PHYSICS & COMBINATORICS OF THE OCTAHEDRON EQUATION : FROM CLUSTER ALGEBRAS TO ARCTIC CURVES

(P. Di Francesco + R. Kedem + R. Soto Garrido)

GGI 5/15

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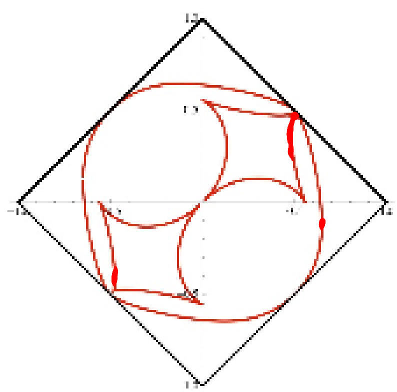
- Octahedron equation and T-systems
- Cluster algebras = definition
- Two examples: Frieze patterns & domino tilings
- The T-system behind Friezes (A_1 case)
- The T-system behind domino tilings (octahedra)
- Arctic curves



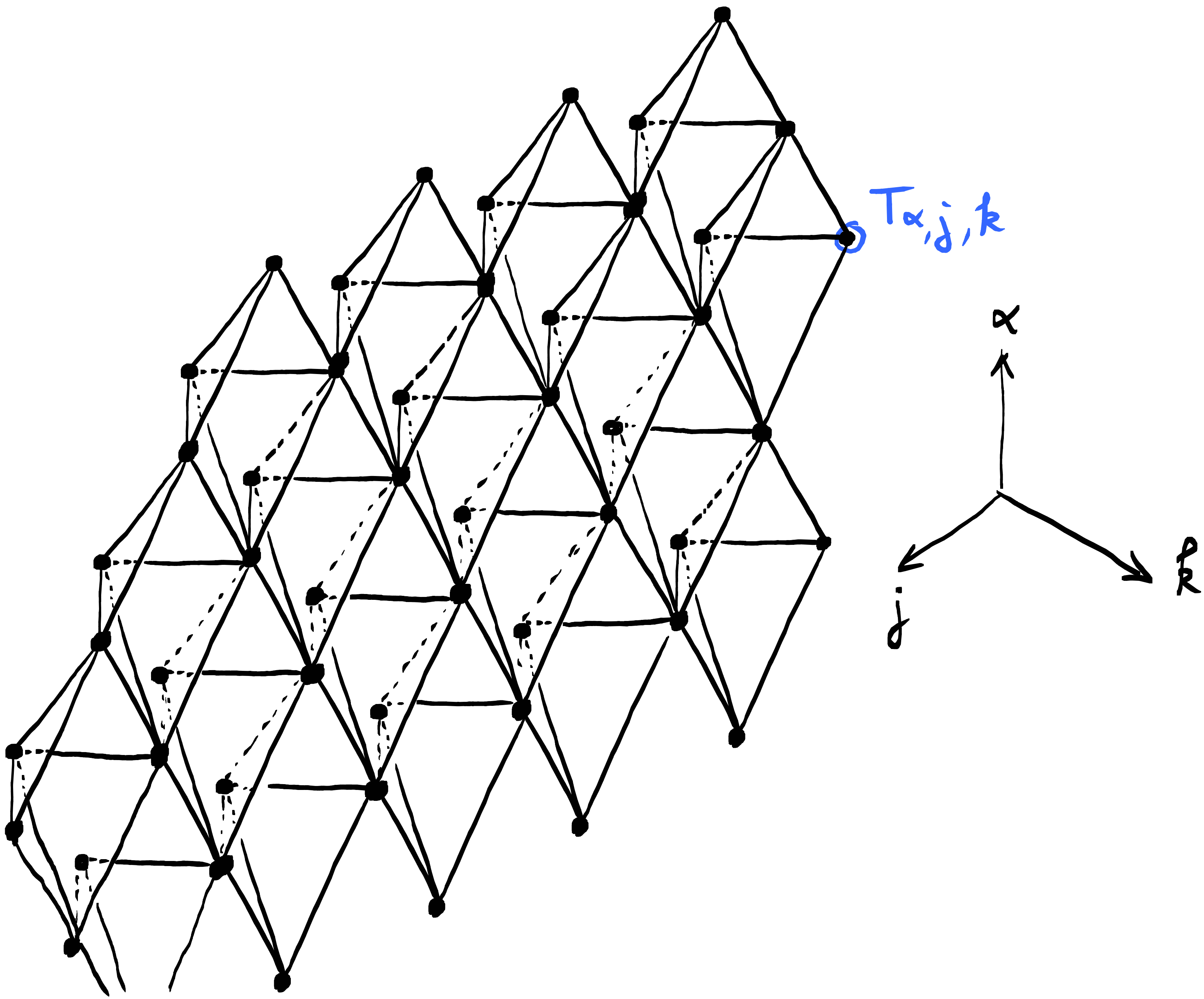
0. OCTAHEDRON EQUATION

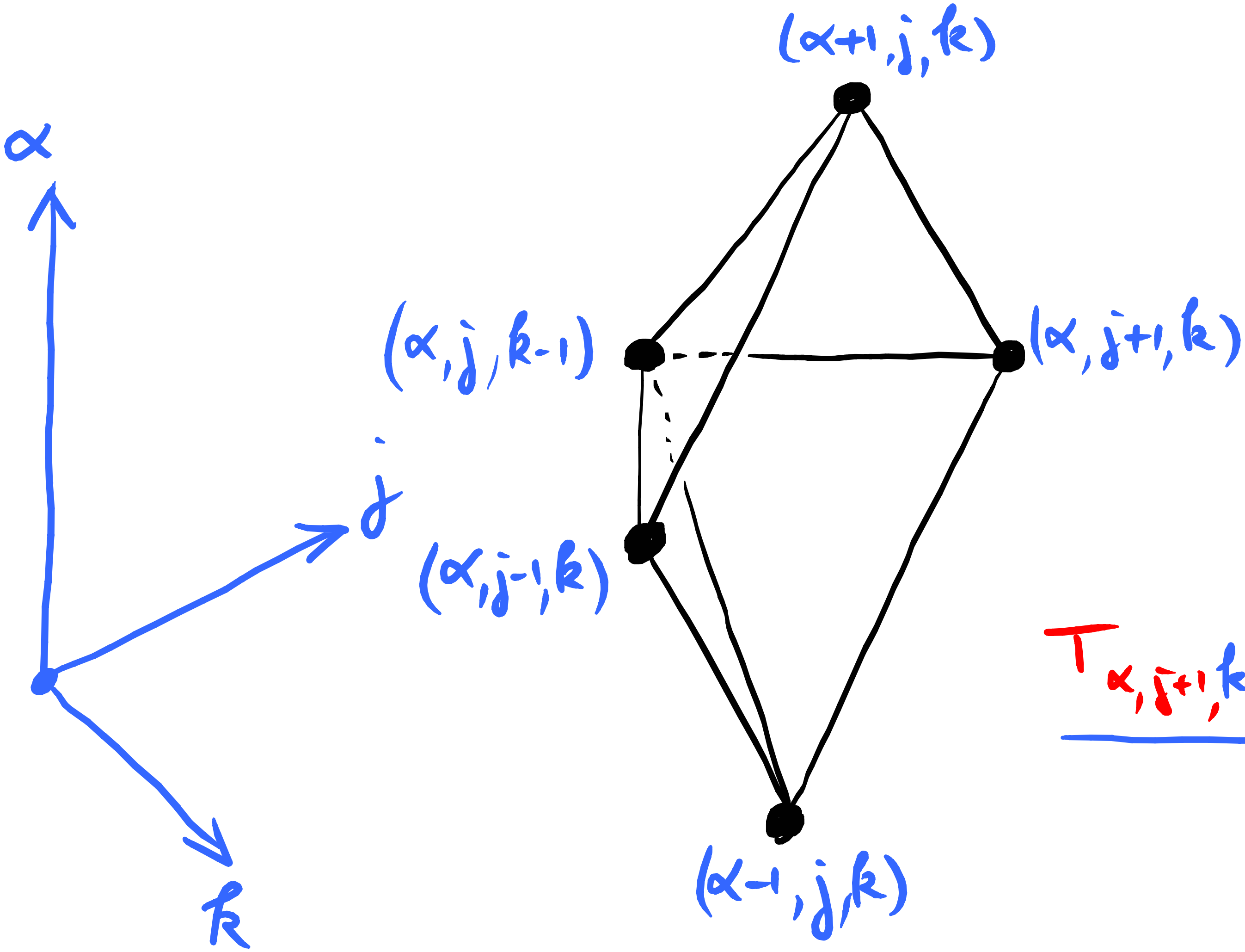
- Dodgson condensation of determinants / Desnanot Jacobi
- Alternating Sign Matrices [MRR]
- Littlewood Richardson rules [KTW]

$$\boxed{\begin{gathered} T_{\alpha,j,k+1}\,T_{\alpha,j,k-1} = T_{\alpha,j+1,k}\,T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\,T_{\alpha-1,j,k} \\ (\alpha,j,k \in \mathbb{Z}) \end{gathered}}$$

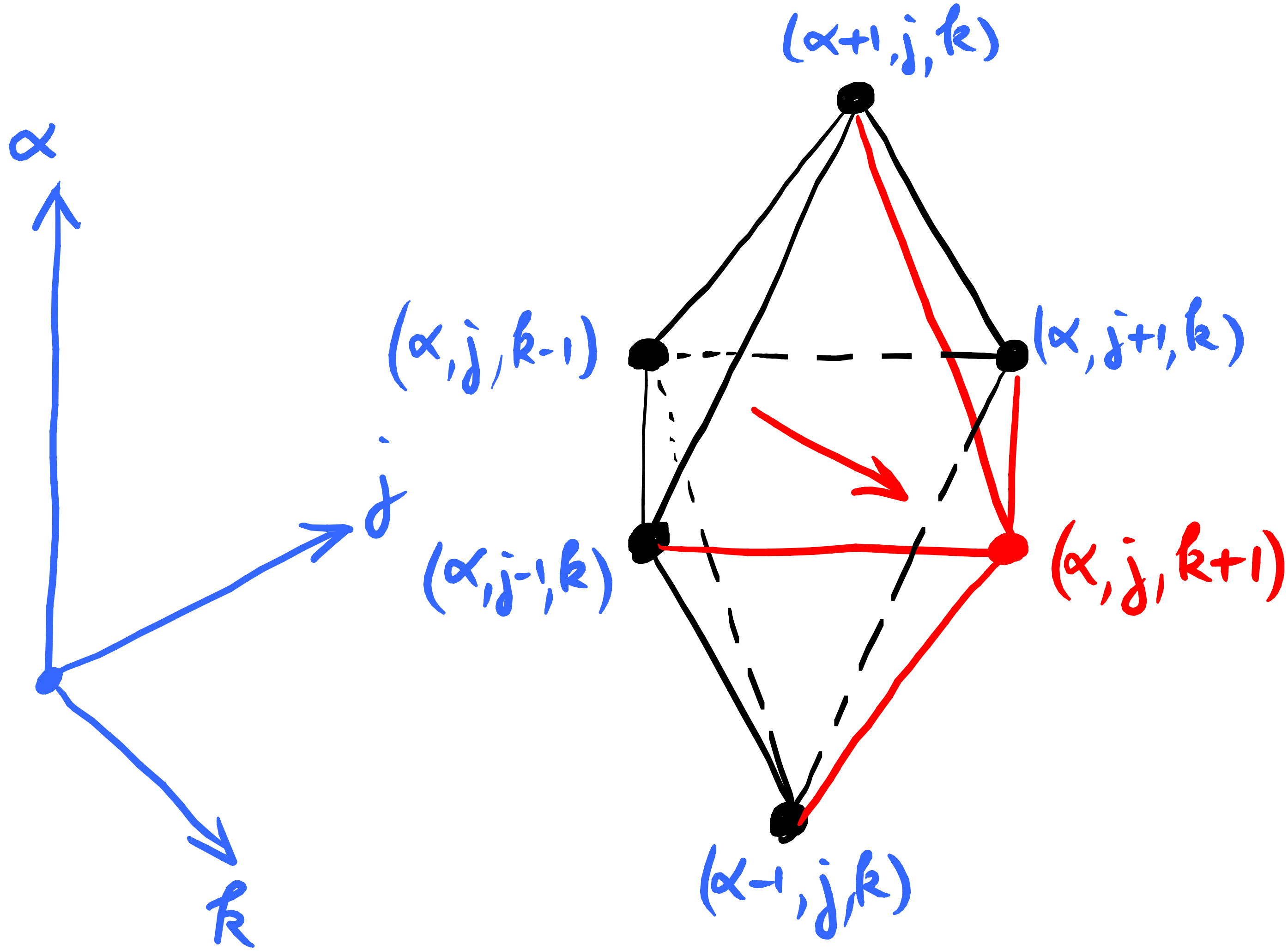
$(\alpha, j) =$ space , $k =$ time $\rightarrow$ 2+1 D

- initial data = "stepped" surface $\{T_{\alpha,j,k_{\alpha,j}}\}$ $\begin{array}{l} |k_{\alpha,j+1}-k_{\alpha,j}|=1 \\ |k_{\alpha+1,j}-k_{\alpha,j}|=1 \end{array}$

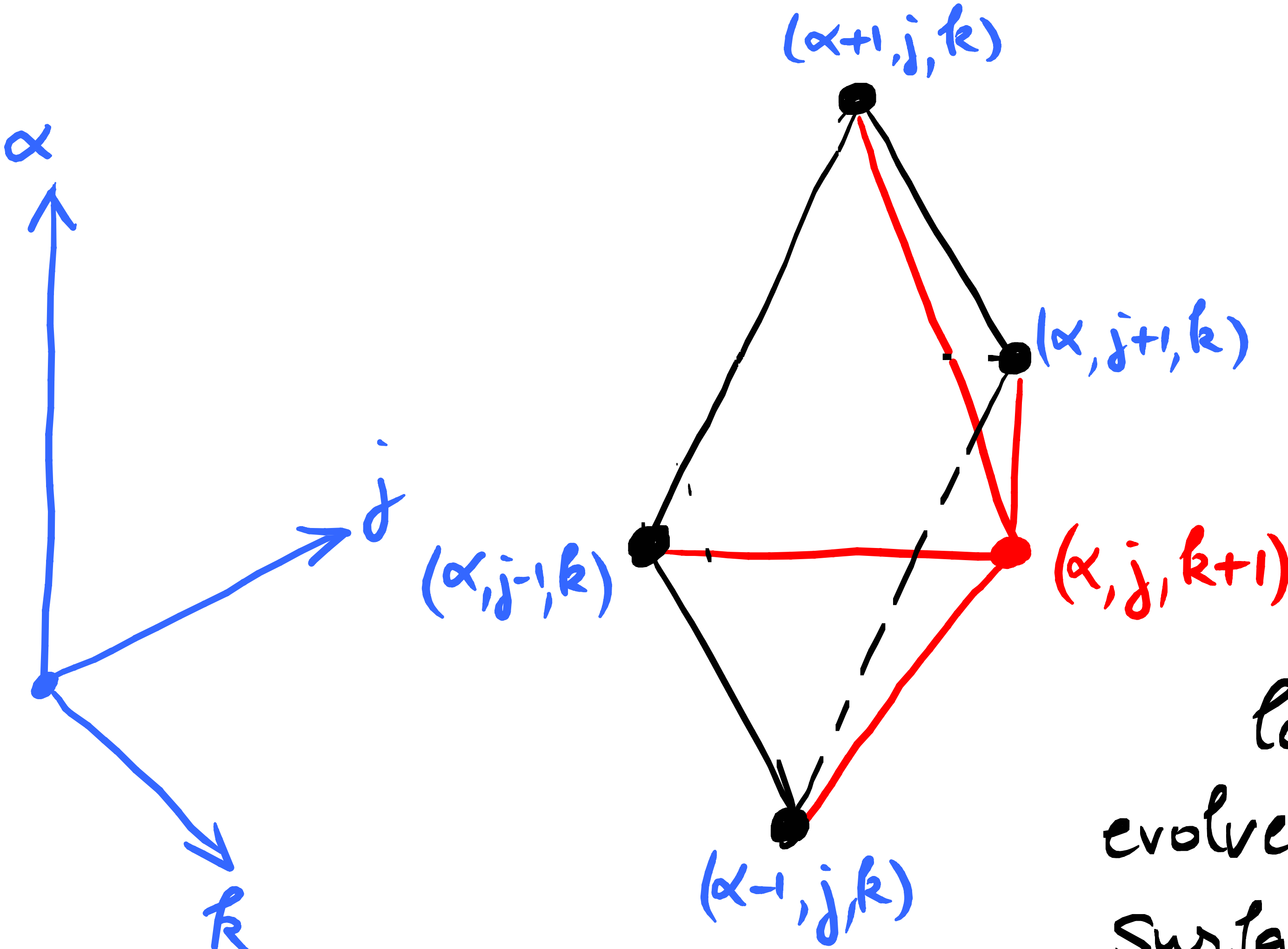

$T_{\alpha,j,k}$
α
j
k



$$\frac{T_{\alpha,j+1,k}\,T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\,T_{\alpha-1,j,k}}{T_{\alpha,j,k-1}}$$



$$= T_{\alpha,j,k+1}$$

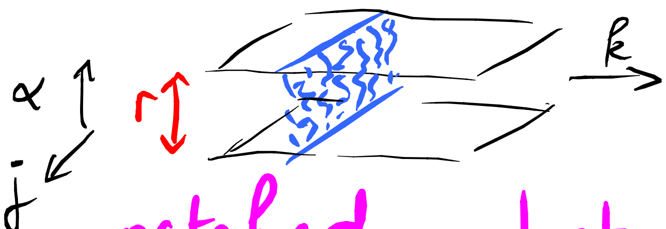


local move that evolves the stepped surface by "adding" an octahedron

1. T-SYSTEM (for A_r)

- transfer matrices of Heisenberg quantum spin chains [KNS]
- Representation theory "q"-characters [N]
- discrete Hirota equation (integrable systems) [LWZ]

$$\boxed{\begin{array}{l} A_r \quad T_{\alpha,j,k+1}\, T_{\alpha,j,k-1} = T_{\alpha,j+1,k}\, T_{\alpha,j-1,k} + T_{\alpha+1,j,k}\, T_{\alpha-1,j,k} \\ \qquad T_{0,j,k} = T_{r+1,j,k} = 1 \qquad\qquad \left(\begin{array}{l}\alpha \in \{1,2,\dots,r\} \\ j,k \in \mathbb{Z}\end{array}\right)\end{array}}$$



octahedron between a floor $\alpha=0$ and a ceiling $\alpha=r+1$

$r=1$ will appear in connection to friezes

2. CLUSTER ALGEBRAS: DEFINITION

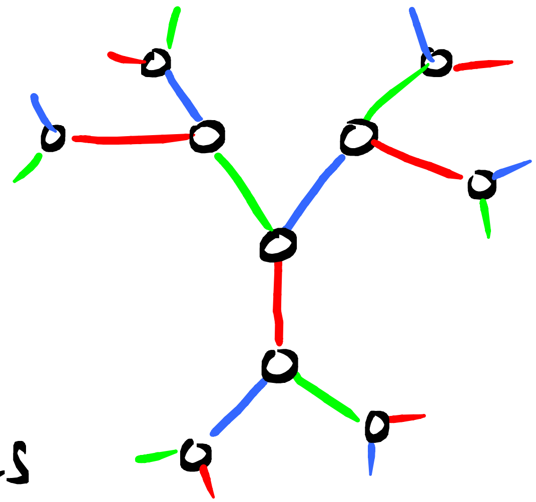
$\mathbb{T}_n$

degree n

infinite tree

w/ labeled edges

(color)



2. CLUSTER ALGEBRAS : DEFINITION

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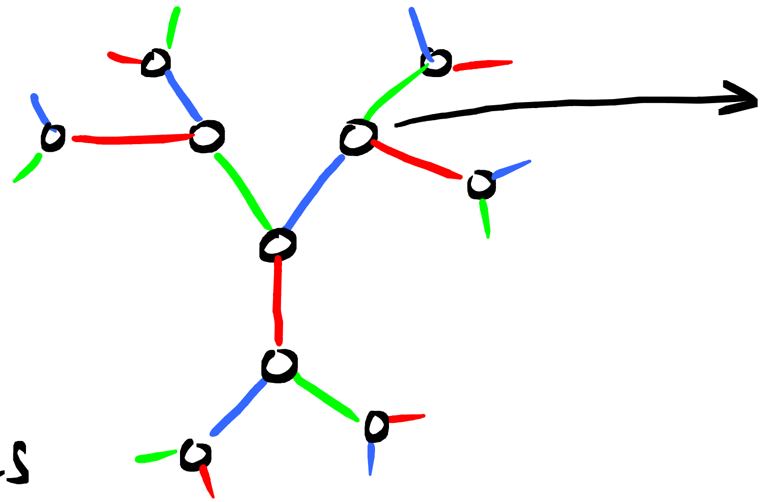


at each vertex, 2 data

1. n-vector $(x_1, \ldots x_n) = \vec{x}$
2. $n \times n$ skew sym matrix

   $B_{ij} \in \mathbb{Z}$ exchange matrix

\+ MUTATION RULES

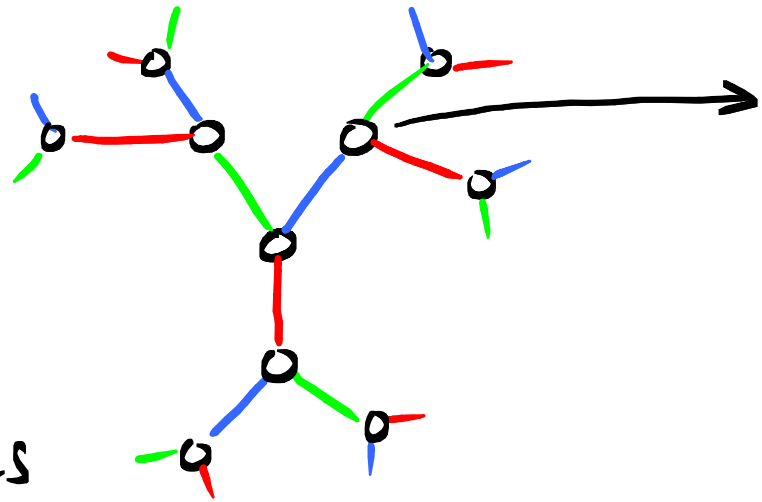
$$(\vec{x}, B) \overset{\mu}{\longleftrightarrow} (\vec{x}', B')$$

→ rank n cluster algebra

= generated by all n-vectors in $\mathbb{T}_n$

2. CLUSTER ALGEBRAS: DEFINITION

$\mathbb{T}_n$ degree n infinite tree w/ labeled edges (color)



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\+ MUTATION RULES

$$(\vec{x}, B) \; \circ\!\!-\!\!\!-\!\!\!-\!\!\circ \; (\vec{x}', B') \qquad \overleftrightarrow{\;\mu_k\;}$$

→ rank n cluster algebra = generated by all n-vectors in $\mathbb{T}_n$

The mutation structure guarantees the Laurent property: $\vec{x}$ at any vertex = Laurent polynomial of $\vec{x}$ at any other vertex.

\+ Positivity Conjecture

MUTATIONS μ_k (in direction $k \in \{1, 2, \dots, n\}$)

• QUIVER MUTATION (B matrix) at vertex k

B-matrix $\iff$ quiver $\left(B_{ij} > 0 \iff i \xrightarrow{B_{ij} \text{ arrows}} j \right)$

no $\circlearrowleft$ no2 $\rightleftarrows$

(i) reflect arrows incident to k

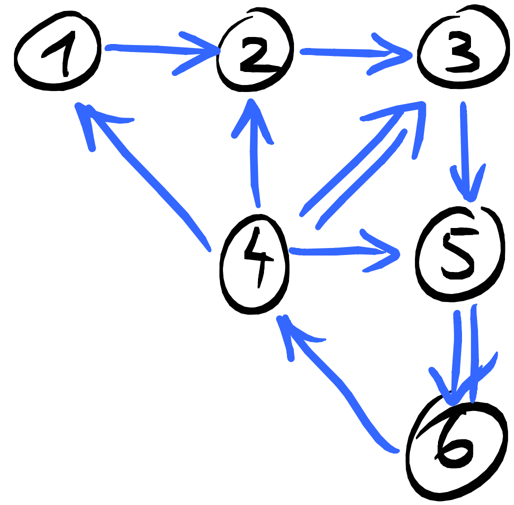
(ii) for each path $i \longrightarrow k \longrightarrow j$ via k, create $i \to j$

short cut
$$\begin{tikzcd} i \arrow[rr, bend right] & k \arrow[l] & j \arrow[l] \end{tikzcd}$$

(iii) $i \rightleftarrows j \Rightarrow i \quad j$ (cancellation)

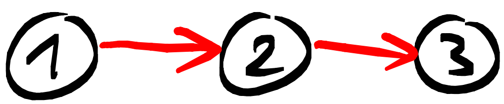
Example

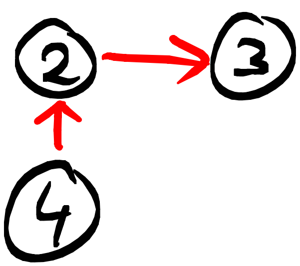
apply μ_2



$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{bmatrix} \end{array}$$

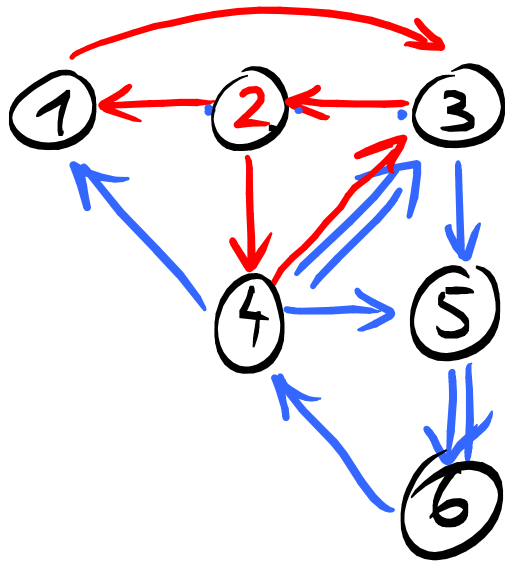
2 length 2 paths thru 2:





Example

apply μ_2



$$B = \begin{array}{c|cccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & & & & & & \\ 2 & & & & & & \\ 3 & & & & & & \\ 4 & & & & & & \\ 5 & & & & & & \\ 6 & & & & & & \end{array}$$

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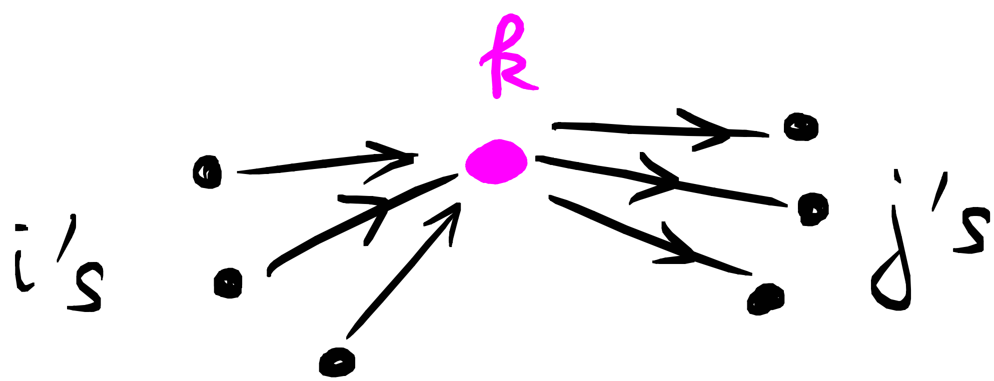
• CLUSTER MUTATION ($\vec{x} = (x_1, x_2 \cdots x_k \, x_{k+1} \cdots x_n)$)

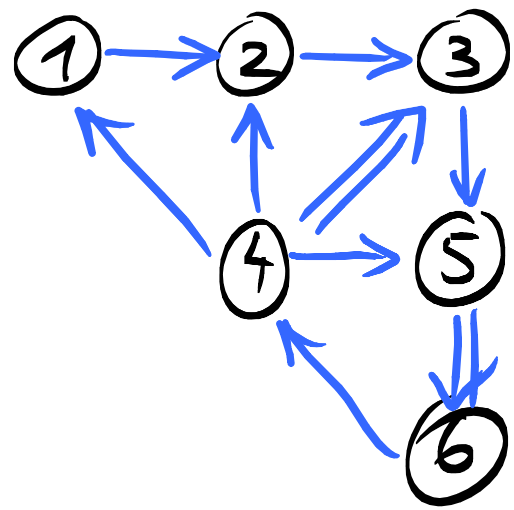
$$\mu_k(x_i) = x_i \quad \text{if } k \neq i$$

$$\mu_k(x_k) = \frac{1}{x_k}\left\{ \prod_{\substack{\text{arrows } i \to k \\ \text{TAILS}}} x_i + \prod_{\substack{\text{arrows } k \to j \\ \text{HEADS}}} x_j \right\}$$



Example

apply μ_5 on $\vec{x}$



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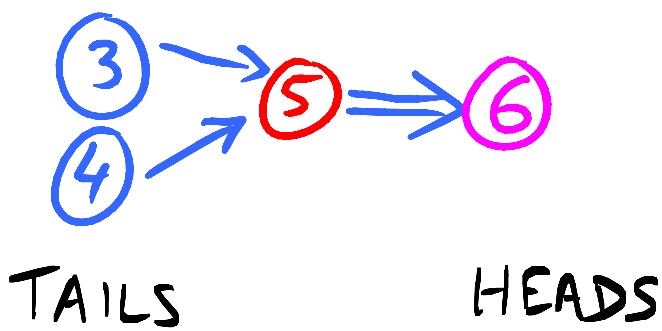
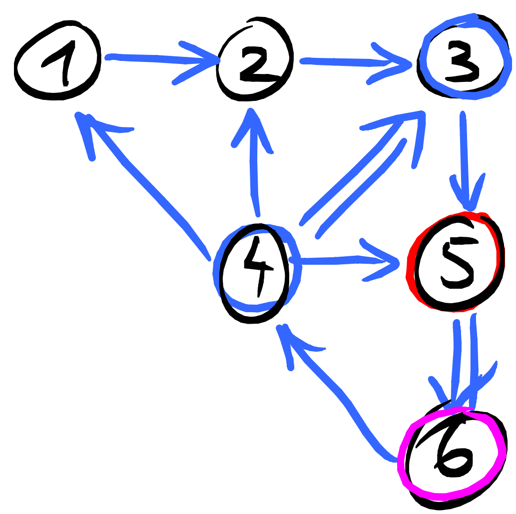
$$\vec{x} = (x_1, x_2, x_3, x_4, x_5, x_6)$$



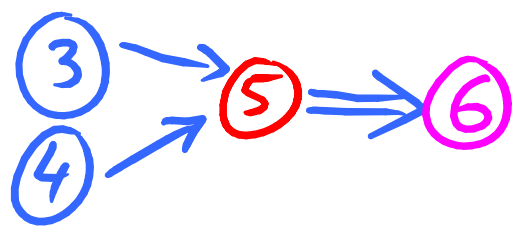
Example

apply μ_5 on $\vec{x}$



$$B = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{bmatrix} \end{array}$$

$$\vec{x} = \left(x_1, x_2, x_3, x_4, \frac{x_3 x_4 + x_6^2}{x_5}, x_6\right)$$



N.B. all μ_i are involutions

PROPERTIES

THM [Fomin Zelevinsky]: $\forall$ sequence $i_1\, i_2 \ldots i_k \in \{1, \ldots n\}$ the mutated cluster $\mu_{i_k} \circ \mu_{i_{k-1}} \cdots \circ \mu_{i_1}(\vec{x})$ is a Laurent polynomial of $\vec{x}$ ($p\ell(x_1, x_1^{-1}, x_2, x_2^{-1} \ldots x_n, x_n^{-1})$)

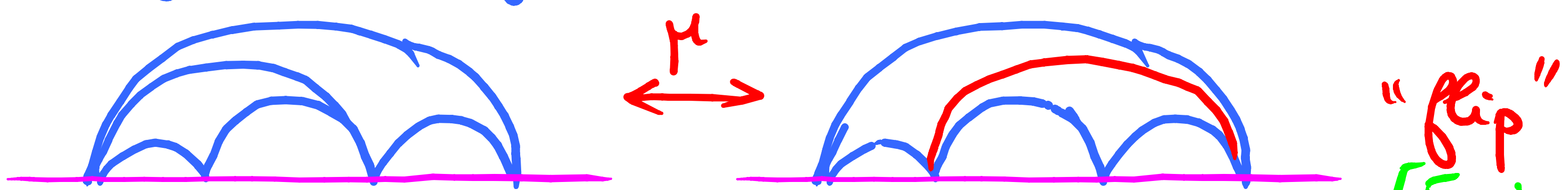
CONJ The polynomial has non-negative integer coefficients (proved for finite rank geometric type).

CLASSIFICATIONS

- finite cluster algebras $\Leftrightarrow$ a mutated quiver is an oriented Dynkin diagram of a classical Lie algebra (ABCDEFG)
- B-finite $\Leftrightarrow$ "Triangulations" + 11 exceptional cases

APPLICATIONS

- Triangulations of Teichmüller space (Positive)



[Fomin-Thurston-Shapiro] [Musiker, Schiffler, Williams]

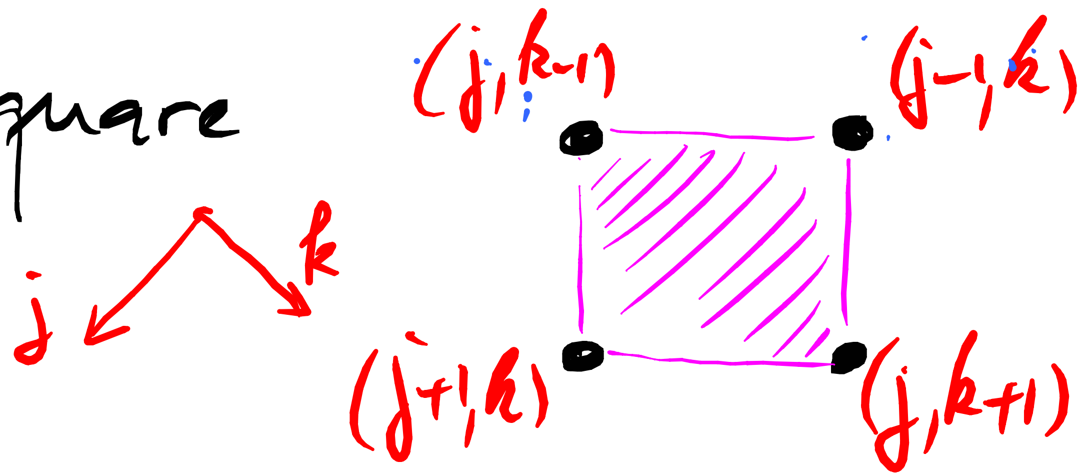
B codes the triangulation $\vec{x}$ = geodesic lengths

- Discrete Integrable systems / somos sequences, pentagram maps [DF, Kedem] [GSV] [article]
- Totally positive GL(n) coordinate patches [Fomin, Zelevinsky]
- Canonical Bases of quantum group [Lusztig, Berenstein, Zelevinsky]
- Triangulated categories [Keller]
- DT invariants of top string theory [Kontsevich-Soibelman]
- Brane Tilings, wall crossing [Franco Eager]
- Supersymmetric Gauge theory [Arkani-Hamed et al]
- New quantum dilogarithm identities [Keller]
- Statistical physics, dimer models [Goncharov-Kenyon DF]

3. A. Example from Combinatorics: Frieze Patterns [Coxeter-Conway]

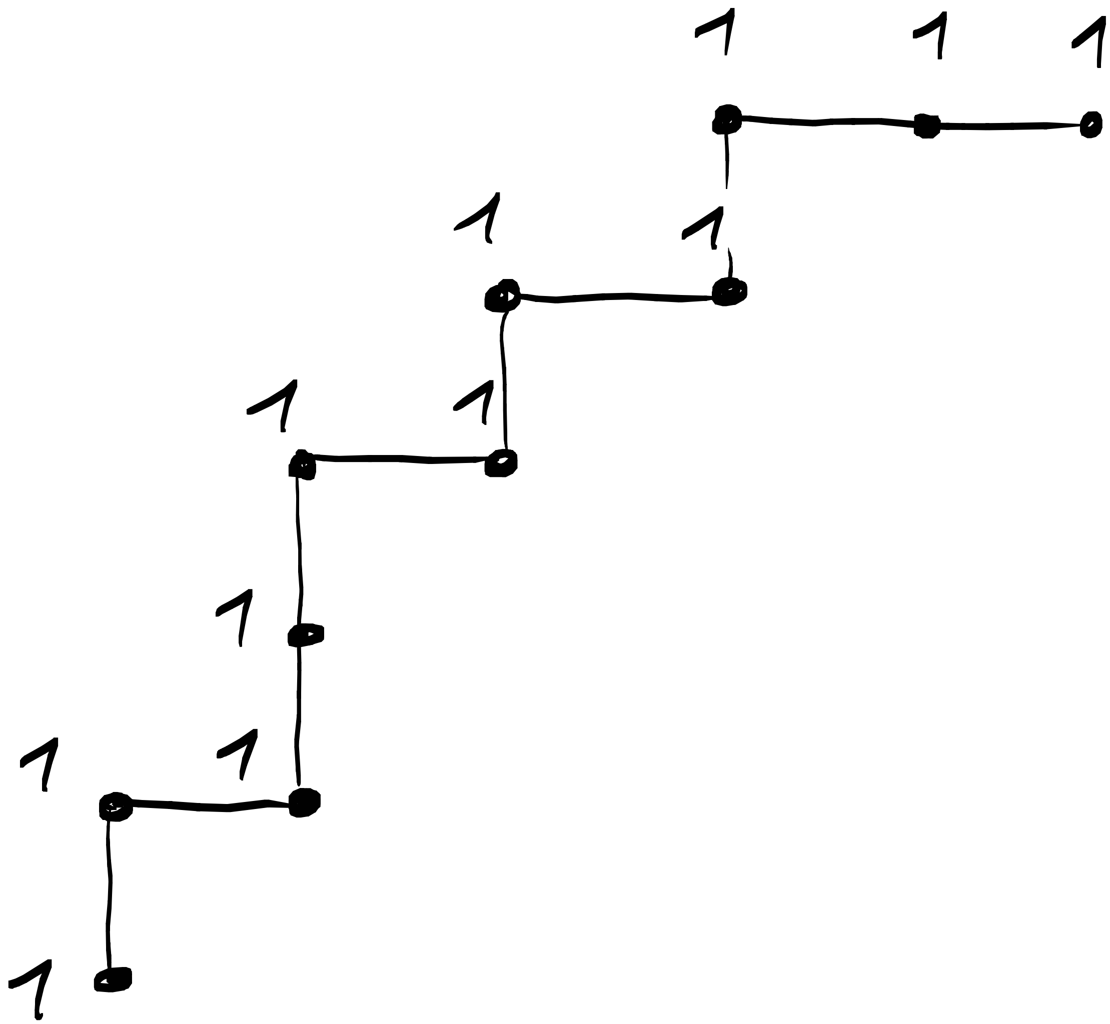
maps $X : \mathbb{Z}^2 \longrightarrow \mathbb{N}$ such that

for each square

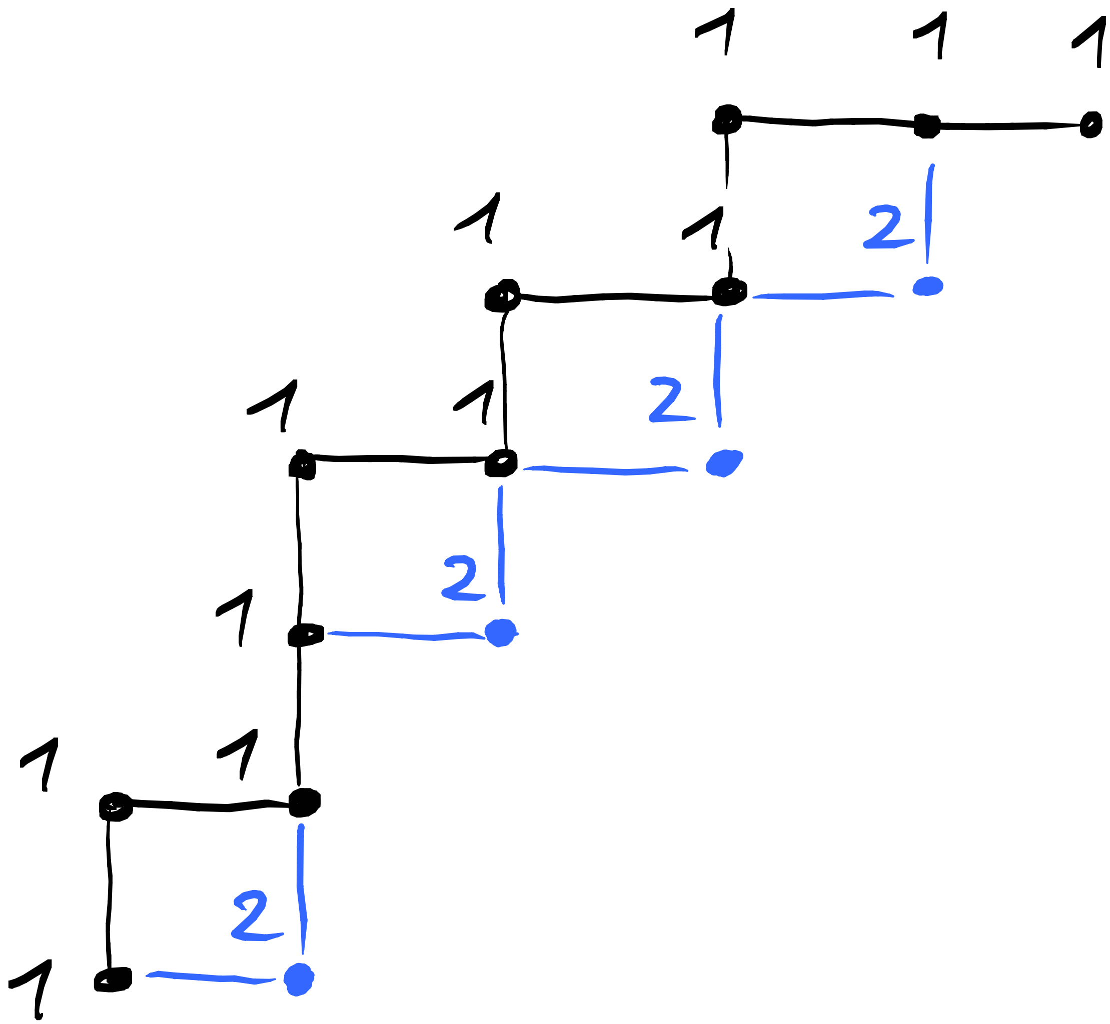


we have
$$\begin{vmatrix} X_{j,k-1} & X_{j-1,k} \\ X_{j+1,k} & X_{j,k+1} \end{vmatrix} = 1$$

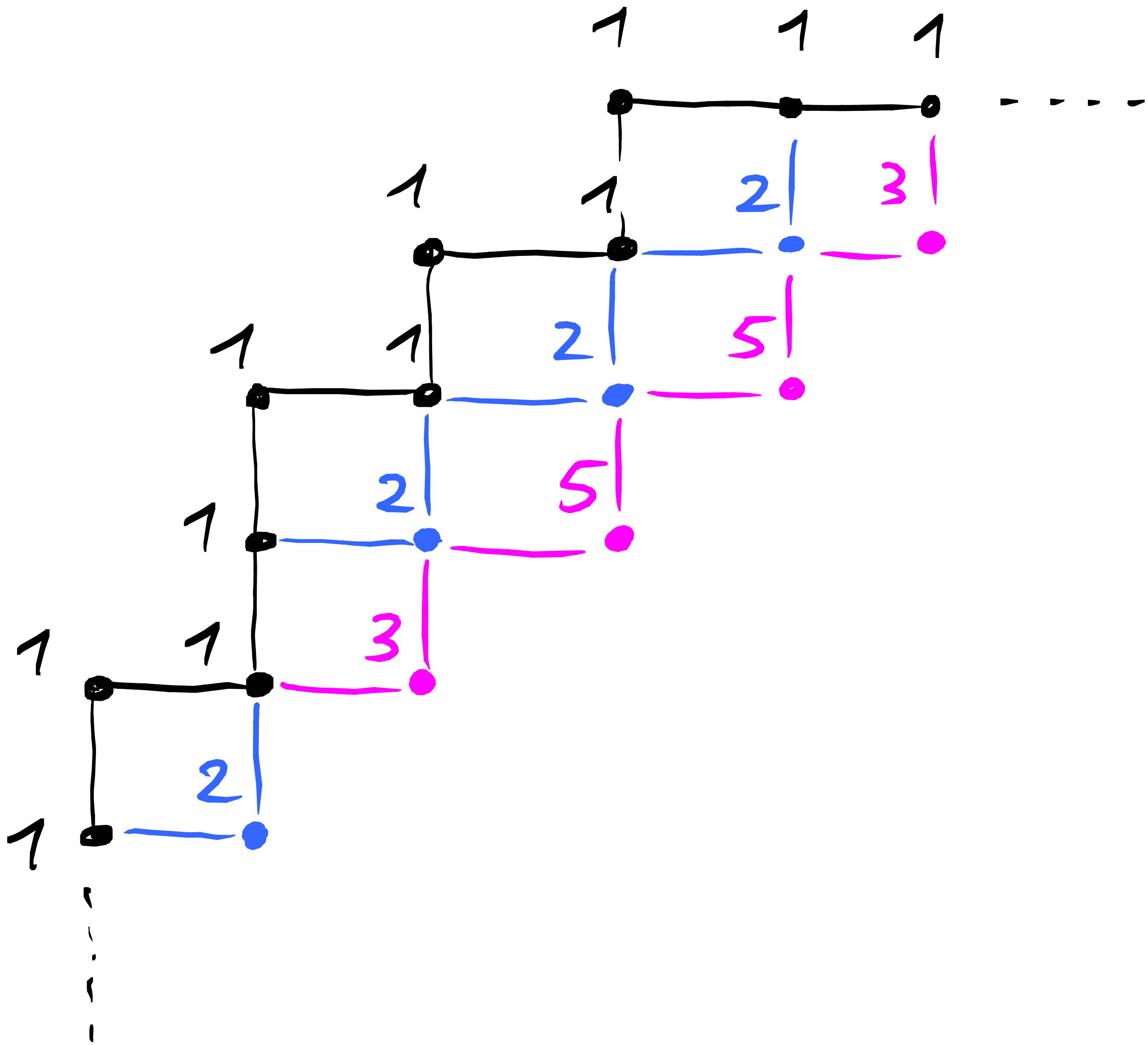
Ex



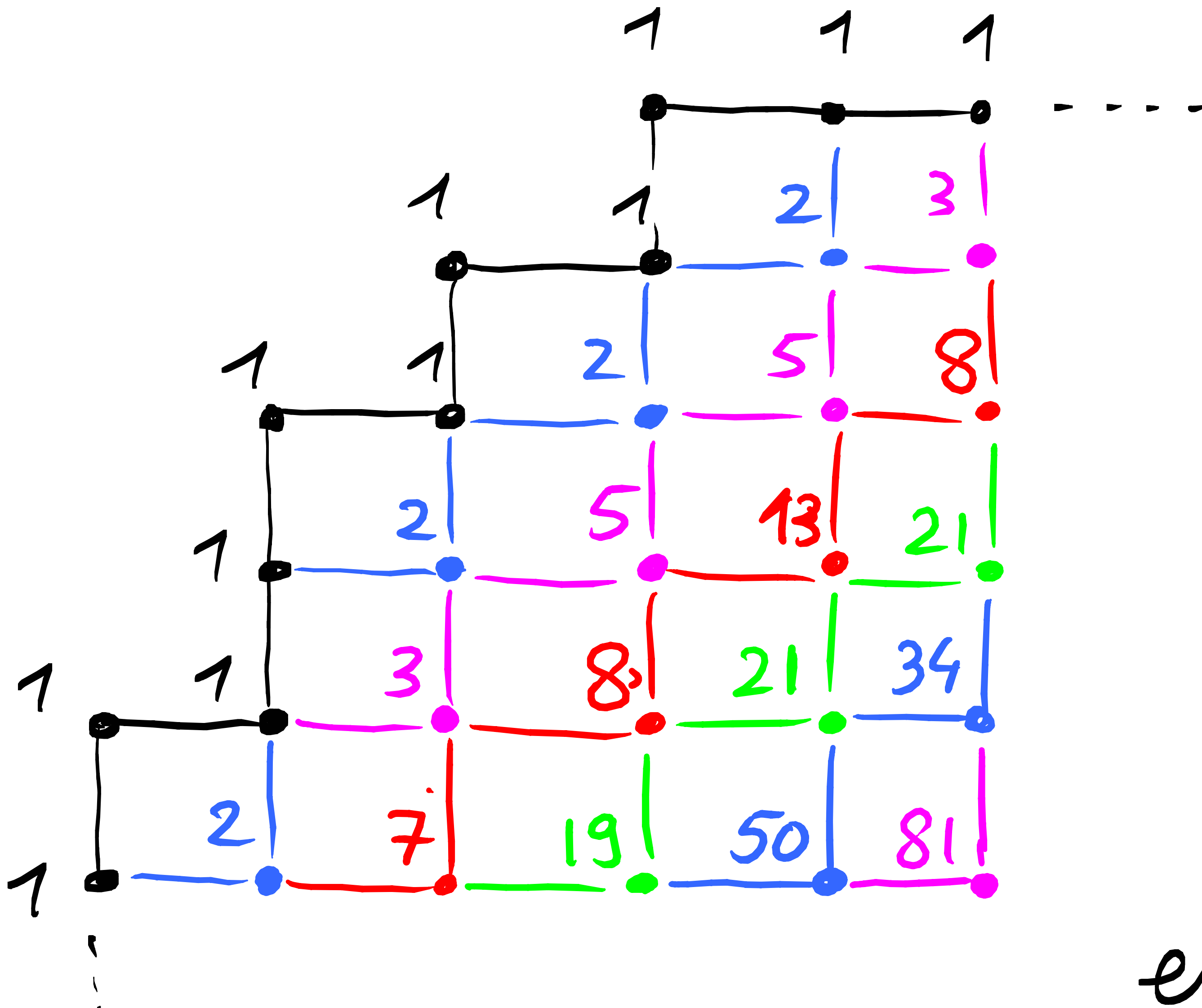
Ex



Ex



Ex

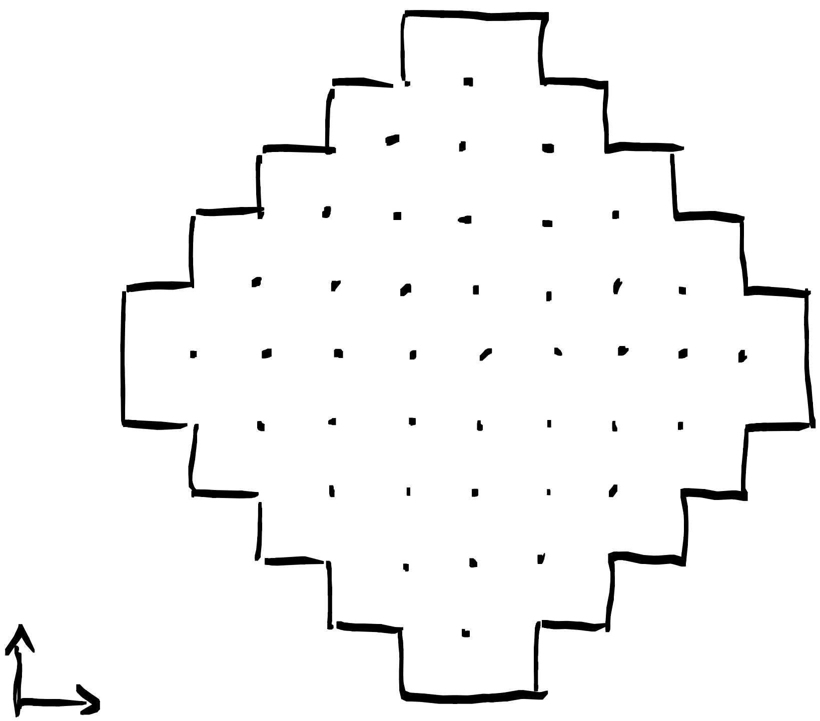


etc...

conservation laws (! -3 -2 !)

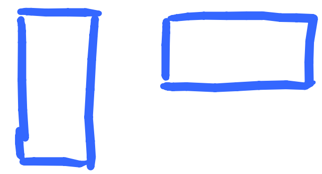
3.B. Example from statistical physics:

Domino Tilings of the Aztec Diamond



$\mathbb{Z}^2$

Dominos $\begin{cases} 1\times 2 \\ 2\times 1 \end{cases}$



→ partition function with weights

= weighted sum of configurations

Arctic curve theorem

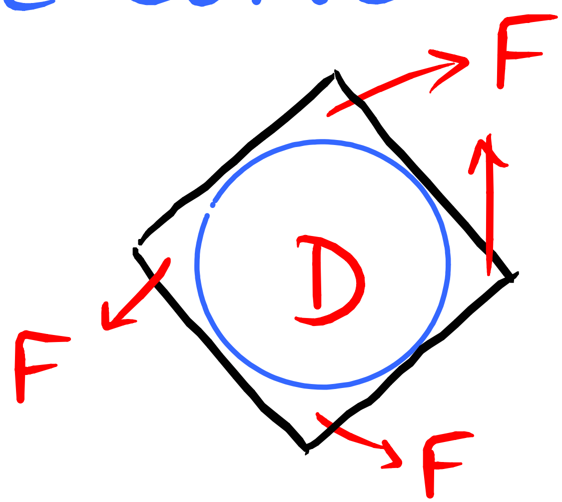
in the continuum limit of large size and small mesh, 2 phases

(1) ordered (frozen) in corners

(2) disordered away from corners

separation = arctic curve

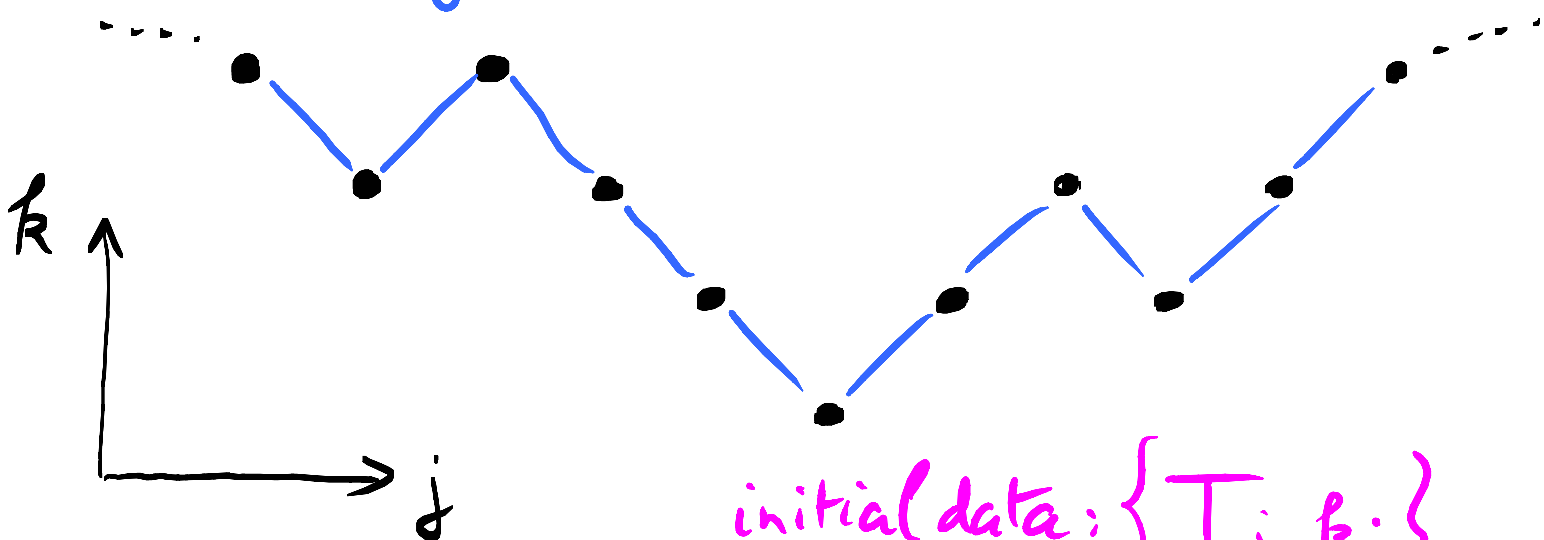
fluctuations = GFF



4. THE T-SYSTEM BEHIND FRIEZES

A_1 T-system: $\boxed{T_{j,k+1}\,T_{j,k-1} = T_{j+1,k}\,T_{j-1,k} + 1}$ $\quad k = \text{time} \in \mathbb{Z}$

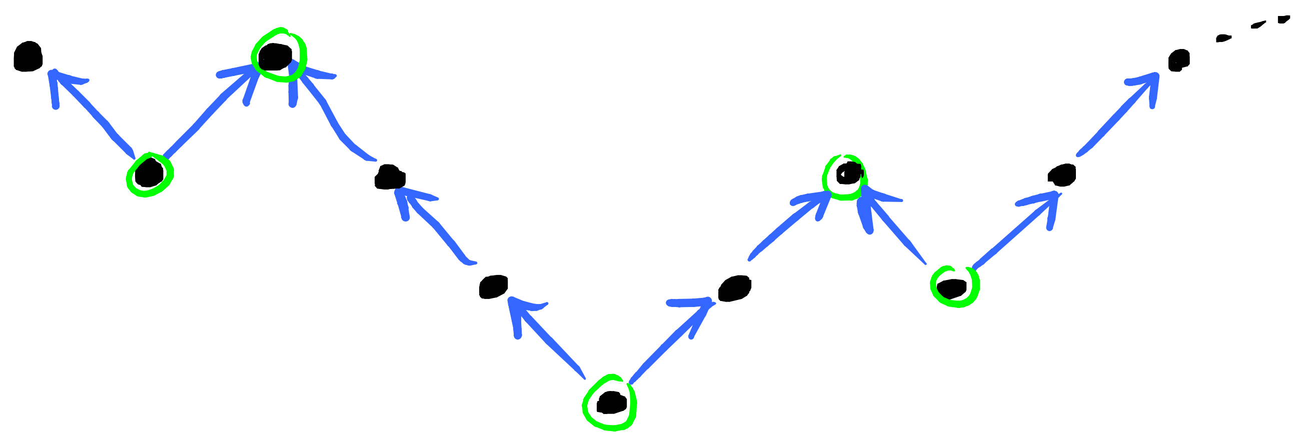
initial data = zig-zag line $\quad |k_{j+1} - k_j| = 1$
(one slice of A_r)



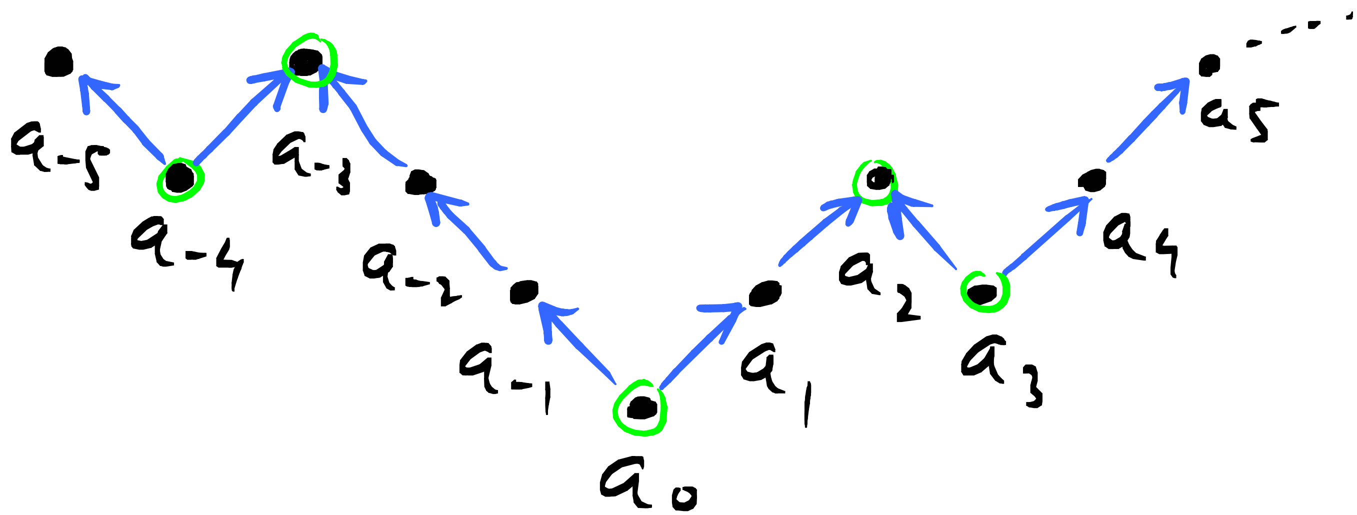
initial data: $\{T_{j,k_j}\}_{j\in\mathbb{Z}}$

THM The A_1 T-system is a mutation in an infinite rank cluster algebra

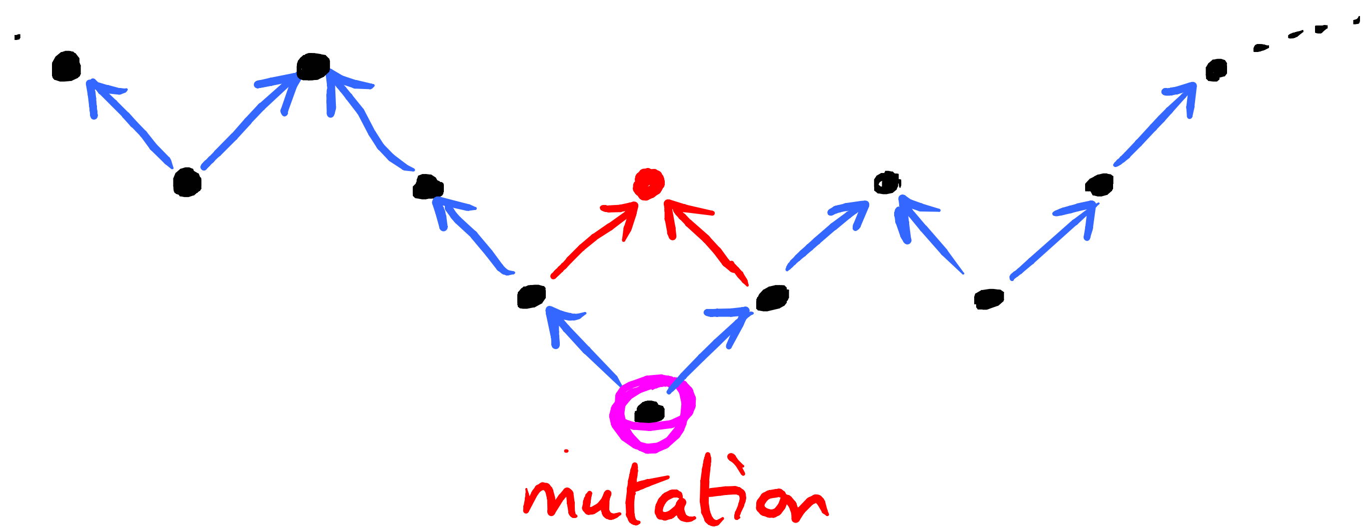
QUIVER



CLUSTER

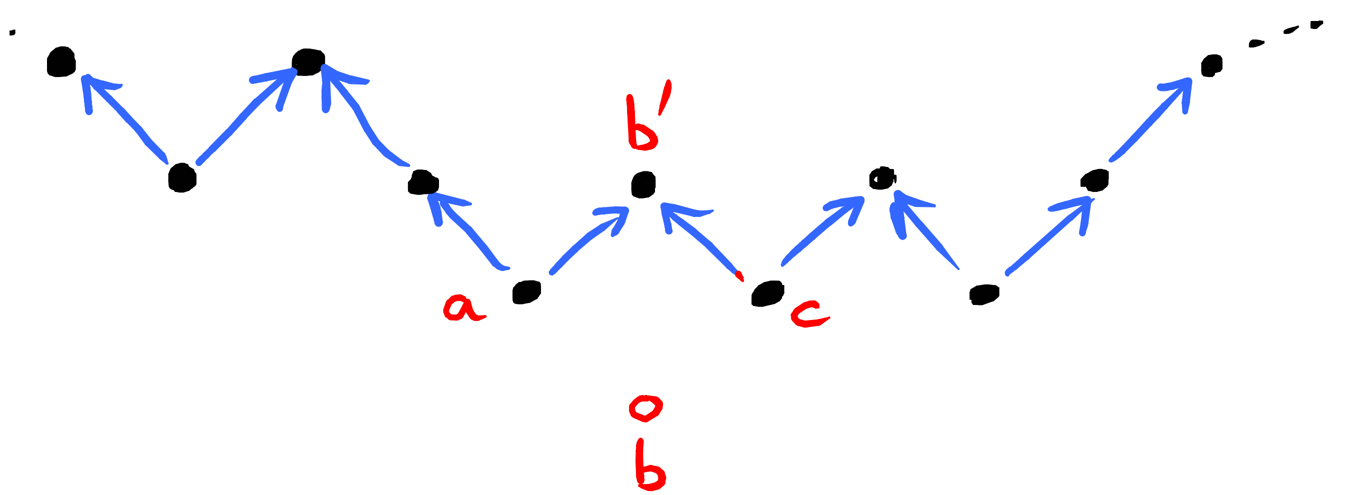


QUIVER MUTATION



Rule: we restrict to only mutations where the two arrows point in or out.
(2 tails or 2 heads only).

CLUSTER MUTATION:



$\Rightarrow$ mutation relation is $\boxed{bb' = ac + 1}$

i.e. det(plaquette) = 1

$$\boxed{\text{Positive Laurent phenomenon} \Rightarrow \text{integrality of Frieze pattern with path-like Boundary condition } 1}$$

Remark: T-system is a discrete integrable system (with infinite dimension)

INTEGRABILITY

Write the eqns as $W_{j,k} = \begin{vmatrix} T_{j,k+1} & T_{j+1,k} \\ T_{j-1,k} & T_{j,k-1} \end{vmatrix} = 1$

Write

$$W_{j,k} - W_{j+1,k-1} = \begin{vmatrix} T_{j,k+1} + T_{j+2,k-1} & T_{j+1,k} \\ T_{j-1,k} + T_{j+1,k-2} & T_{j,k-1} \end{vmatrix} = 0$$

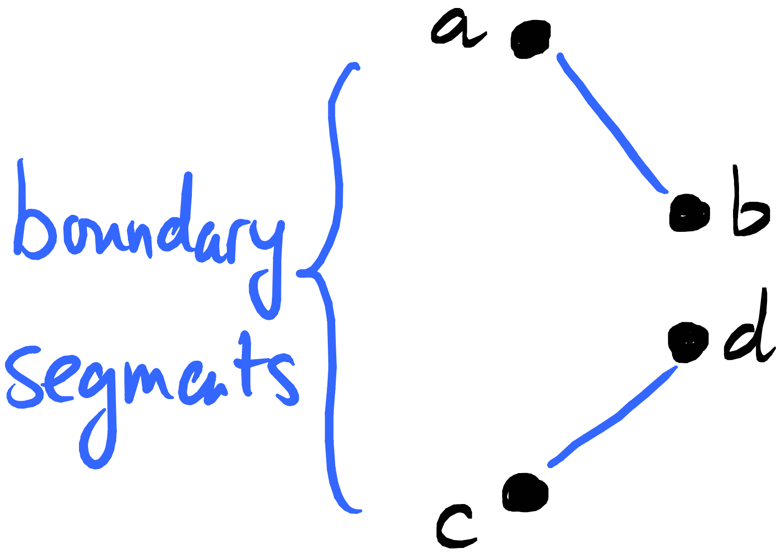
$$\Rightarrow \quad \exists\, c_{j,k} : \quad \left.\begin{array}{l} T_{j,k+1} + T_{j+2,k-1} = c_{j,k}\, T_{j+1,k} \\ T_{j-1,k} + T_{j+1,k-2} = c_{j,k}\, T_{j,k-1} \end{array}\right\} \Rightarrow \begin{array}{c} c_{j,k} \\ \shortparallel \\ c_{j-1,k-1} \\ \shortparallel \\ \boxed{c(j-k)} \end{array}$$

$$\boxed{T_{j,k+1} - c(j-k)\, T_{j+1,k} + T_{j+2,k-1} = 0}$$

Solution?

- Find explicit formulas for T_{jk} as a function of T_{j,k_j} along the initial data path $(j, k_j)_{j\in\mathbb{Z}}$
- Check Laurent positivity
- Interpret result

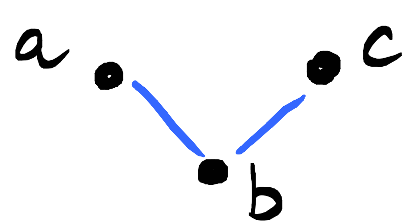
MATRIX REPRESENTATION:



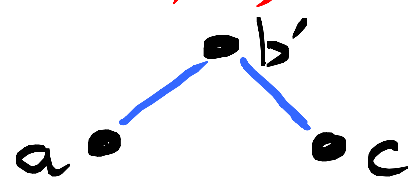
$$\longrightarrow \quad D(a,b) = \begin{pmatrix} \frac{a}{b} & \frac{1}{b} \\ 0 & 1 \end{pmatrix}$$

$$\longrightarrow \quad U(c,d) = \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix}$$

$D(a,b)\,U(b,c)$

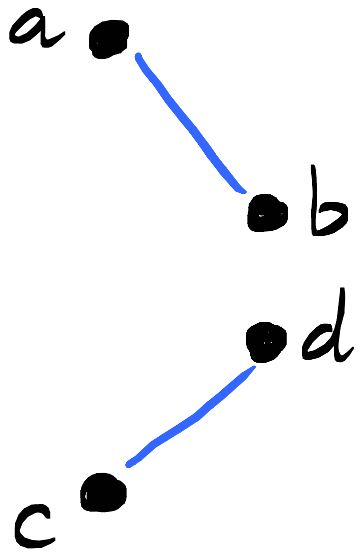


$U(a,b')\,D(b',c)$



MATRIX REPRESENTATION:

boundary segments



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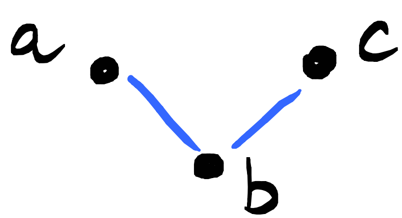
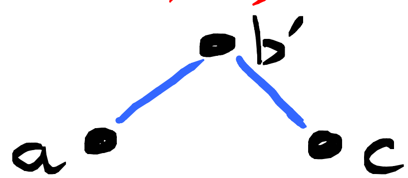
$$\longrightarrow \quad U(c,d) = \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix}$$

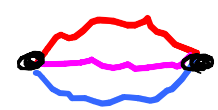
$$D(a,b)\,U(b,c) = U(a,b')\,D(b',c) \quad \Leftrightarrow$$



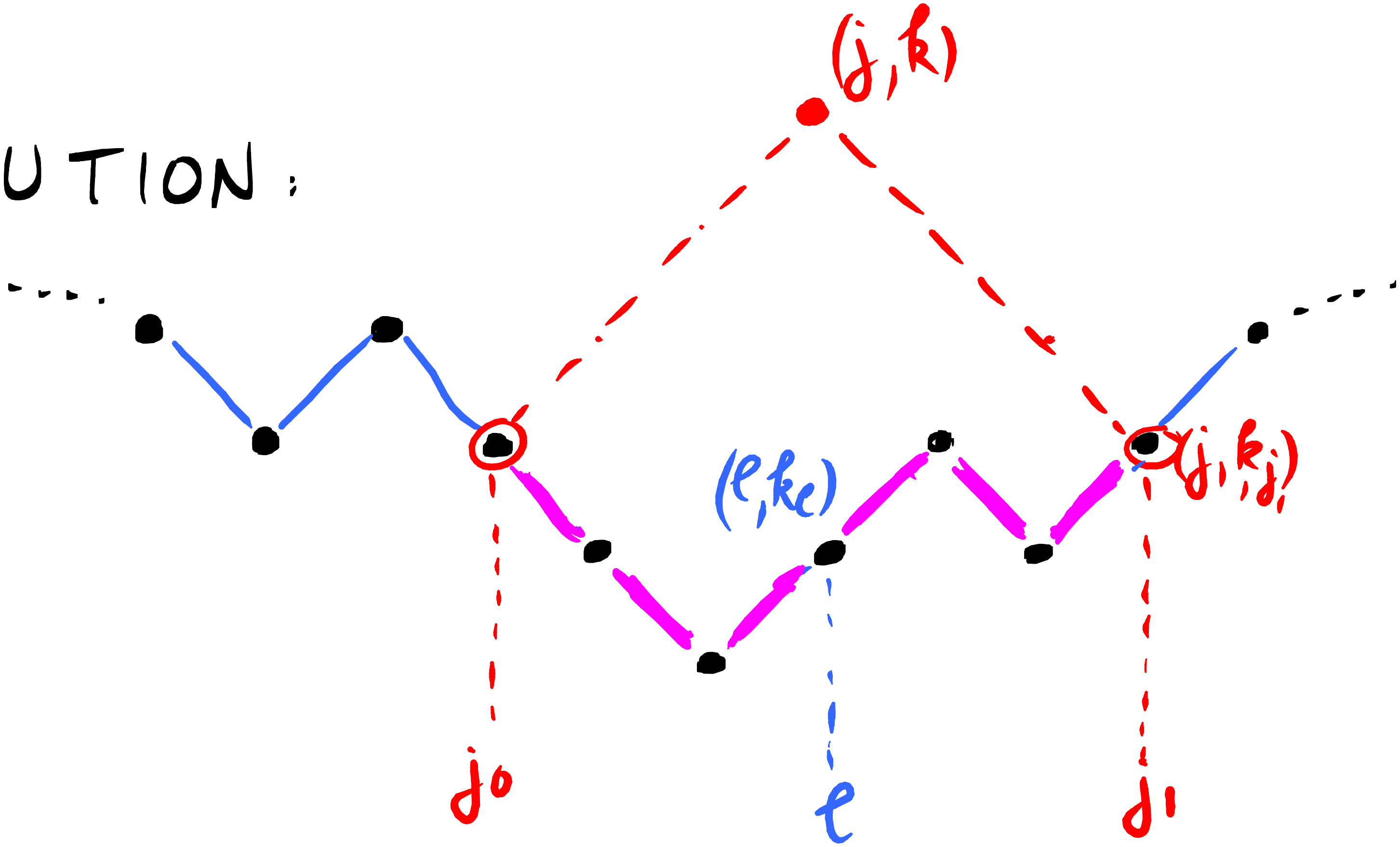


$$\boxed{bb' = ac+1}$$

(Flat $GL(2)$ connection) $\longleftrightarrow$ (integrability)



SOLUTION:



$$\frac{T_{j,k}}{T_{j_1,k_{j_1}}} = \left[\prod_{l=j_0}^{j_1-1} \begin{Bmatrix} D \\ U \end{Bmatrix} (k_l, k_{l+1}) \right]_{1,1} \leftarrow \text{(1,1) element}$$

"Tranfer matrix"

<u>Note:</u> (1) the arguments of D, U are values of T_{j,k_j} from the initial data

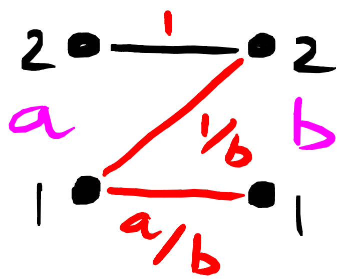
(2) entries are all >0 Laurent monomials

$\Rightarrow$ LAURENT POSITIVITY

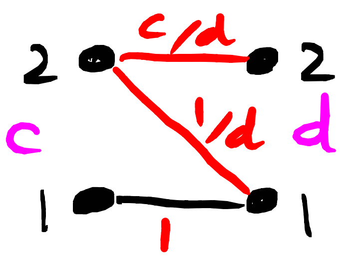
NETWORK FORMULATION

weighted graphs (oriented left→right)

$$D(a,b) = \begin{pmatrix} \frac{a}{b} & \frac{1}{b} \\ 0 & 1 \end{pmatrix} =$$

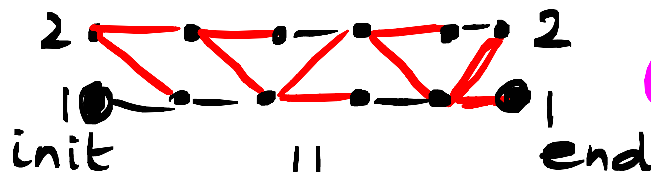


$$U(c,d) = \begin{pmatrix} 1 & 0 \\ \frac{1}{d} & \frac{c}{d} \end{pmatrix} =$$



$$(U\ U\ D\ U\ D)_{1,1}$$

$$\equiv$$



Product

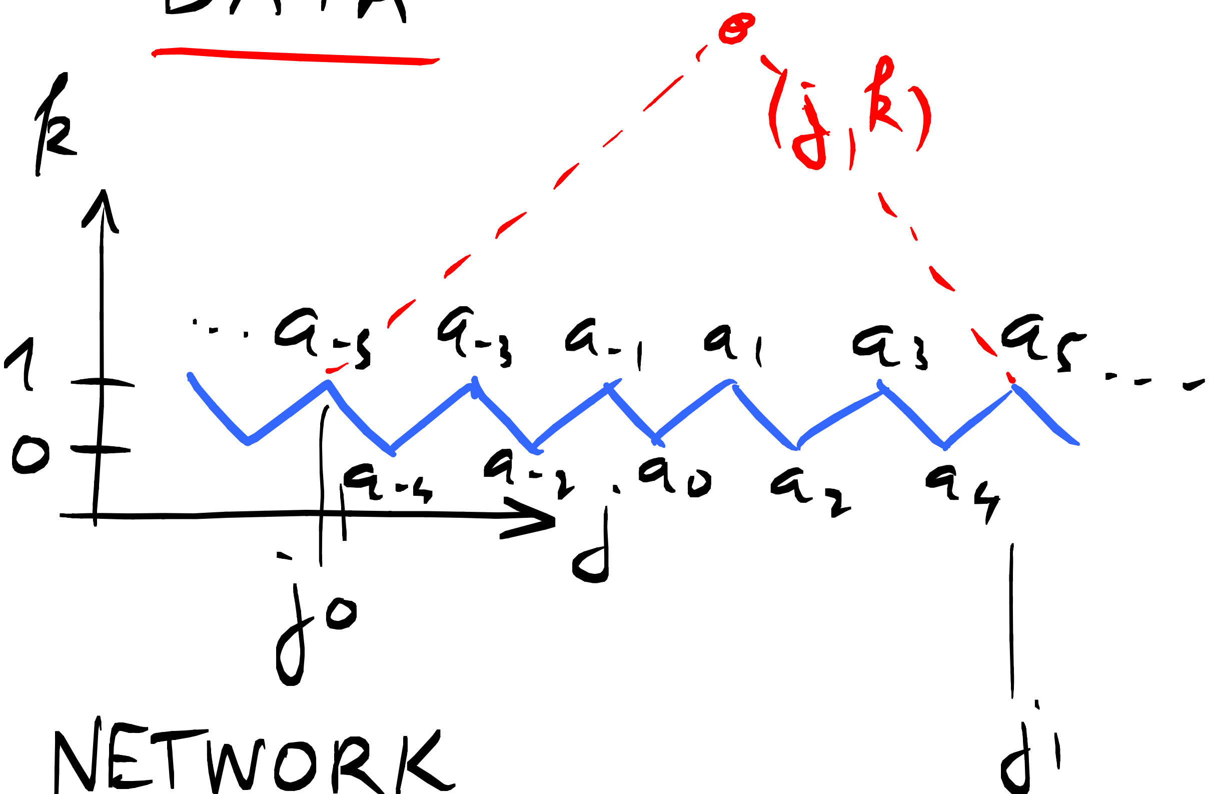
↓

Concatenate

$$\|$$

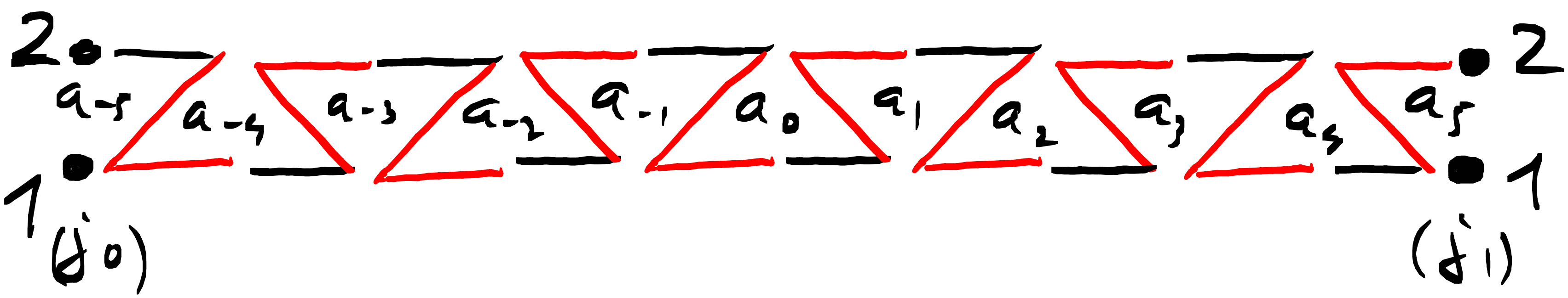
PF for paths $1 \to 1$ on the network.

PARTICULAR CASE: THE "FLAT" INITIAL DATA

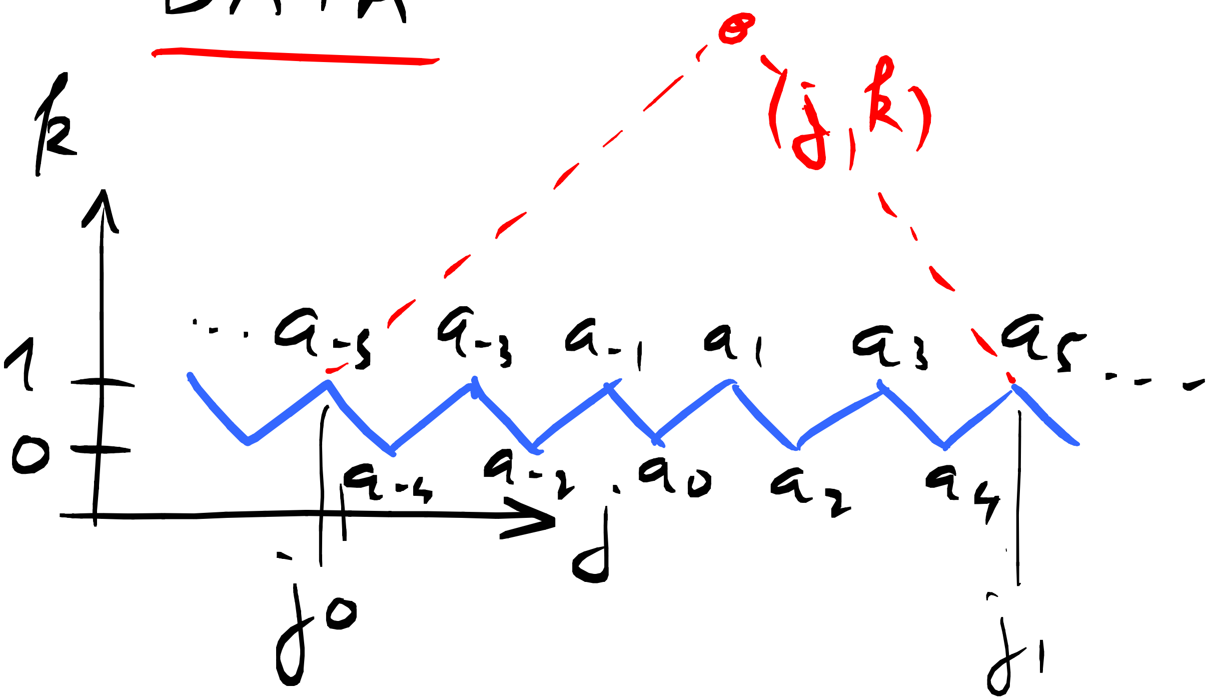


$$T_{jk} = \left(\prod_{j_0}^{j_1 - 1} DU \right)_{1,1} a_{j_1}$$

NETWORK



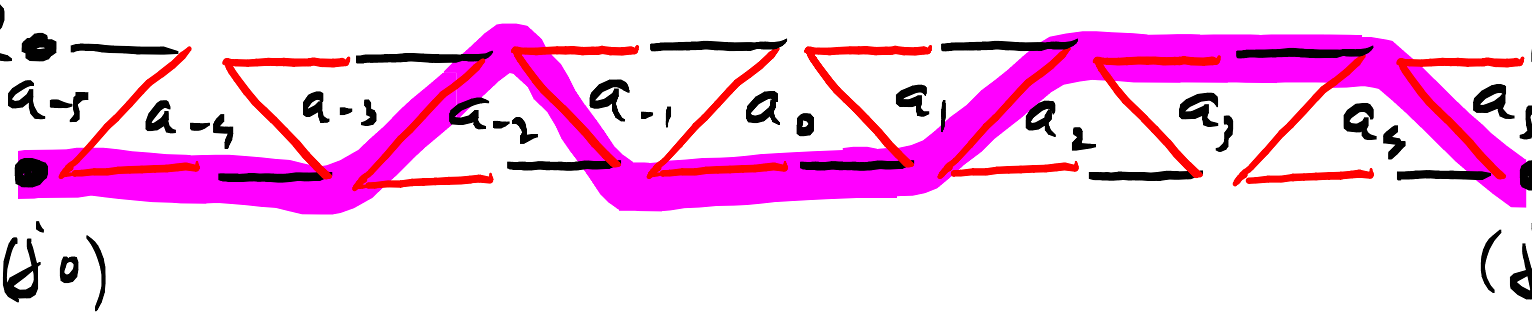
PARTICULAR CASE: THE "FLAT" INITIAL DATA



$$T_{jk} = \left(\prod_{j_0}^{j_1-1} DU\right)_{1,1} a_{j_1}$$

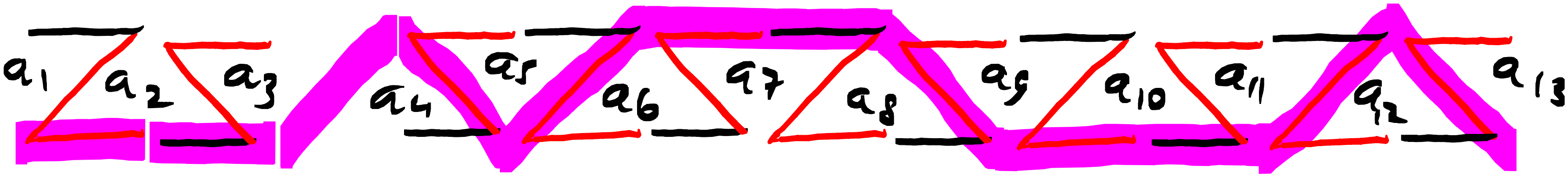
NETWORK

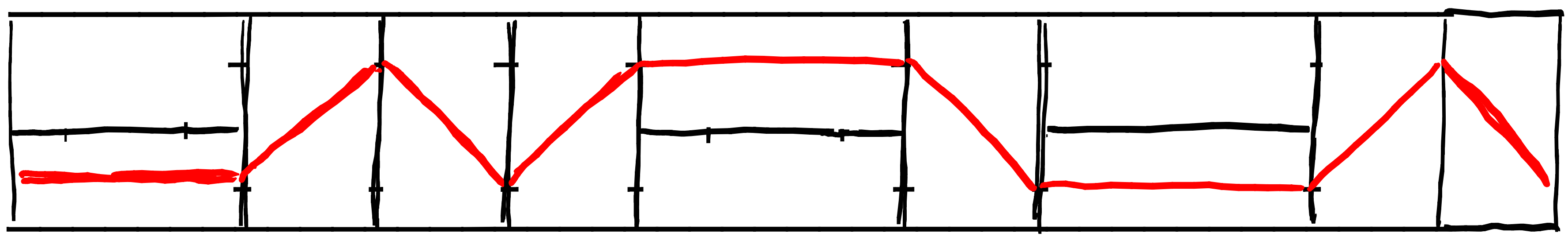
$T_{jk} = \sum_{\text{paths}}$

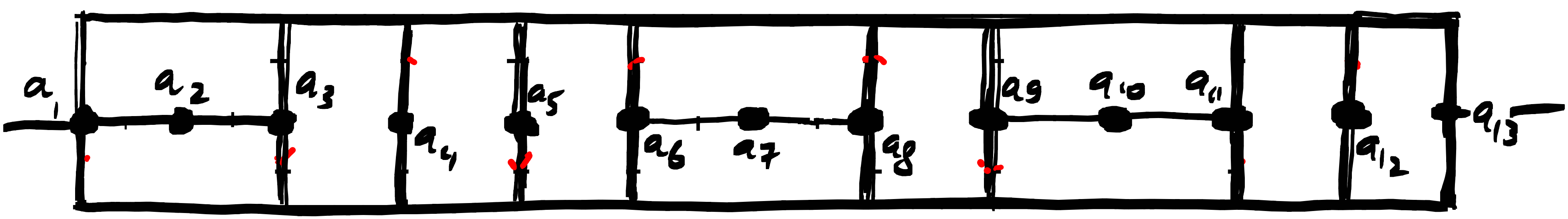


$\times a_{j_1}$

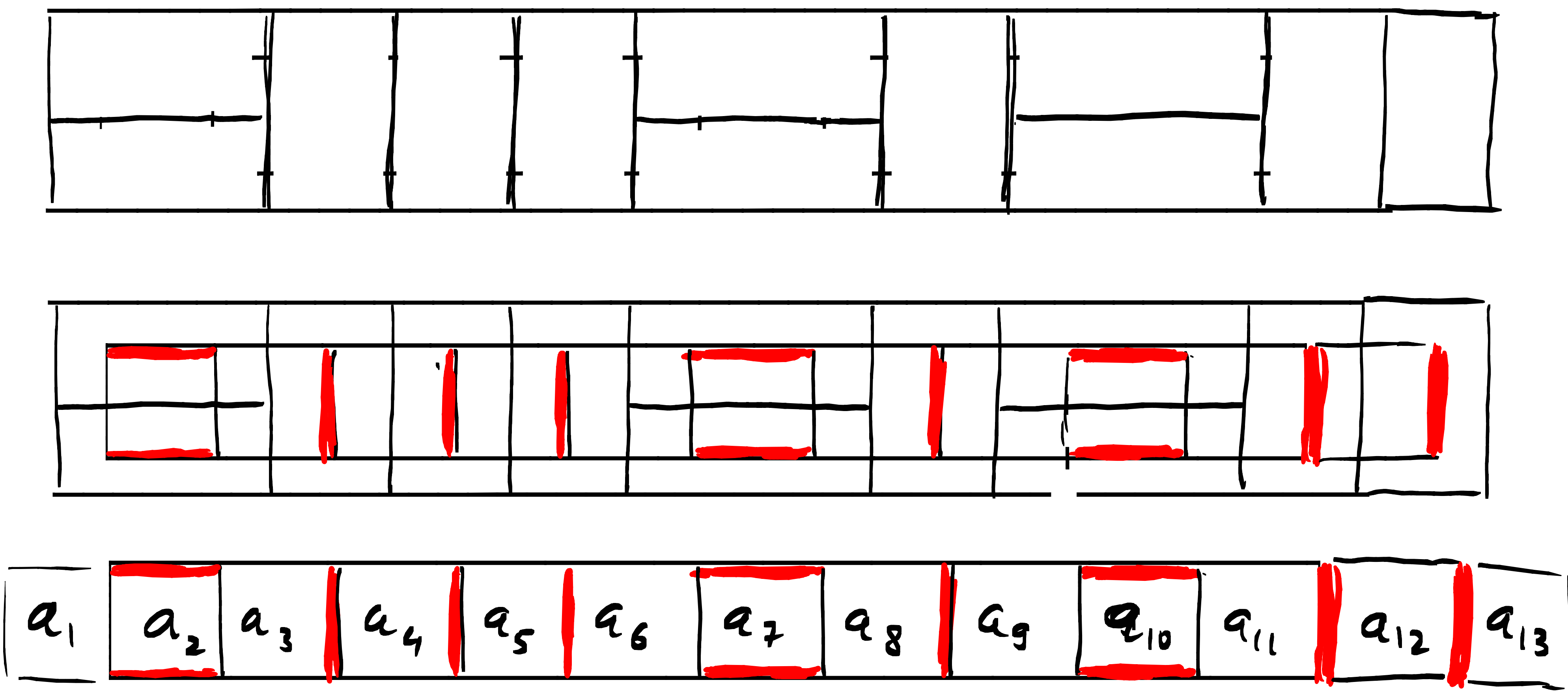
FROM PATHS TO DOMINO TILINGS







From Dominos to Dimers



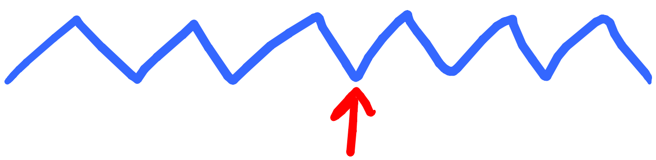
$$\text{Weight} = \prod a_i^{\,1-D_i}$$

dimers on square i

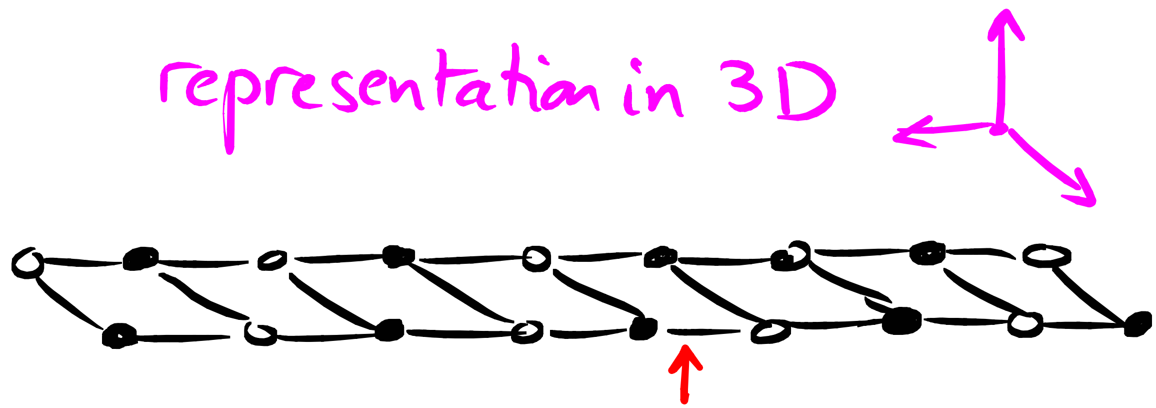
CONCLUSION:

- We can think of D, U as transfer matrices for tiling/dimer model.
- Laurent positivity $\Leftrightarrow$ positivity of the Boltzmann weights of the dimers
- Coefficients count dimer configurations

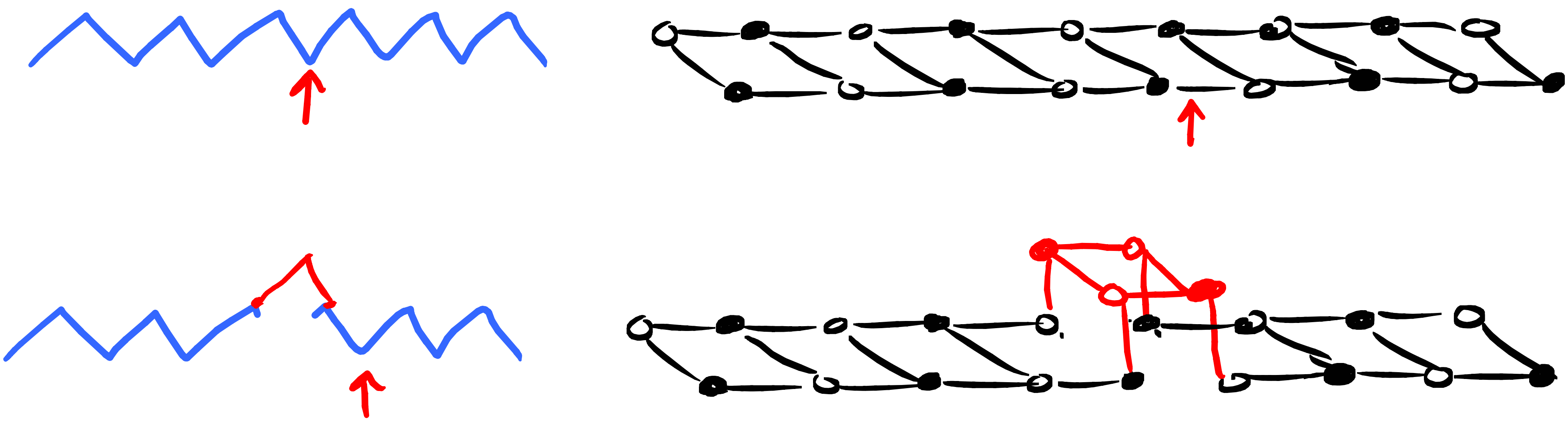
MUTATIONS ?

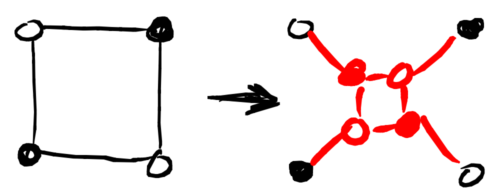


representation in 3D

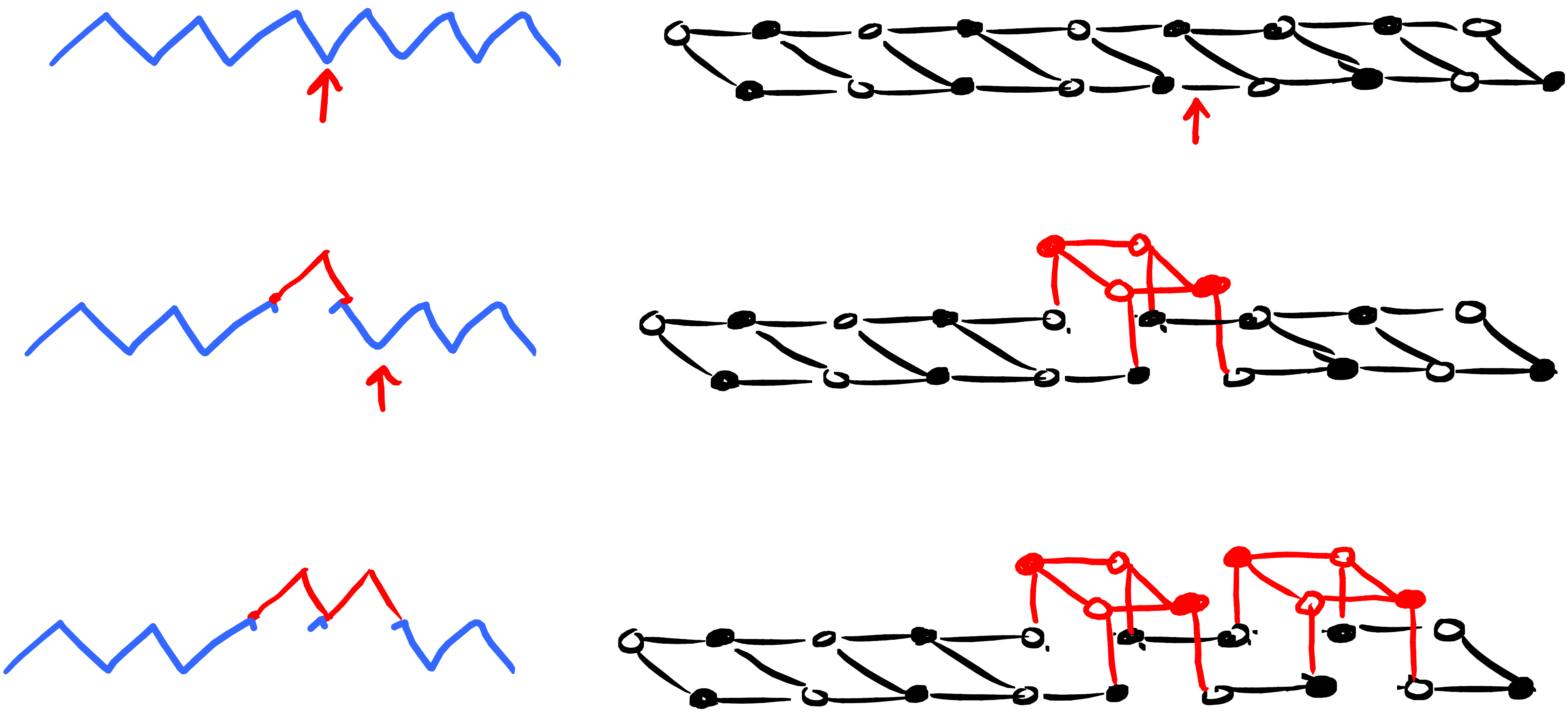


MUTATIONS ?

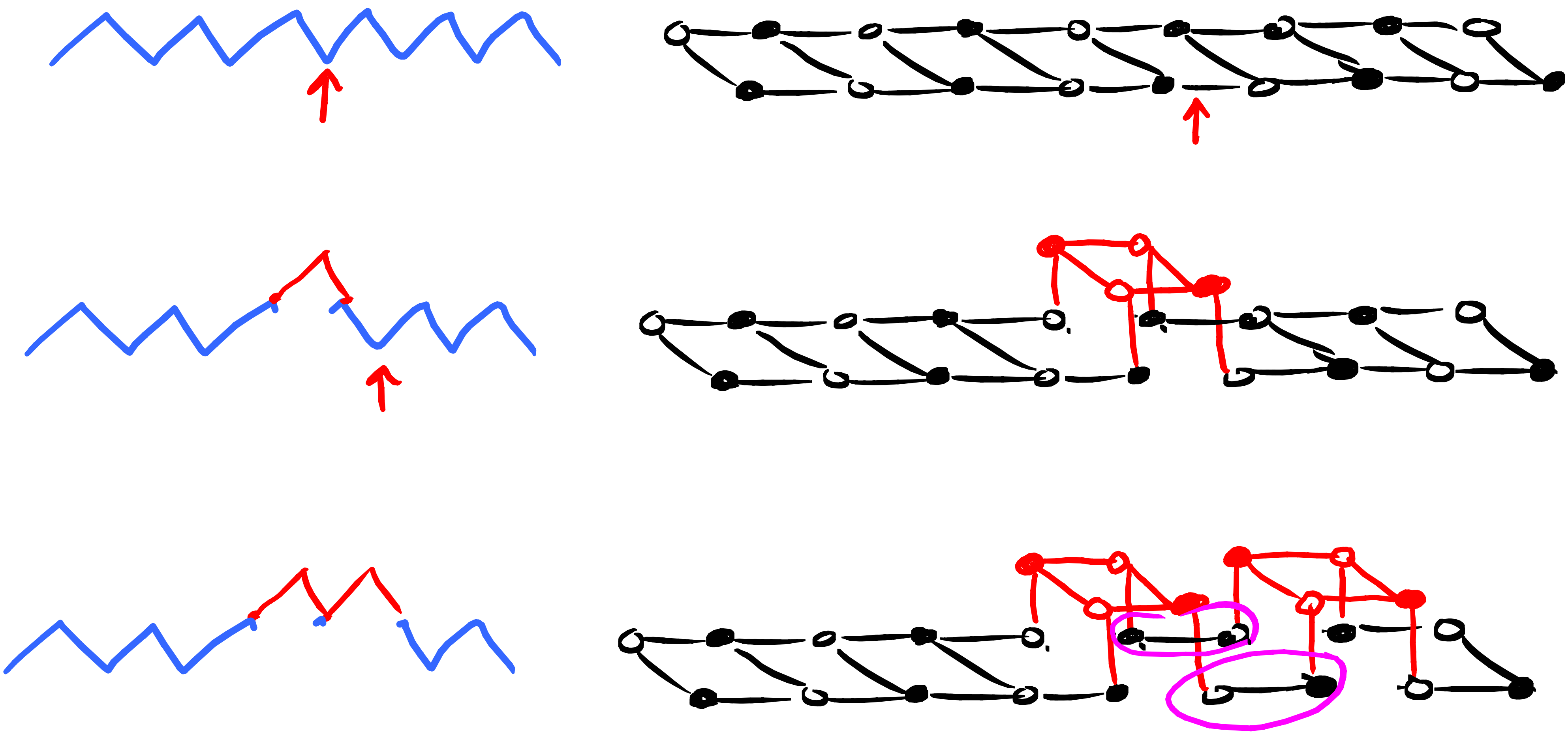


"Urban Renewal" 

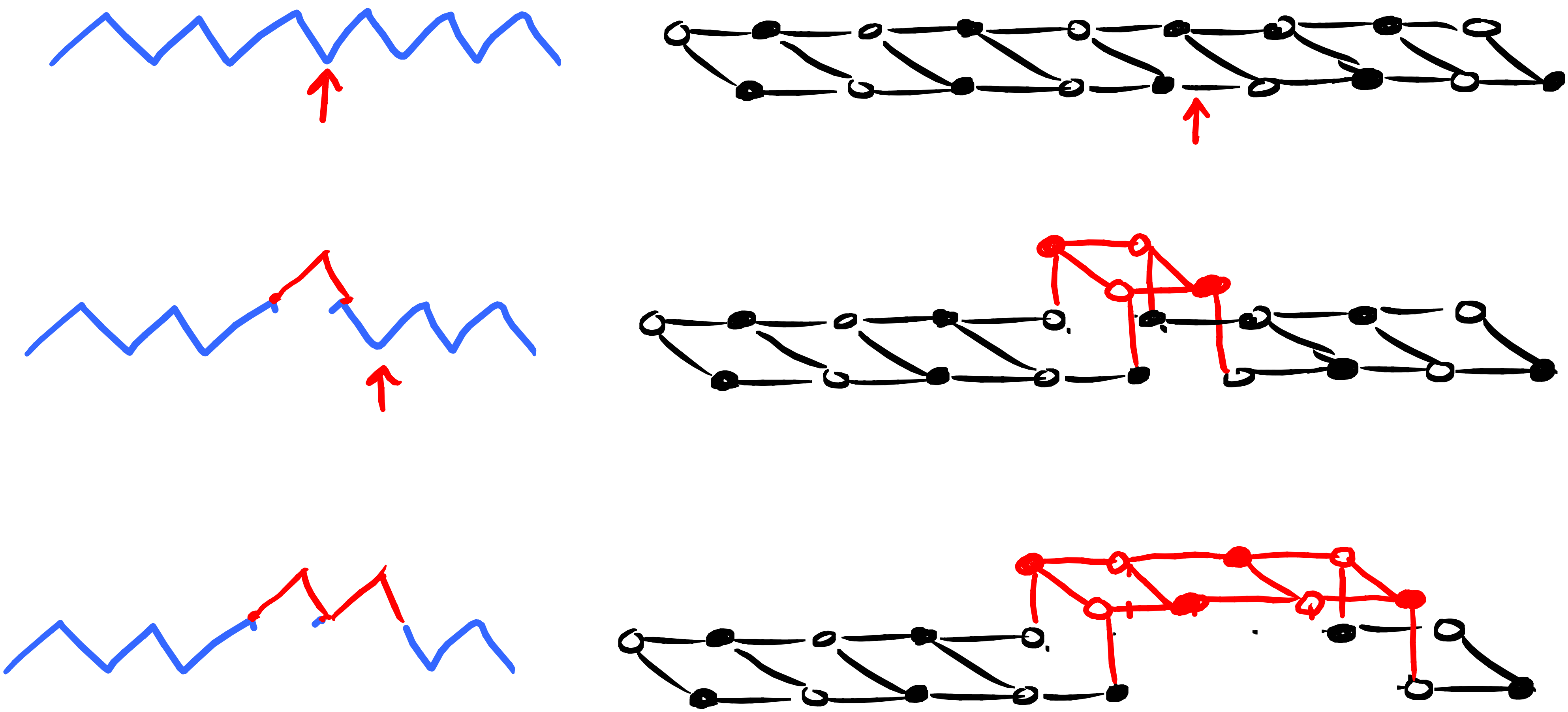
MUTATIONS ?



MUTATIONS ?



MUTATIONS ?



etc.

weights :

$$w\left(\boxed{a}\right) = a^{1-d}$$

$$w\left(\boxed{\ \ a\ \ }\right) = a^{2-d}$$

d = # dimers araund the square / hexagon

$\boxed{\text{THM for any given initial data}\quad T_{ij} = \sum_{\substack{\text{dimers on}\\ \text{3D ladder}\\ \text{graph}}} \prod(\text{face weights})}$

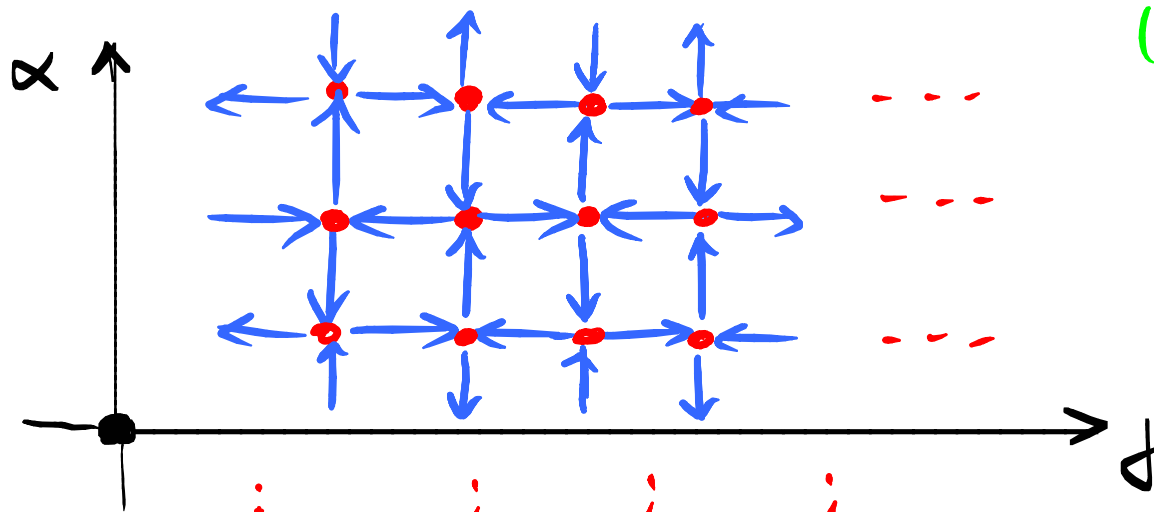
"reverse quantum gravity" Z = invariant (surface + weights)

5. OCTAHEDRON eqn from Cluster Alg. to Dimers

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:



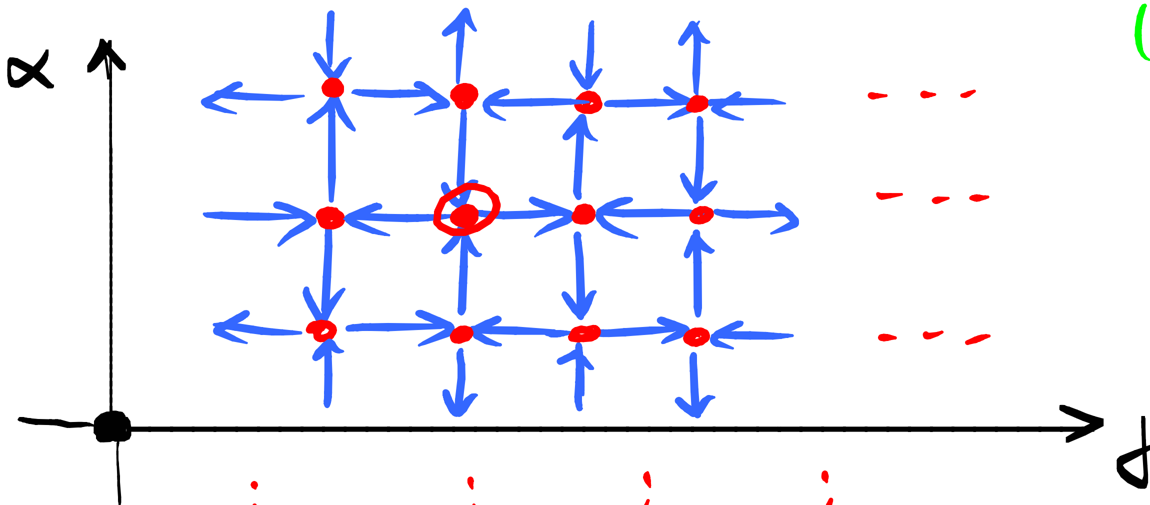
Cluster:

$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & T_{1,-1,0} & T_{1,0,1} & T_{1,1,0} & T_{1,2,1} & \cdots \\
\cdots & T_{0,-1,1} & T_{0,0,0} & T_{0,1,1} & T_{0,2,0} & \cdots \\
\cdots & T_{-1,-1,0} & T_{-1,0,1} & T_{-1,1,0} & T_{-1,2,1} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:
mutation



Cluster :

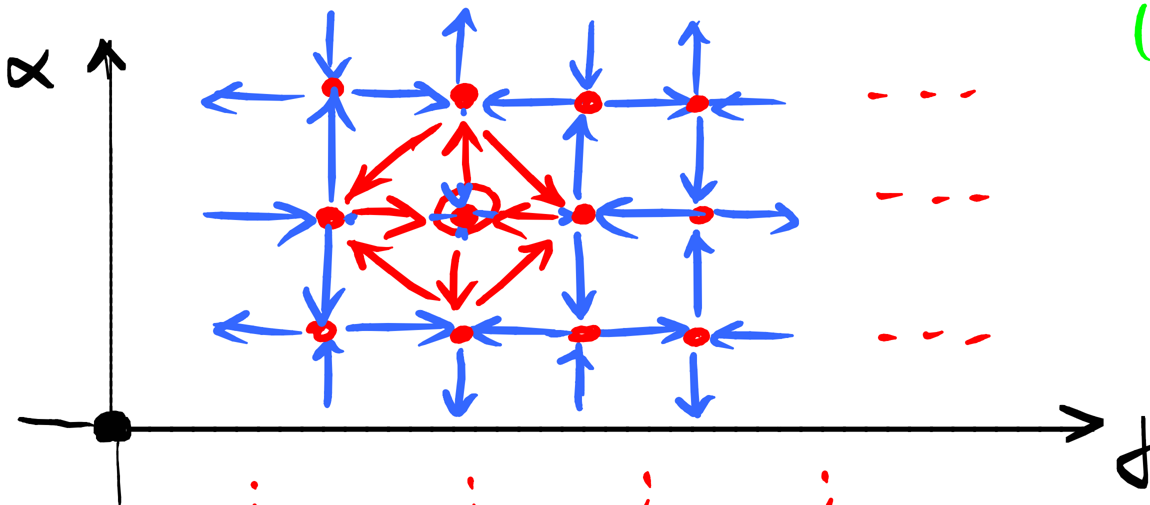
$$\begin{array}{cccccc}
 & \vdots & \vdots & \vdots & \vdots & \\
\cdots & T_{1-10} & T_{101} & T_{110} & T_{121} & \cdots \\
\cdots & T_{0-11} & T_{000} & T_{011} & T_{020} & \cdots \\
\cdots & T_{-1-10} & T_{-101} & T_{-110} & T_{-121} & \cdots \\
 & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

THM The octahedron move is a mutation in an infinite rank Cluster Algebra

(DF Kedem 09)

Quiver:

mutation

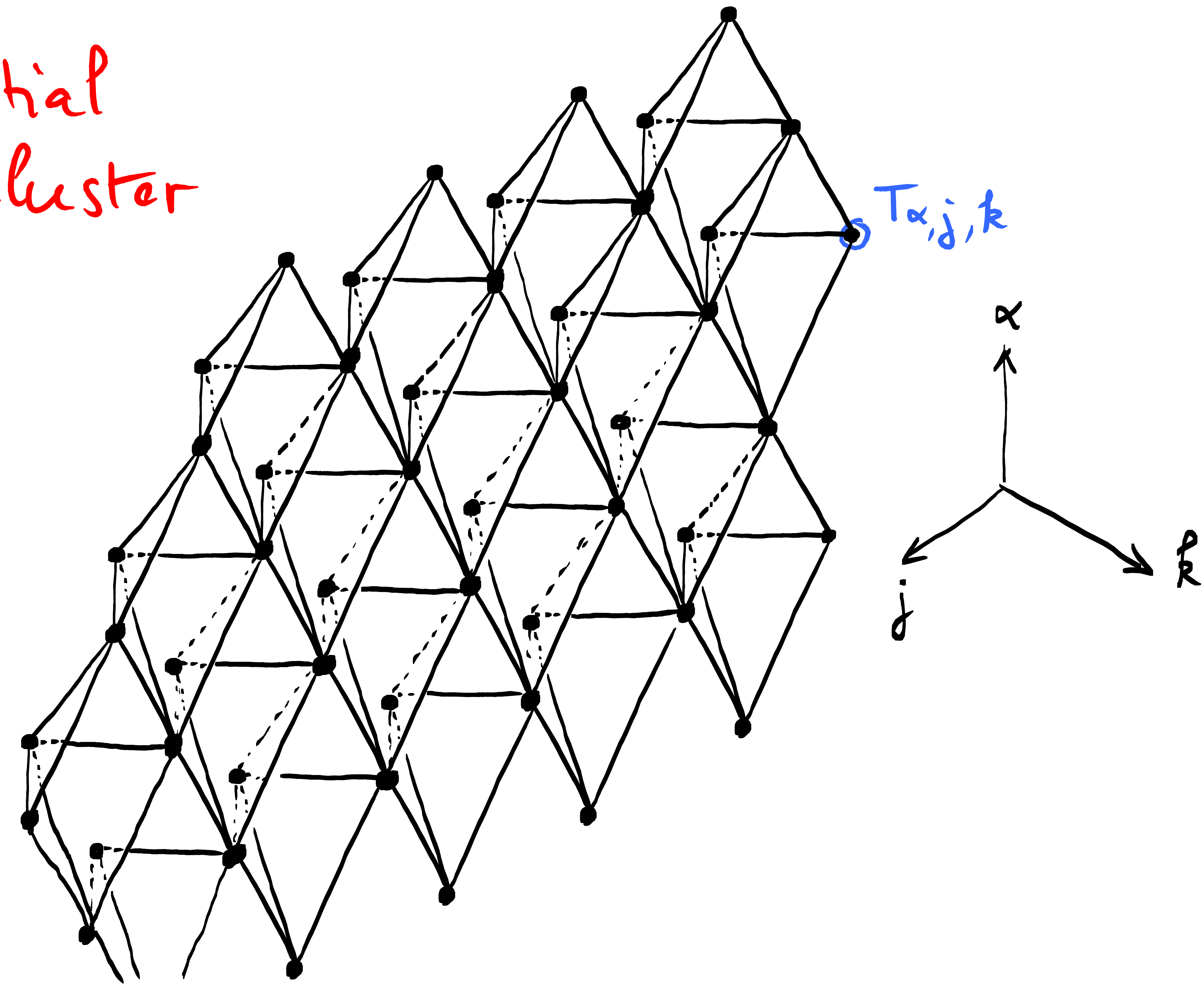


Cluster:

$$\begin{array}{cccccc} & \vdots & \vdots & \vdots & \vdots & \\ \cdots & T_{1-10} & T_{101} & T_{110} & T_{121} & \cdots \\ \cdots & T_{0-11} & \circ & T_{011} & T_{020} & \cdots \\ \cdots & T_{-1-10} & T_{-101} & T_{-110} & T_{-121} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \end{array}$$

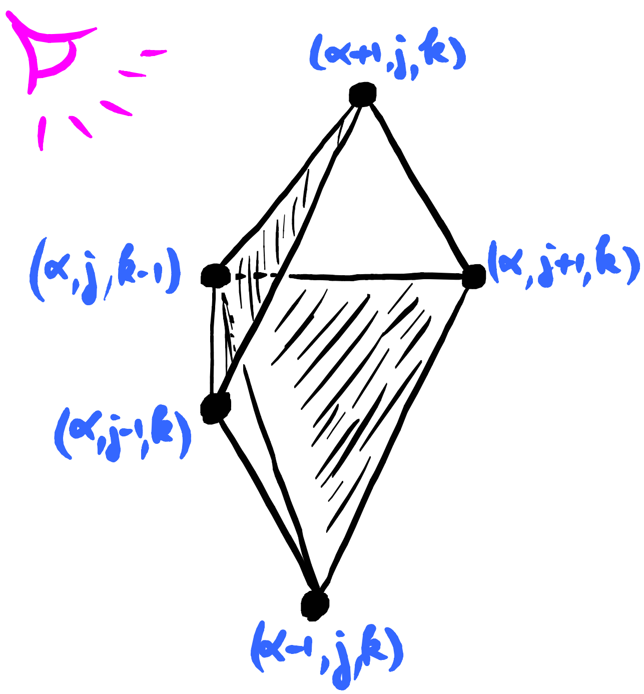
$$\frac{T_{0-11}T_{011}+T_{101}T_{-101}}{T_{000}}$$

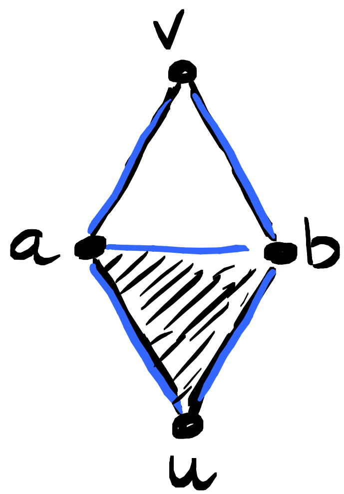
initial cluster



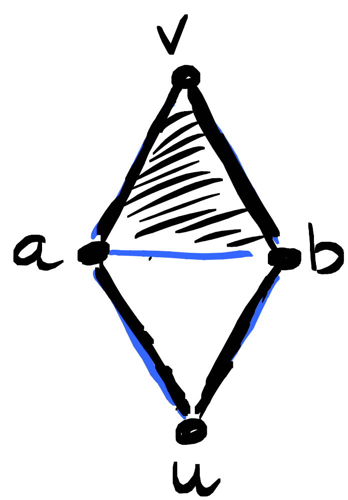
POSITIVITY:

EXACT SOLUTION BY MATRIX REPRESENTATION

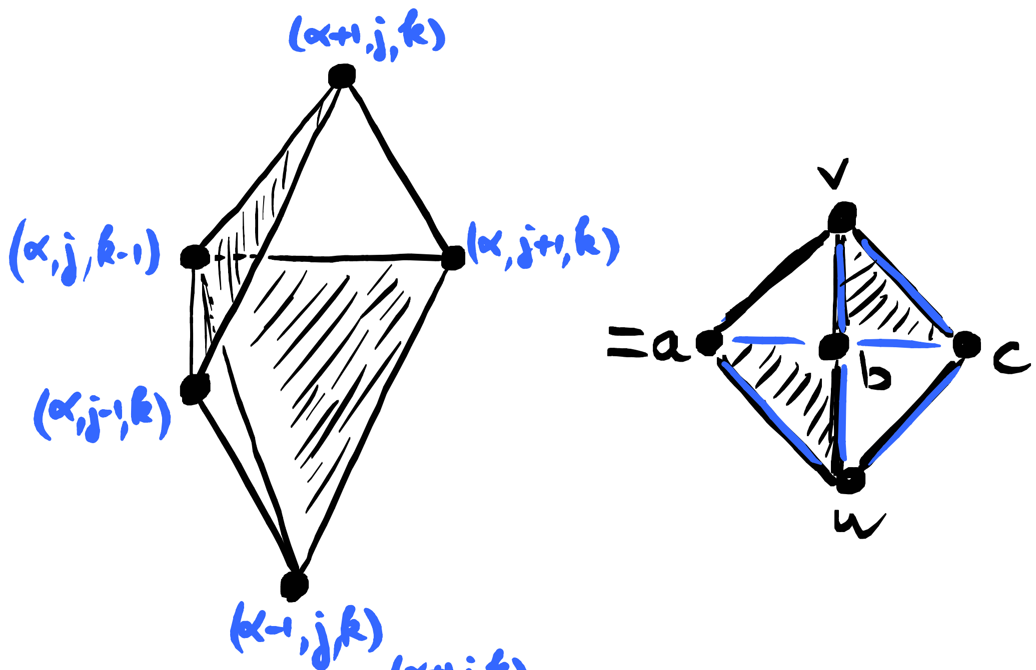




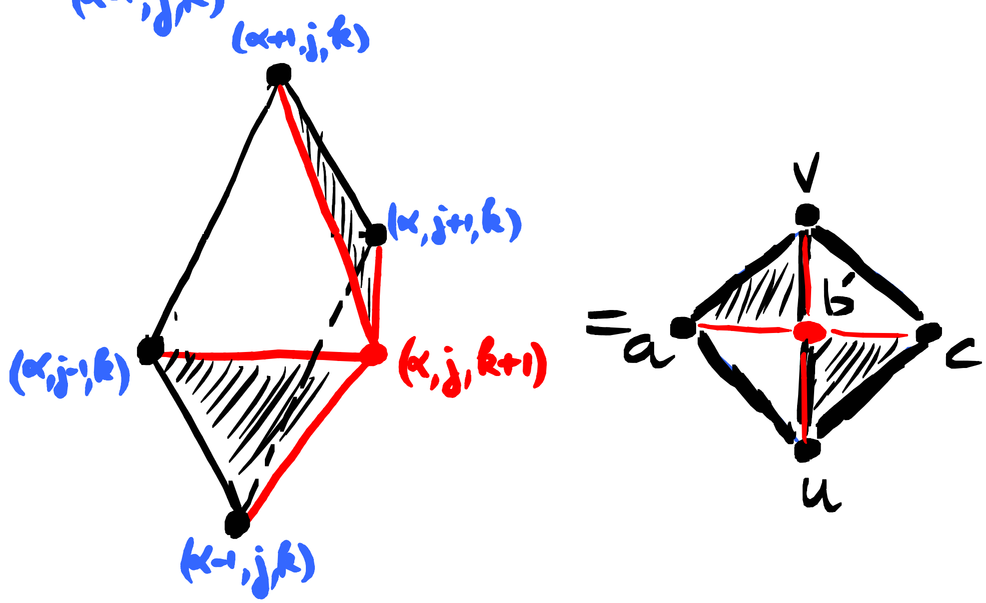
$$\rightarrow \quad D(u,a,b) = \begin{pmatrix} \frac{a}{b} & \frac{u}{b} \\ 0 & 1 \end{pmatrix}$$



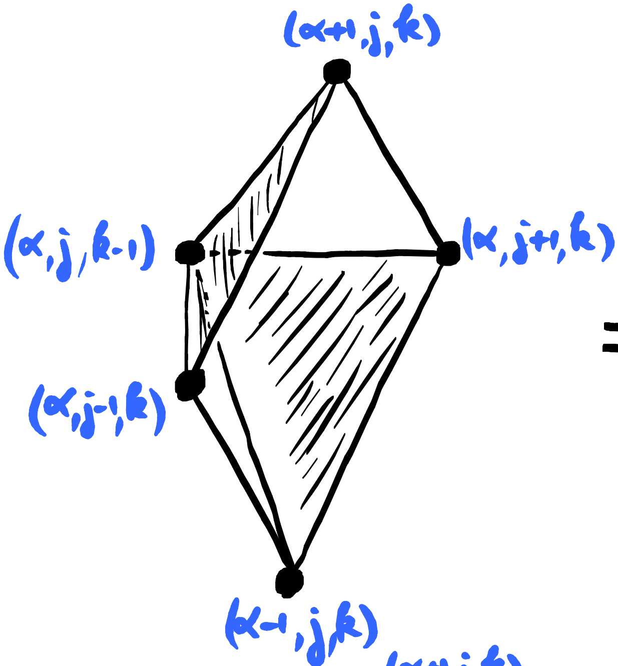
$$\rightarrow \quad U(a,b,v) = \begin{pmatrix} 1 & 0 \\ \frac{v}{b} & \frac{a}{b} \end{pmatrix}$$

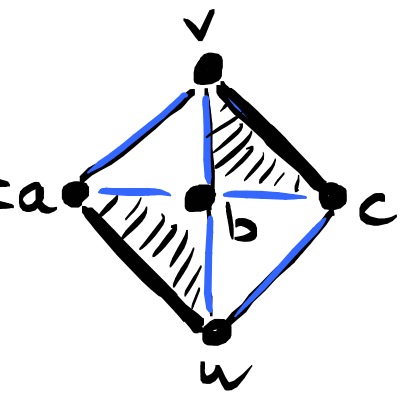


$$= D(u,a,b)\,U(b,c,v)$$



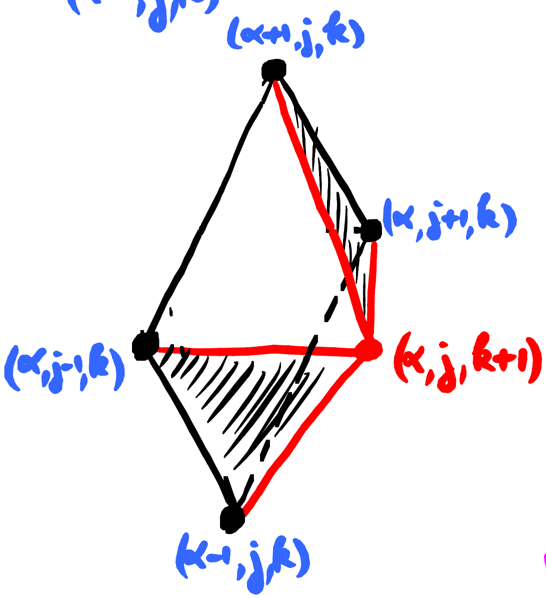
$$= U(a,b',v)\,D(u,b',c)$$

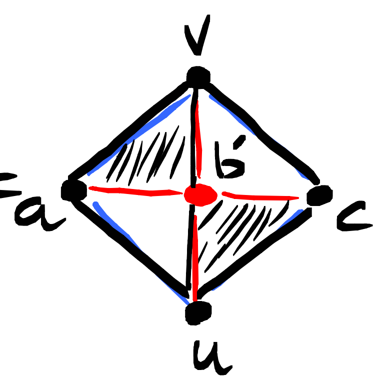


= 

$= D(u,a,b)\,U(b,c,v)$

$\|$ $\quad \Leftrightarrow \quad \boxed{bb' = ac + uv}$



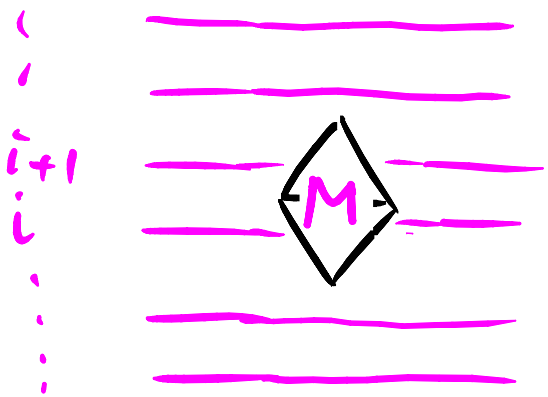
= 

$= U(a,b',v)\,D(u,b',c)$

"Flat connection"

(related to Yang-Baxter eq)

- Attach to the initial data stepped surface a product of D, U matrices:

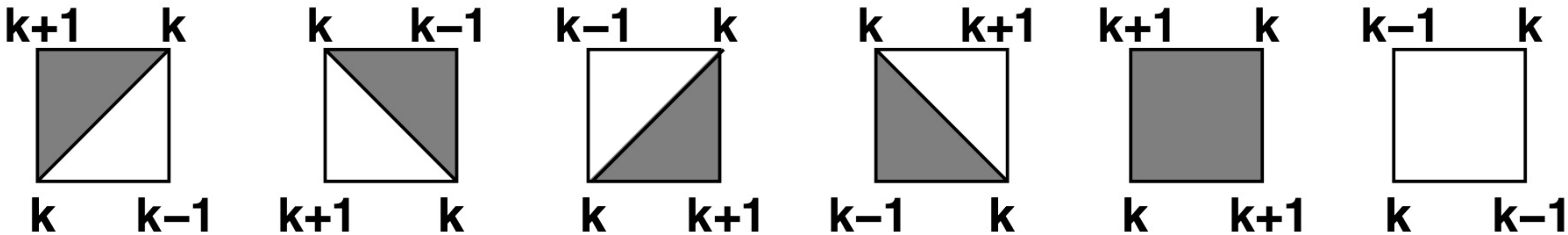
$M_i =$ 

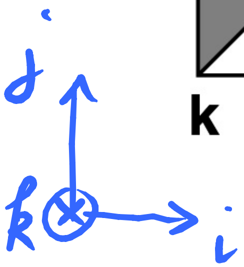
$$= \begin{array}{c|c} & \begin{array}{cc} i & i+1 \end{array} \\ \begin{array}{c} i \\ i+1 \end{array} & \begin{pmatrix} 1 & & & & & 0 \\ & 1 & & & & \\ & & 1 & & & \\ & & & \boxed{M} & & \\ & & & & 1 & \\ & 0 & & & & \ddots \\ & & & & & & 1 \end{pmatrix} \end{array}$$

- Product rule: $M_i \cdot P_j$ iff $\langle M \rangle$ to the left of $\langle P \rangle$

- well-defined for any initial data stepped surface

Rules:



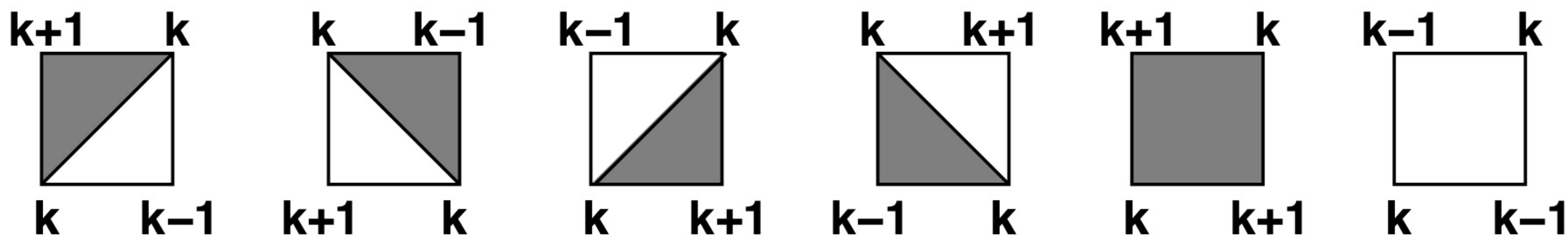


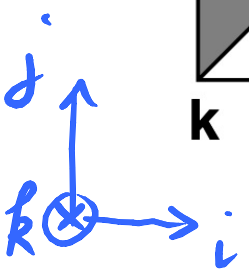
Tetrahedron ambiguity

top bottom

$$\boxed{\text{Stepped surface} = \{\text{vertices}\}\ \text{but Triangulation not unique!}}$$

Rules:

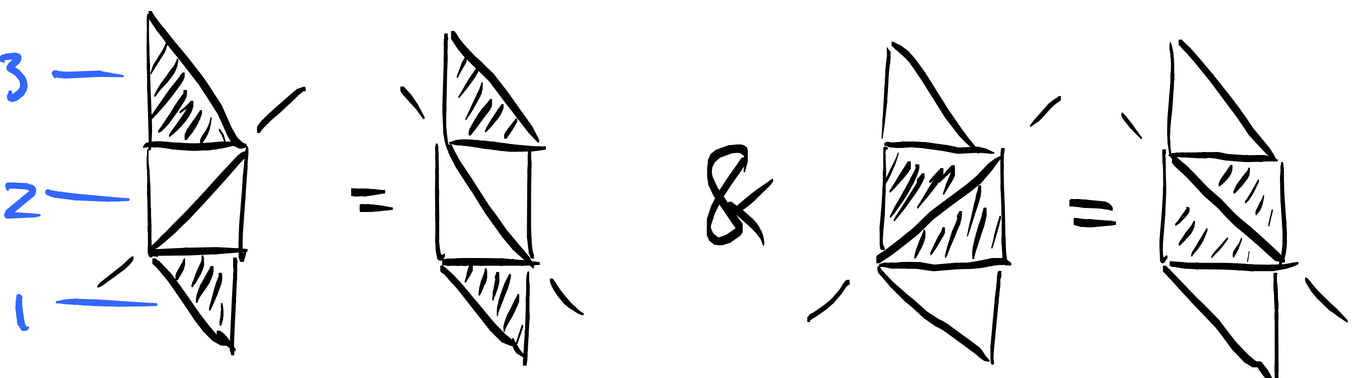




Tetrahedron ambiguity

top bottom

The matrix reps does not see this

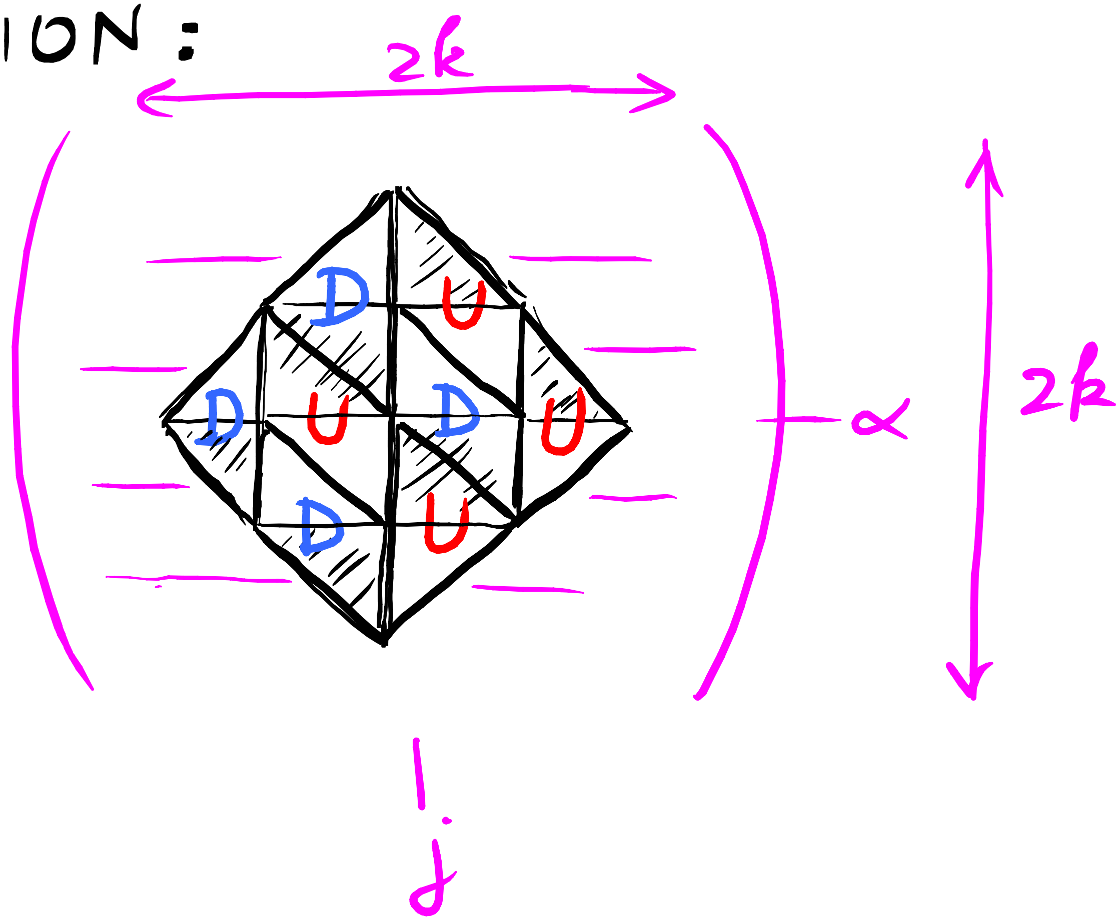


$$U_{23}V_{12} = V'_{12}U'_{23} \qquad V_{23}U_{12} = U'_{12}V'_{23}$$

$\Rightarrow$ $\boxed{\text{Matrix product depends only on surface not Triangulation}}$

SOLUTION:

Principal Minor $[1,k]\times[1,k]$

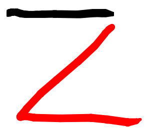
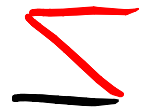


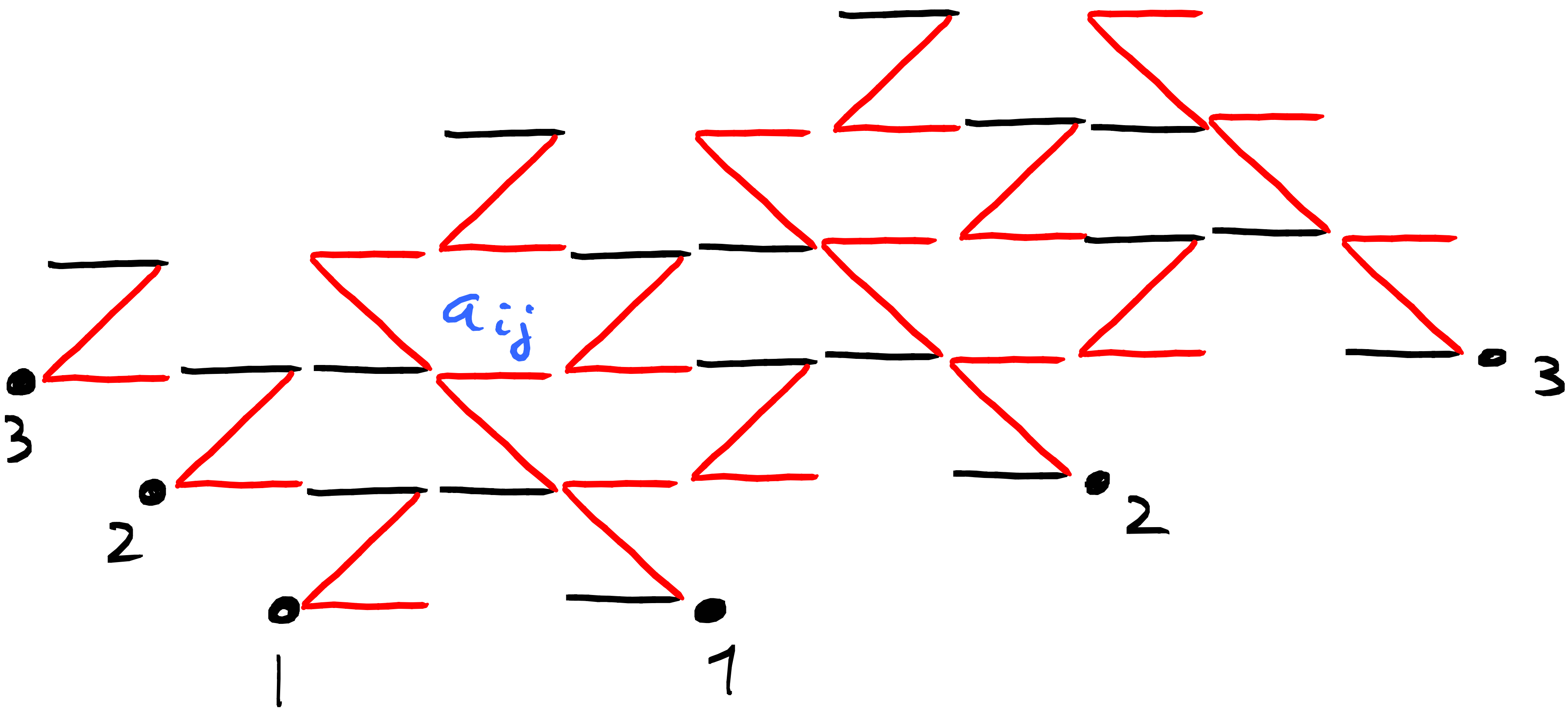
$\sim T_{\alpha,j,k+1}$

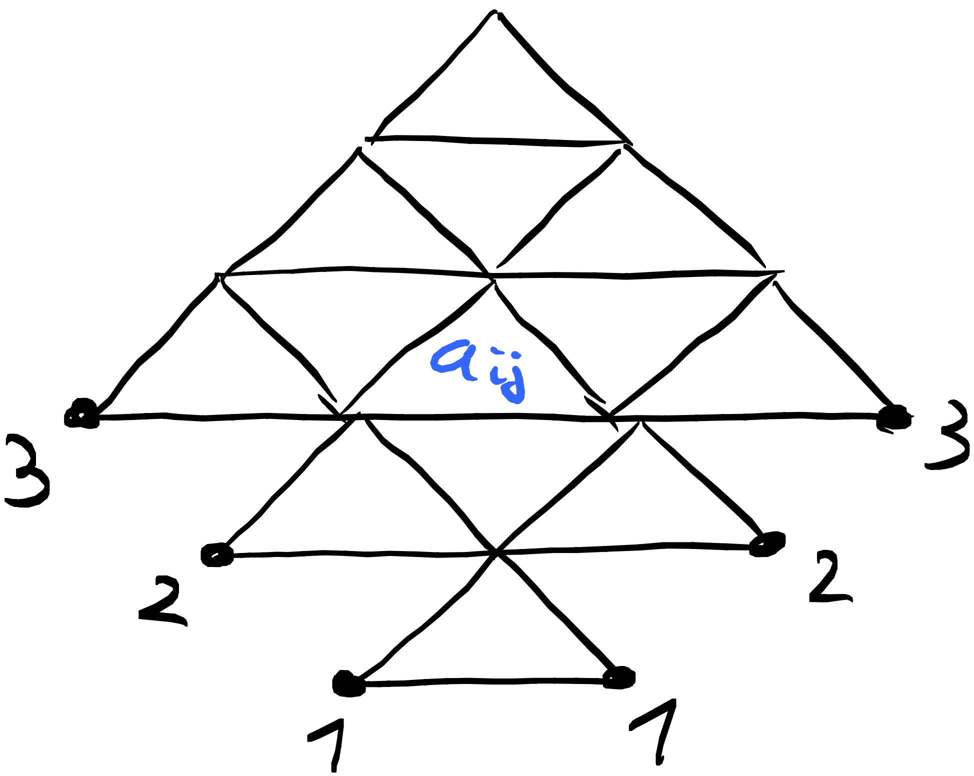
POSITIVITY:

entries of D, U are ≥ 0 monomials of initial data $\Rightarrow$ Laurent Positivity

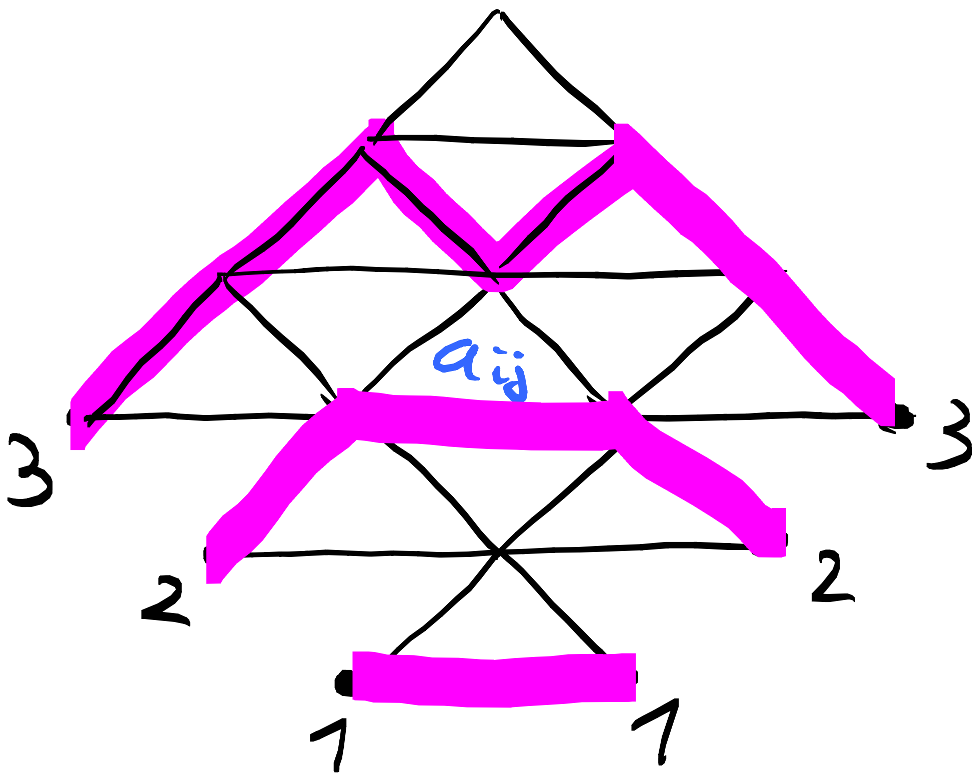
NETWORK FORMULATION (FLAT CASE)

$D =$  $U =$ 





By Gessel Viennot:
principal minor =
$\sum$ non-intersecting
paths $(1,2,..k) \to (1,2,..k)$

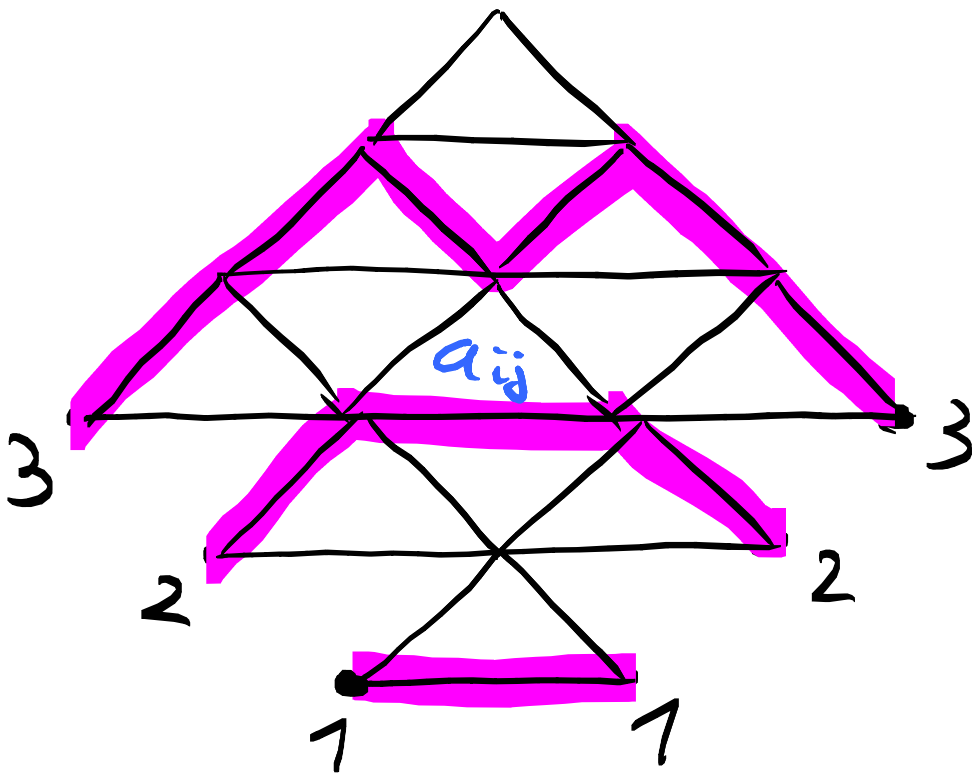


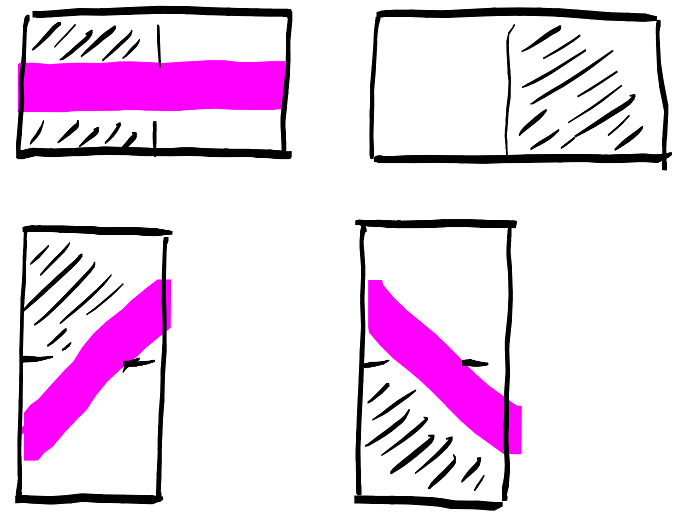
By Gessel Viennot:

principal minor $=$

$\sum$ non-intersecting paths $(1,2,..k) \to (1,2,..k)$

FROM NETWORK PATHS TO DOMINOS

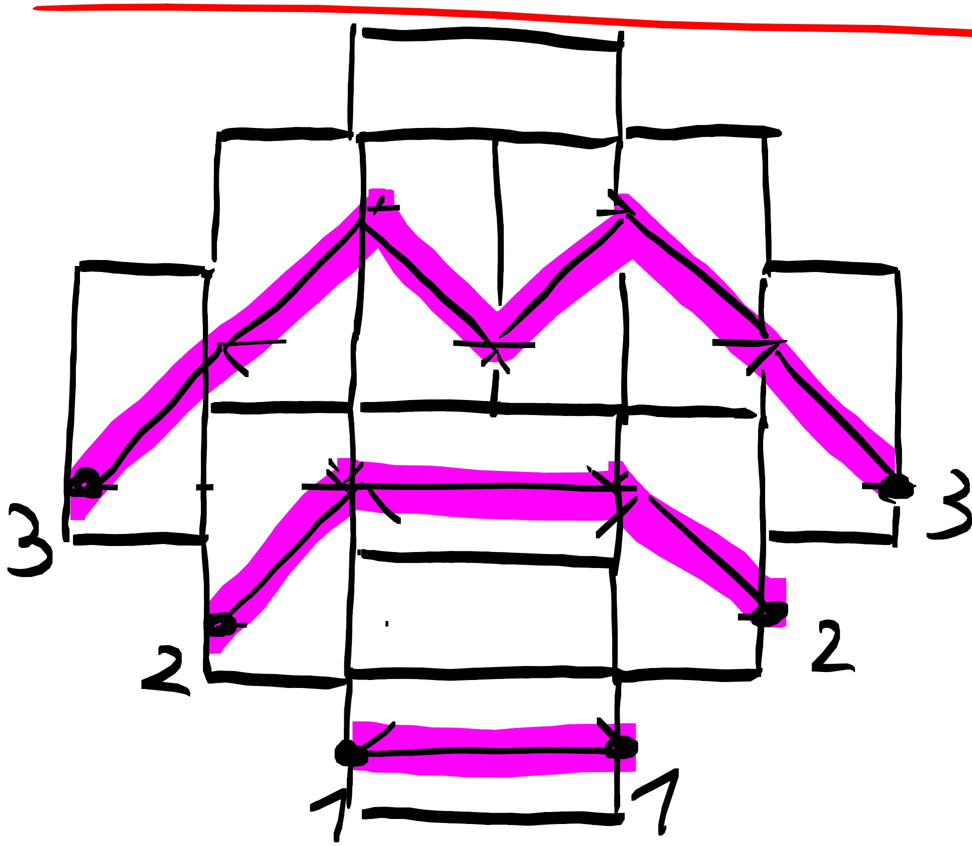


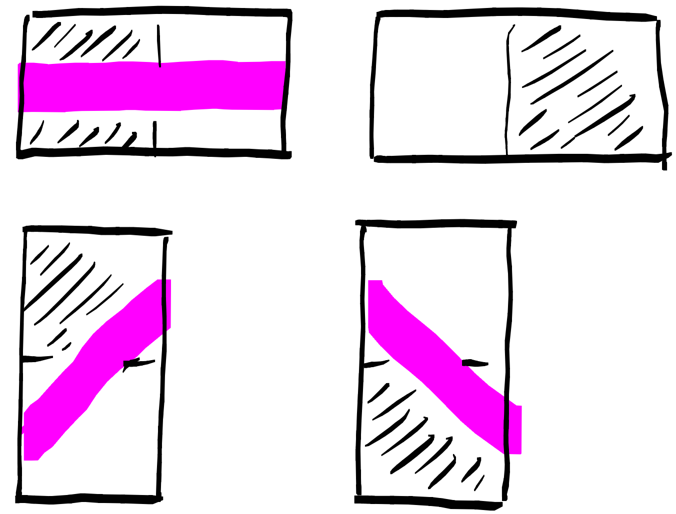


bijection

path – tiling

FROM NETWORK PATHS TO DOMINOS

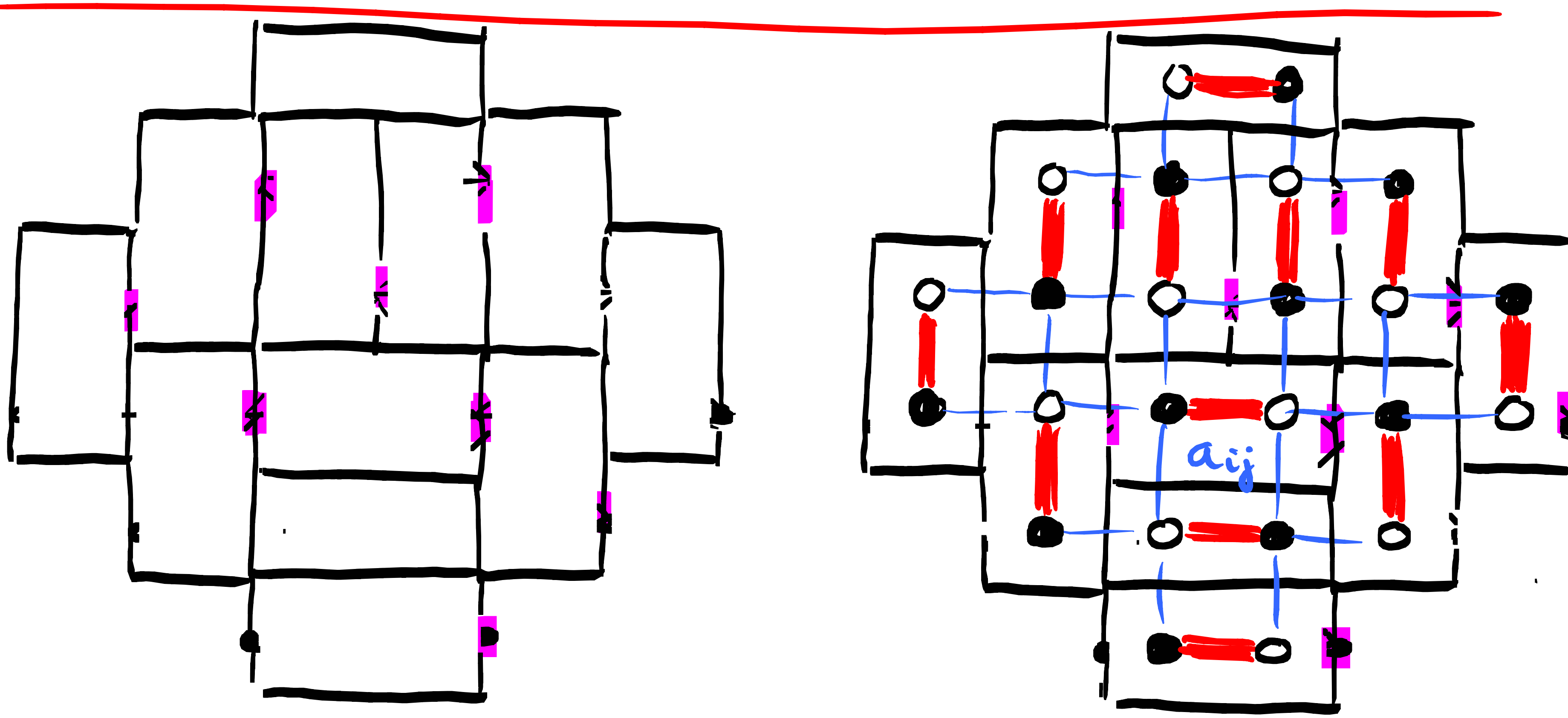




Domino Tilings of the $k \times k$ Aztec Diamond!

(+ weights)

FROM DOMINOS TO DIMERS



Dimer Coverings of the k×k Aztec Graph

$$\text{weight}\left(\boxed{a}\right) = a^{1-D}$$

D. ← (# dimers around the face)

SUMMARY:

T_{ijk} = partition function of dimer coverings of the $k \times k$ Aztec Graph with weights Laurent monomials in the initial data. $\left(= \prod a_{ij}^{1-D_{ij}}\right)$

[Speyer, DF Kedem]

6. ARCTIC CURVES

A. UNIFORM CASE

- Consider the solution with initial data $\boxed{\begin{array}{r} T_{ij0}=T_{ij1}=1 \\ T_{001}=x \end{array}}$
  (at $x=1$: $T_{ijk}=2^{k(k-1)/2}$)
- Define $\rho_{ijk}=\frac{\partial}{\partial x}\operatorname{Log} T_{ijk}\Big|_{x=1}$ $=\langle 1-D_{00}\rangle$ (susceptibility)
- Differentiate octahedron eqn : $\frac{\partial}{\partial x}\left(TT=TT+TT\right)\Big|_{x=1}$

  Then : $\boxed{2(\rho_{ijk+1}+\rho_{ijk-1})=\rho_{i+1jk}+\rho_{i-1jk}+\rho_{ij+1k}+\rho_{ij-1k}}$
- Define gen. function $\rho(x,y,z)=\sum_{ijk\geq 0}x^iy^jz^k\rho_{ijk}$

$$\boxed{\rho(x,y,z)=\frac{z}{1+z^2-\frac{1}{2}z\left(x+\frac{1}{x}+y+\frac{1}{y}\right)}}$$

- Singularities from the denominator: $\left\{ \begin{array}{l} x \to 1 - t x \\ y \to 1 - t y \\ z \to 1 + t(ux + vy) \\ t \to 0 \end{array} \right.$

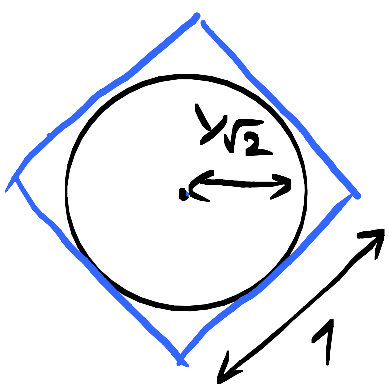
  probes $\rho_{uk,vk,k}$ as $k \to \infty$

- Series expansion in t:

$$1 + z^2 - \frac{z}{2}(x + x^{-1} + y + y^{-1}) \approx \frac{t^2}{2}\Big(\underbrace{4uvxy + (2u^2-1)x^2 + (2v^2-1)y^2}_{P(x,y)}\Big)$$

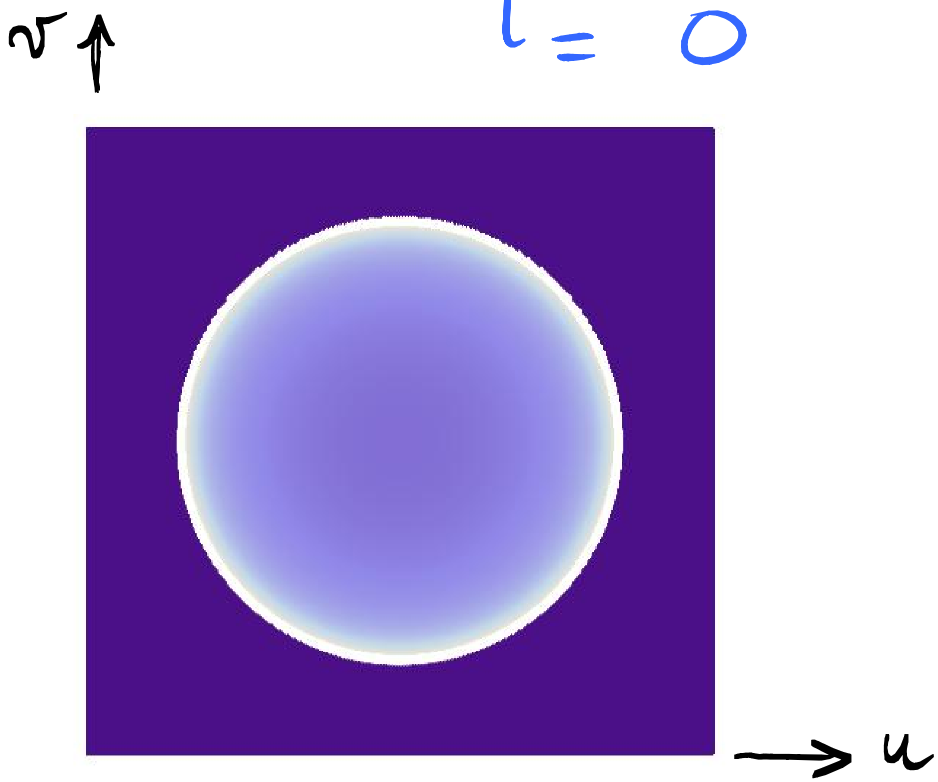
- Singularity locus: $P(x,y) = 0$ & $\frac{\partial P}{\partial x}(x,y) = 0$

$$\Longleftrightarrow \quad \boxed{2(u^2+v^2) - 1 = 0} \quad \text{ARCTIC CIRCLE}$$



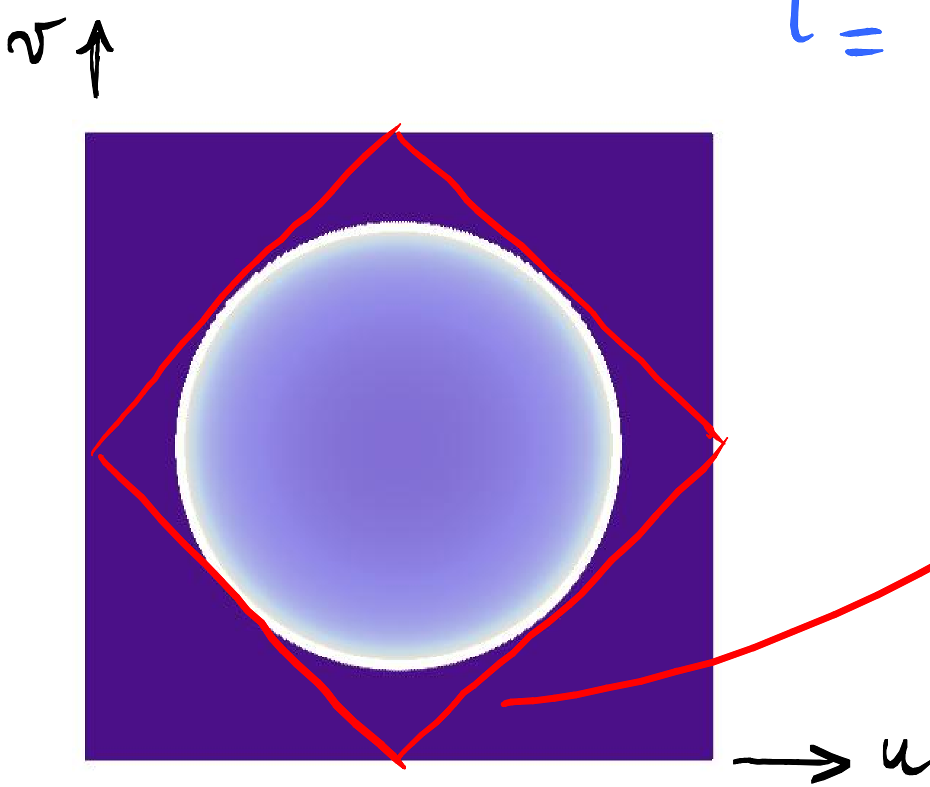
Behavior of ρ_{ijk} for $\frac{i}{k} \sim u$ $\frac{j}{k} \sim v$ $k \to \infty$

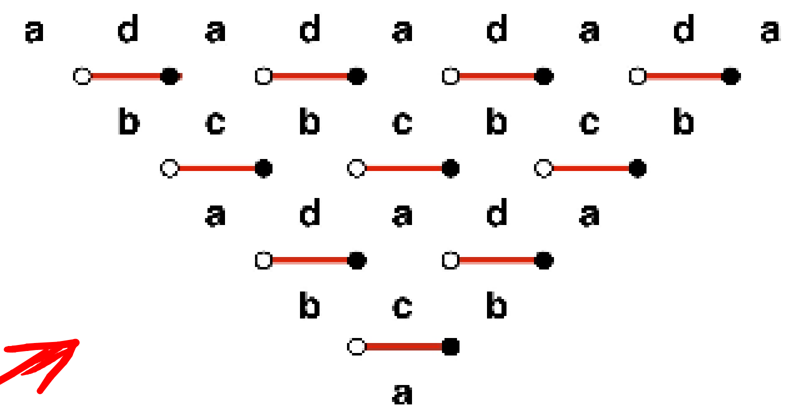
$$\rho(u,v) = \lim_{k\to\infty} k\,\rho_{i,j,k} \begin{cases} = \dfrac{2}{\pi} \dfrac{1}{\sqrt{1-2(u^2+v^2)}} & (u^2+v^2 < \frac{1}{2}) \\ = 0 & (\text{otherwise}) \end{cases}$$



Behavior of ρ_{ijk} for $\frac{i}{k} \sim u \quad \frac{j}{k} \sim v \quad k \to \infty$

$$\rho(u,v) = \lim_{k\to\infty} k\, \rho_{i,j,k} \begin{cases} = \dfrac{2}{\pi} \dfrac{1}{\sqrt{1-2(u^2+v^2)}} & (u^2+v^2<\tfrac{1}{2}) \\ = 0 & (\text{otherwise}) \end{cases}$$

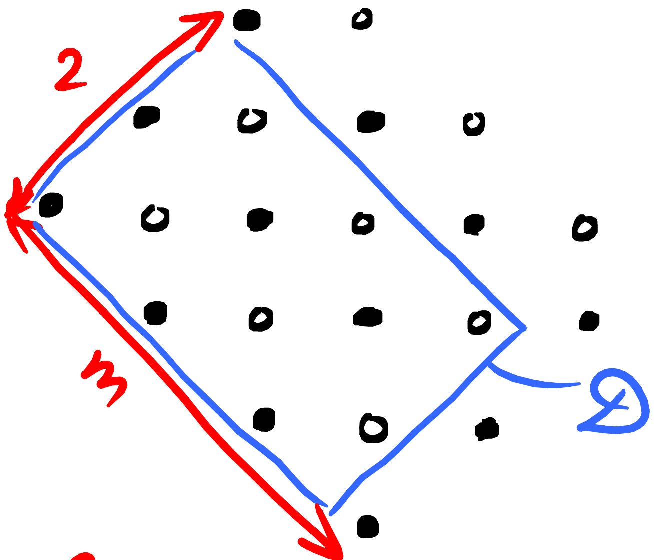




frozen phase (corners)

$\rho = \langle 1-D \rangle = 0$

B. Periodic initial data $2\times m$



$$\begin{cases} T_{i+2,j+2,k} = T_{ijk} \\ T_{i+m,j-m,k} = T_{ijk} \end{cases} \qquad k=0,1$$

The octahedron relation has an exact solution

- T_{ijk} = explicit monomial of $T_{ij0}, T_{ij1}, T_{ij2}, T_{ij3}$ within the fundamental domain $\mathcal{D}$
- Introduce $\rho_{ijk} = \frac{\partial}{\partial x} \log(T_{ijk}) \Big|_{x=1} \quad (T_{001}=x)$

Solution

$$\theta_{i,j,k} = T_{i+\lfloor \frac{k}{2} \rfloor, j+\lfloor \frac{k}{2} \rfloor, k \bmod 2}$$

$$x_i = \frac{c_i d_{i+1} + c_{i+1} d_i}{a_i b_i} \qquad \text{and} \qquad y_i = \frac{a_{i-1} b_i + a_i b_{i-1}}{c_i d_i} \qquad (i \in \mathbb{Z})$$

$$u_{n,i} = \prod_{\ell=0}^{n-1} (x_{i-\ell-1})^{\frac{n+1}{2} - \left| \frac{n-1}{2} - \ell \right|} \qquad v_{n,i} = \prod_{\ell=0}^{n-1} (y_{i-\ell-1})^{\frac{n+1}{2} - \left| \frac{n-1}{2} - \ell \right|}$$

$$\boxed{T_{i,j,k} = u_{k-1, \frac{i-j+k-1}{2}} \, v_{k-2, \frac{i-j+k-1}{2}} \, \theta_{i,j,k}}$$

$$\boxed{\begin{aligned} L_{i,j,k} &= \frac{T_{i+1,j,k} T_{i-1,j,k}}{T_{i,j,k+1} T_{i,j,k-1}} = \delta^{[4]}_{i+j+k,0} \left(\delta^{[2]}_{k,0} \frac{a_\alpha b_{\alpha-1}}{a_\alpha b_{\alpha-1} + a_{\alpha-1} b_\alpha} + \delta^{[2]}_{k,1} \frac{c_{\beta+1} d_\beta}{c_\beta d_{\beta+1} + c_{\beta+1} d_\beta} \right) \\ &\quad + \delta^{[4]}_{i+j+k,2} \left(\delta^{[2]}_{k,0} \frac{a_{\alpha-1} b_\alpha}{a_\alpha b_{\alpha-1} + a_{\alpha-1} b_\alpha} + \delta^{[2]}_{k,1} \frac{c_\beta d_{\beta+1}}{c_\beta d_{\beta+1} + c_{\beta+1} d_\beta} \right) \\ R_{i,j,k} &= \frac{T_{i,j+1,k} T_{i,j-1,k}}{T_{i,j,k+1} T_{i,j,k-1}} = 1 - L_{i,j,k} \end{aligned}}$$

• Differentiate octahedron eqn $\frac{\partial}{\partial x}\left(TT = TT + TT\right)\Big|_{x=1}$

$$\Rightarrow \quad \rho+\rho = \underbrace{\frac{TT}{TT}}_{L}(\rho+\rho) + \underbrace{\frac{TT}{TT}}_{R=1-L}(\rho+\rho)$$

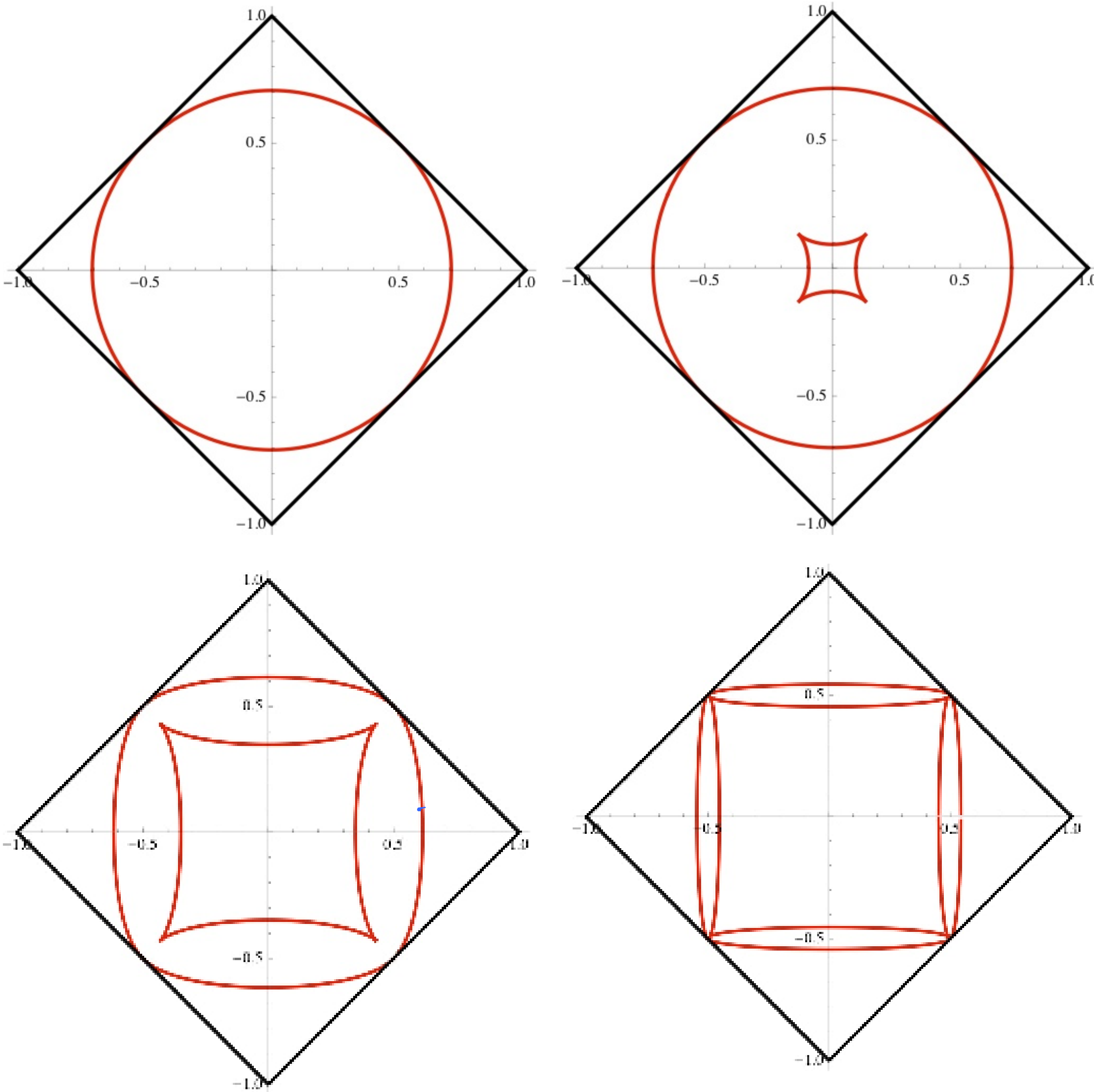
$\Rightarrow$ linear recursion for ρ_{ijk} w/ periodic coefficients

$$\boxed{L_{ijk} = L_{i+2,j+2,k} = L_{i+m,j-m,k} = L_{i+1,j+1,k+2}}$$

fundamental domain has $4m$ points

$\Rightarrow$ generating series $\rho(xyz)$ has for denominator the det of a $4m \times 4m$ matrix $\in \mathbb{Z}[x,x^{-1},y,y^{-1},z,z^{-1}]$

Arctic curve = singularity locus.



m = 2

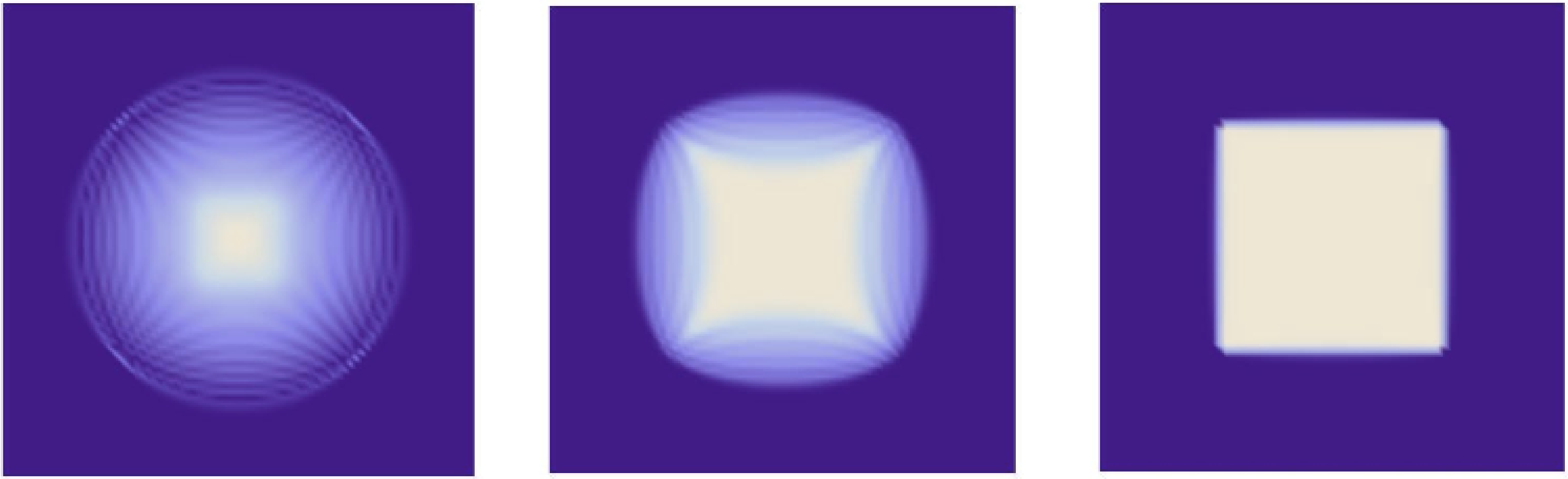
2×2 period case

1 parameter

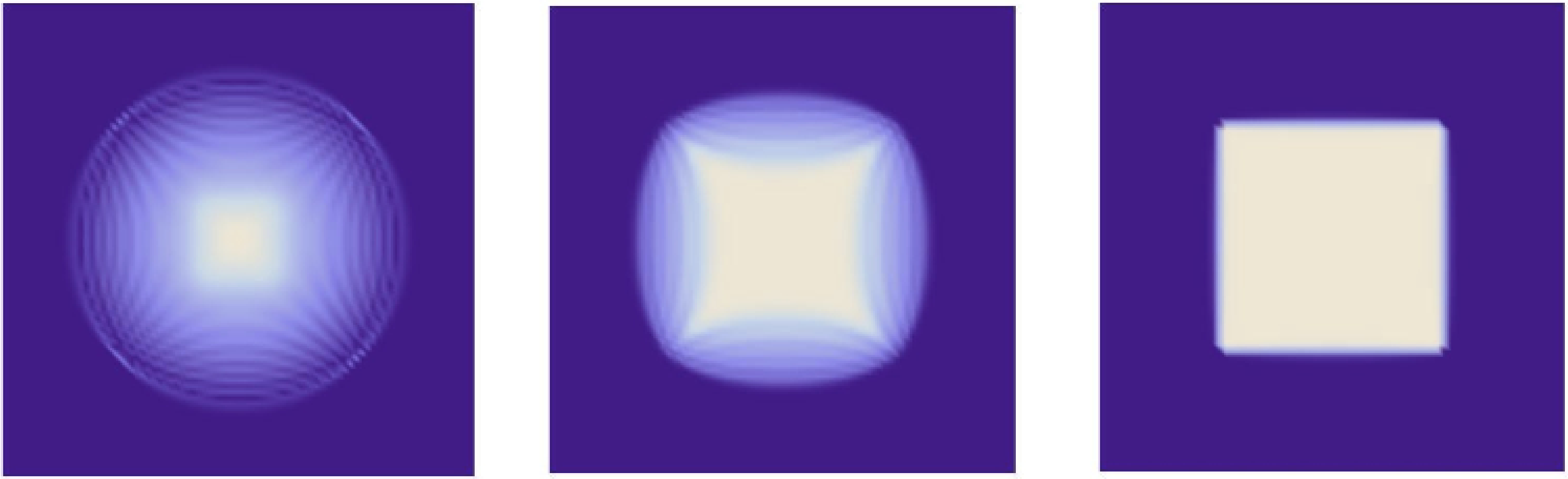
center phase = facet g=

corner = crystal g=

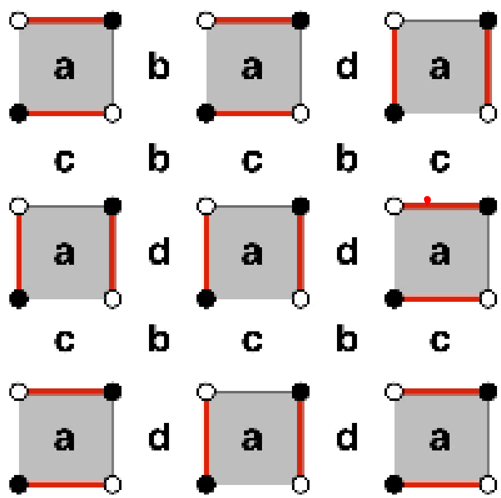
intermediate = disordered

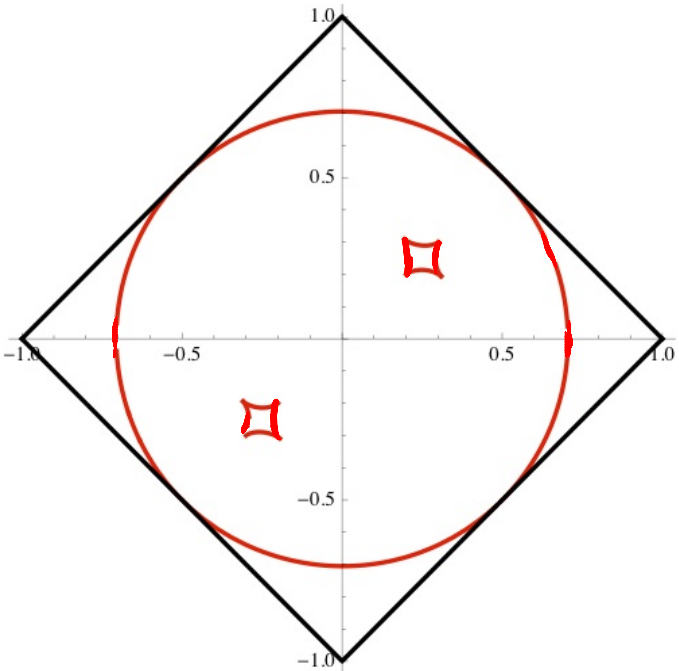


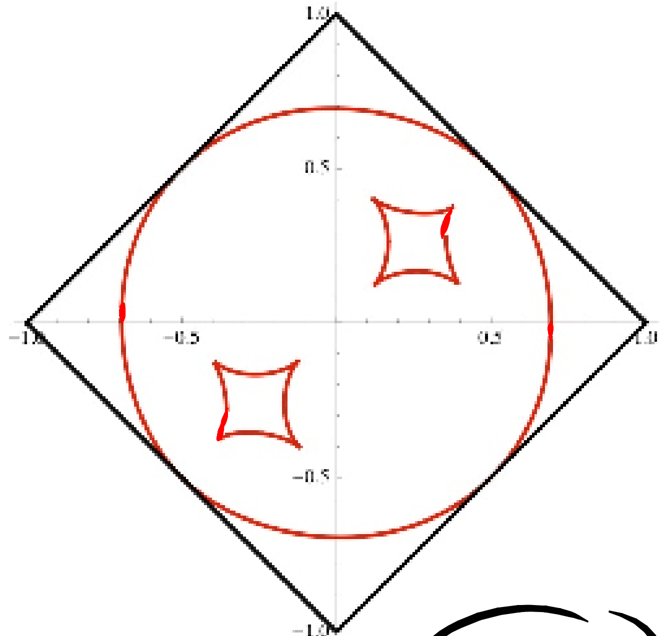
value of $g(u,v) = \lim_{k\to\infty} k\, g_{ijk}$: 3 phases $\begin{cases} \text{frozen (corners).} \\ \text{disordered} \\ \text{facet (center)} \end{cases}$

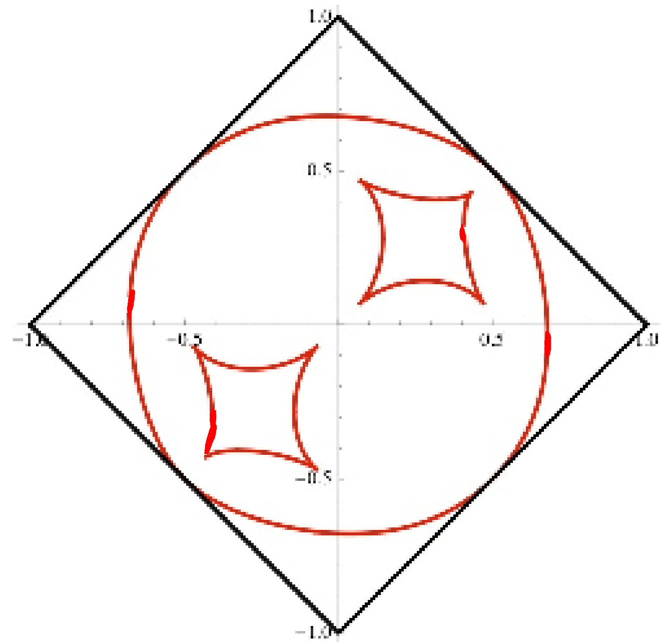


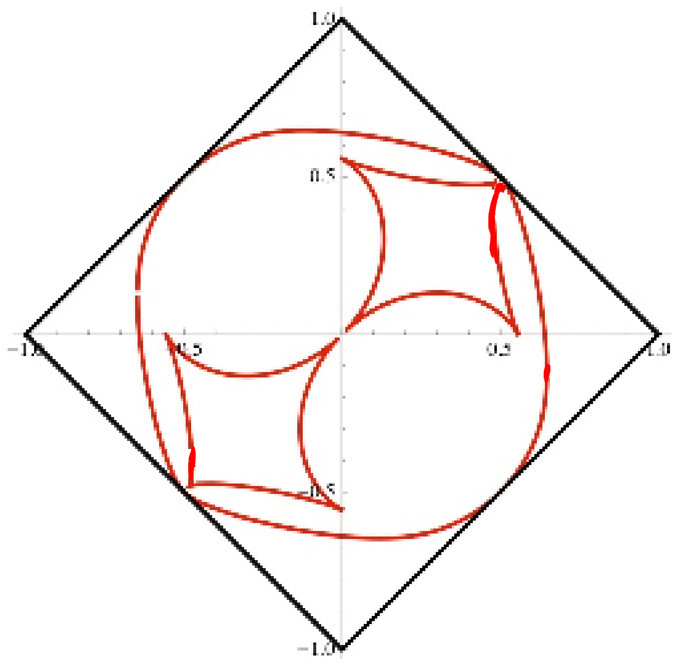
value of $g(u,v) = \lim_{k\to\infty} k\, g_{ijk}$: 3 phases $\begin{cases} \text{frozen (corners)} \\ \text{disordered} \\ \text{facet} \end{cases}$







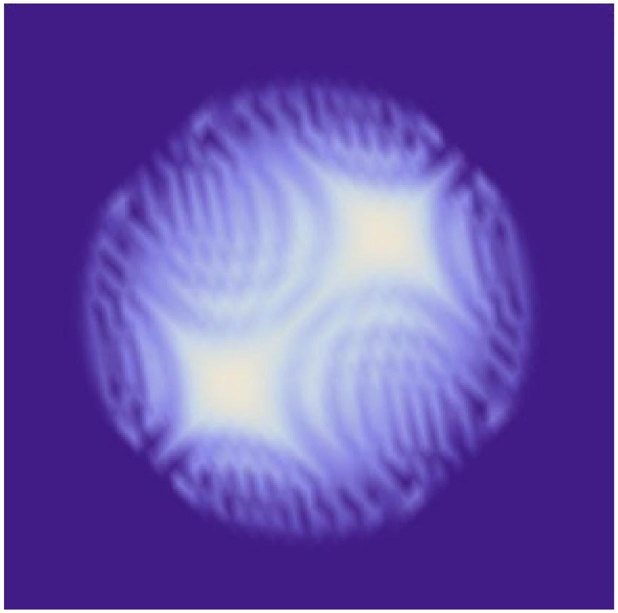


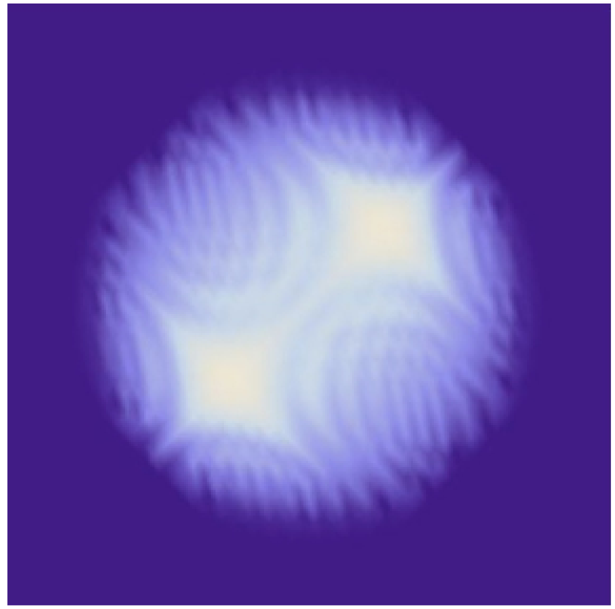


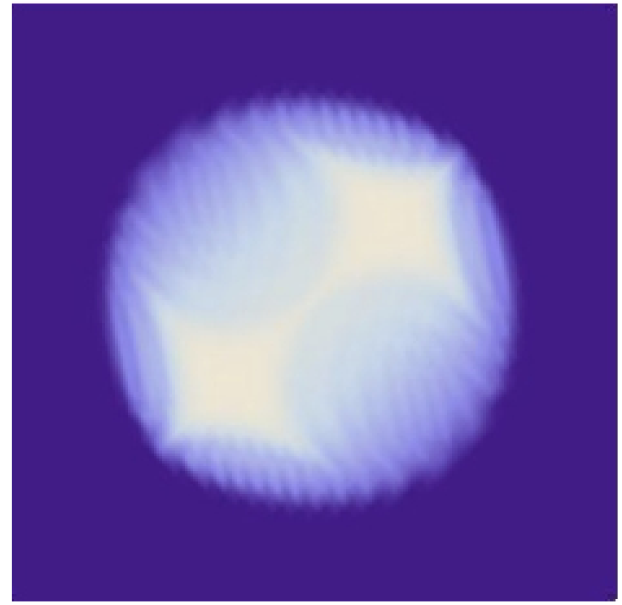
$m = 3$

Period 2×3

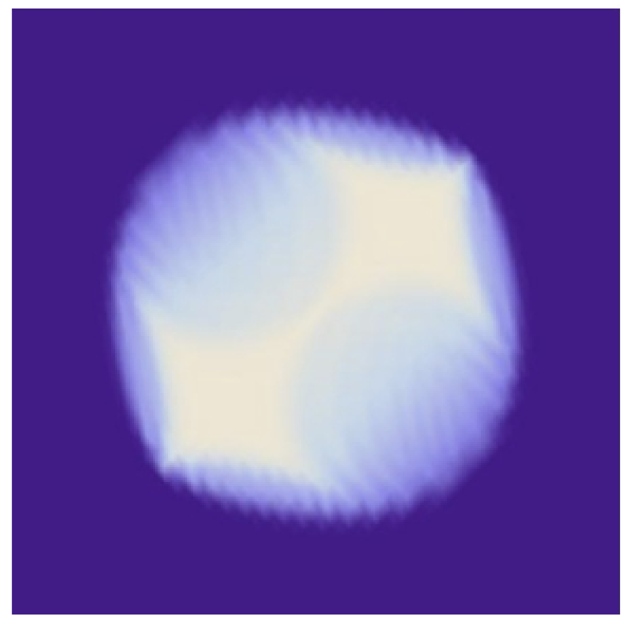
$\left\{ \begin{array}{ll} \text{2 facet phases} & g = 1 \\ \text{4 crystal phases} & g = 0 \\ \text{1 disordered phase} & \end{array} \right.$







value of $\rho(u,v) = \lim_{k \to \infty} k \rho_{ijk}$



CONCLUSION

Discrete Integrable Systems

Cluster Algebras

CONCLUSION

Discrete Integrable Systems

Initial Data

Cluster Algebras

CONCLUSION

Discrete Integrable Systems

Initial Data

Cluster Algebras

Laurent Positivity

- Gives a simple explanation for the positive Laurent phenomenon of CA

- Other clusters?

  ≡ other stepped surfaces / initial data

  ≡ Dimer part. fctns on other graphs.

PDF [math-ph/1307.0095]

- q-deformation: generalized λ-determinants
  Cluster Algebras with coefficients
- TILINGS / DIMER MODELS

→ easy derivations of arctic curves
(by differentiating the octahedron relation)

→ "Cluster Integrable" models

[Kenyon, Goncharov, Pemantle 12]

[PDF+R.Soto Garrido arXiv:1402.4493[math-ph]]

[PDF+Soto Garrido + Lapa in progress]

- Quantum version:

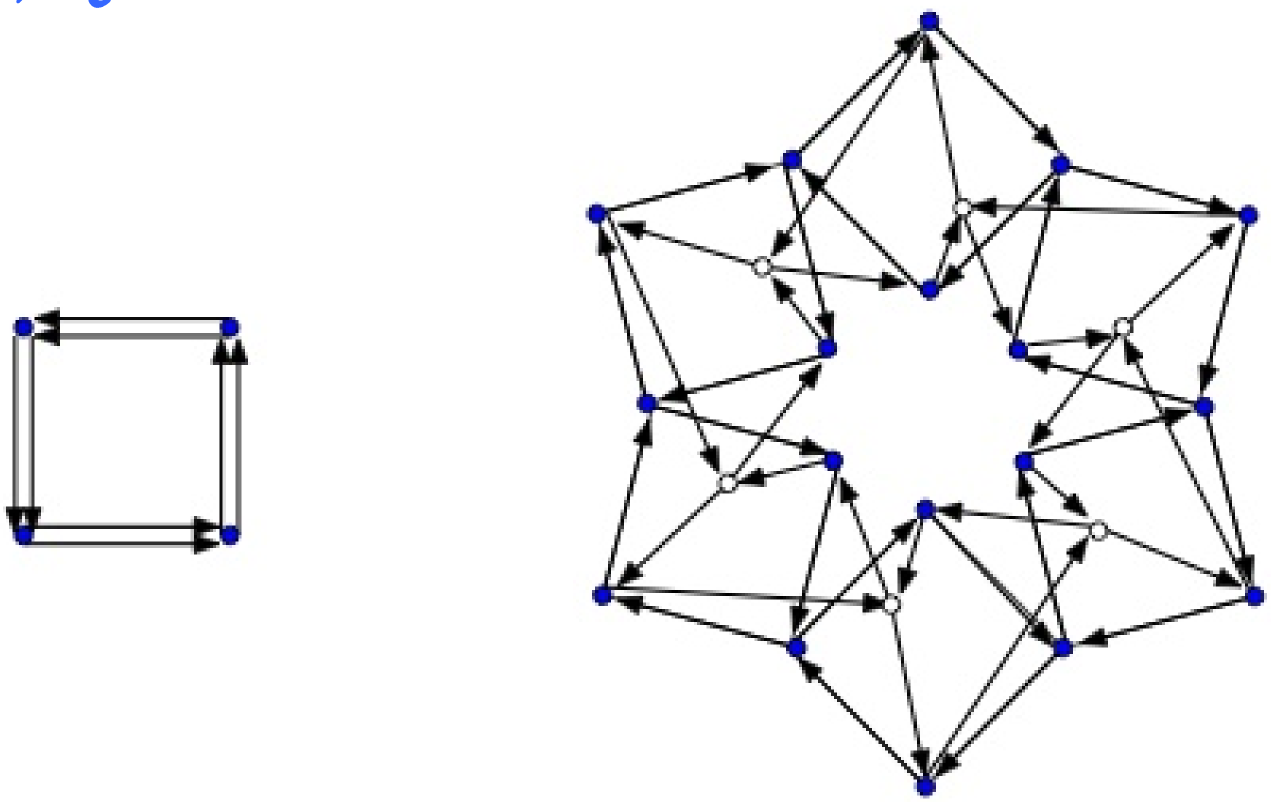
fusion products, CFT... [PDF+Kedem]

[PDF in progress]

- Non-Commutative [PDF 14]

- Folded Cluster algebra is y-finite

$$y_j = \prod_i x_i^{B_{ij}}$$



(a) $m=2$

(b) $m=6$