} \cup \cdots \cup O_k^{(n)})}{\mathrm{M}_f(\Omega_n \setminus O_1'^{(n)} \cup \cdots \cup O_k'^{(n)})} \to \frac{\exp(-E)}{\exp(-E')},$$

where E' is the heat energy of the system obtained from S by moving the point heat sources to positions $a_1', \ldots, a_k'$.