Graded tensor and q-Whittaker functions

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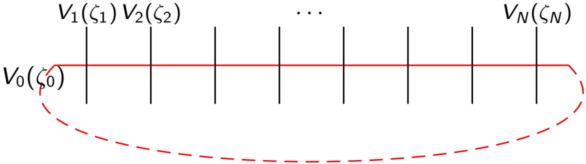
GGI 2014

Outline

The Hilbert space

Hilbert space of the generalized Heisenberg spin chain:

- Choose a set of representations of the Yangian $Y(\mathfrak{g})$ $\{V_i(\zeta_i)\}$;
- Auxiliary space representation V_0 and periodic boundary conditions:



The Hilbert space is $\mathcal{H} \underset{\mathfrak{g}-\mathrm{mod}}{\simeq} V_1 \otimes \cdots \otimes V_N$.

Example: XXX spin chain: $\mathfrak{g} = \mathfrak{sl}_2$, $V_i \simeq \mathbb{C}^2$.

Example: If $\mathfrak{g} = \mathfrak{sl}_n$ choose $V_i \simeq V(k_i\omega_{j_i})$ for each site i, where $k_i \in \mathbb{N}$ and ω_j is a fundamental weight.

Grading on the Hilbert space

- For combinatorial purposes, it is sufficient to consider $\mathfrak{g}[t]$ modules (for $\mathfrak{g} = \mathfrak{sl}_n$ they are evaluation modules isomorphic to V_i).
- The algebra $\mathfrak{g}[t] \simeq \mathfrak{g} \otimes \mathbb{C}[t]$ acts on $V(\zeta)$ and on $\mathcal{F} = V_1(\zeta_1) \otimes \cdots \otimes V_n(\zeta_N)$ by the coproduct;
- There is a filtration on $\mathcal{F}$ compatible with the grading of $U(\mathfrak{g}[t])$ (Feigin, Loktev).
- We call the associated graded space $\mathcal{F}^*$ and the generating function for graded components the **graded character** $ch_{q,\mathbf{z}}\mathcal{F}^*$ ($\mathbf{z} \leftrightarrow \mathfrak{h}$-grading; $q \leftrightarrow t - grading$).
- There are several known formulas for these characters, starting with work by Kirillov-Reshetikhin on completeness of Bethe ansatz solutions.

Examples

- **Example:** For the XXX spin chain, $|\mathfrak{F}| = 2^N$ and

$$ch_{q,\mathbf{z}}\mathfrak{F}^* = \sum_{\lambda:|\lambda|=N,\ell(\lambda)\leq 2} K_{\lambda,1^N}(q) S_\lambda(z_1, z_2)$$

with $z_1 z_2 = 1$.

-$S_\lambda(\mathbf{z})$ is the Schur function or character of irreducible rep of $\mathfrak{sl}_2$;
-$K_{\lambda,\mu}(q)$ is a Kostka polynomial.

- **Example:** For $\mathfrak{g} = \mathfrak{sl}_n$ and $V_i = V(\mu_i\omega_1)$ symmetric power reps of $\mathfrak{sl}_n$,

$$ch_{q,\mathbf{z}}\mathfrak{F}^* = \sum_{\lambda:|\lambda|=N,\ell(\lambda)\leq n} K_{\lambda,\mu}(q) S_\lambda(z_1, \cdots z_n)$$

with $z_1 \cdots z_n = 1$.

Physical interpretation of grading 1: CFT limit

- Recall: XXX spin chain has CFT limit (WZW). The (chiral) Hilbert space is a level-1 $\widehat{\mathfrak{sl}}_2$-module. Define $\mathcal{F}^*(N)$ to be the graded tensor product of N factors of $V(\omega_1)$.

$$\lim_{N\to\infty} \mathrm{ch}_{q,\mathbf{z}}\mathcal{F}^*(2N) = \mathrm{ch}\, V(\Lambda_0)$$

  the character of the affine vacuum module, level 1.

- For $\mathfrak{g} = \mathfrak{sl}_n$ choose $V_i = V(\omega_1)$, then

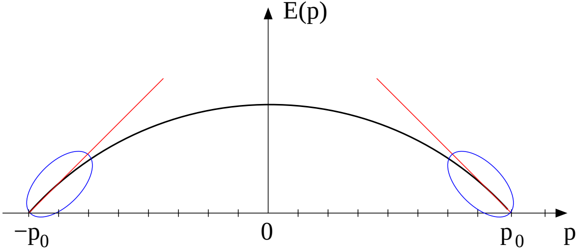
$$\lim_{N\to\infty} \mathrm{ch}_{q,\mathbf{z}}\mathcal{F}^*(nN) = \mathrm{ch}\, V(\Lambda_0)$$

  .

- For higher level, choose $V_i = V(k\omega_1)$.
- Remark: The finite product is an affine Demazure module in these cases.

Physical interpretation of grading 2: Lattice model

- In conformal limit, the partition function Z is dominated by order $1/N$ excitations: Massless quasi-particles, with linearized energy function, $E(P_i) \simeq v|(P_i - P_0)|$. ($P$=momentum and v=Fermi velocity).



- Periodic system: Momenta P_i are **quantized** in units of $\frac{2\pi}{N}$: $\Longrightarrow$ Dominant contribution to the chiral partition function is a series in $q = \exp(\frac{-2\pi v}{kNT})$.

Combinatorics of Bethe ansatz equations.

Our main character: The polynomial $\chi_{\mathbf{n}}(q,\mathbf{z})$

Our function of interest in this talk is the graded character of $\mathcal{F}^*$

$$\chi_{\mathbf{n}}(q,\mathbf{z}) = \mathrm{ch}_{q,\mathbf{z}}\mathcal{F}^*_{\mathbf{n}}$$

where $\mathcal{F}_{\mathbf{n}}$ is the tensor product of $n_i^{(\alpha)}$ modules with highest weight $i\omega_\alpha$:

$$\mathcal{F}_{\mathbf{n}} \simeq \bigotimes_{i\geq 1} \bigotimes_{\alpha=1,\dots,n-1} V(i\omega_\alpha)^{n_i^{(\alpha)}}.$$

and $\mathcal{F}^*_{\mathbf{n}}$ is the graded space associated to it.

- **Example:** If $n_i^\alpha = 0$ for all $i > 1$, we call $\chi_{\mathbf{n}}(q,\mathbf{z})$ a level-1 character.

An unreasonably nice expression for the graded character

Theorem: (DFK11) The graded character $\chi_{\mathbf{n}}(q, \mathbf{z})$ can expressed as a **constant term identity** in terms of solutions of the quantum Q-system.

Next, we explain:

- The Q-system;
- Its discrete integrable structure;
- The natural quantization;
- The constant term identity.

The Q-system for $\mathfrak{g} = \mathfrak{sl}_n$

Schur polynomials corresponding to rectangular Young tableaux

$$Q_k^{(\alpha)} := S_{(k)^\alpha}(\mathbf{z}), \quad z_1 \cdots z_n = 1$$

satisfy a **Discrete dynamical system** in "time" variable k:

$$Q_{k+1}^{(\alpha)} Q_{k-1}^{(\alpha)} = (Q_k^{(\alpha)})^2 - Q_k^{(\alpha+1)} Q_k^{(\alpha-1)}, \quad \alpha \in \{1, ..., n-1\}, k \geq 1$$

with **Boundary conditions:**

$$Q_k^{(0)} = Q_k^{(n)} = 1 \quad \text{for all } k \in \mathbb{Z}$$

and **Initial data:**

$$\{Q_0^{(\alpha)} = 1,\ Q_1^{(\alpha)} = e_\alpha(\mathbf{z}), \alpha \in [1, n-1]\}.$$

(Proof: Snake lemma from cluster algebras.)

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with **Boundary conditions:**

$$Q_k^{(0)} = Q_k^{(n)} = 1 \quad \text{for all } k \in \mathbb{Z}$$

Consider the same equation with generic initial data; then
Lemma: $Q_k^{(\alpha)}$ ($k \in \mathbb{Z}$) is a Laurent polynomial of $\{Q_0^{(\alpha)}, Q_1^{(\alpha)}\}$.
(Proof: Snake lemma from cluster algebras.)

Discrete integrability

Discrete Wronskian matrix:

$$W_k^{(\alpha)} = \begin{pmatrix} Q_k & Q_{k+1} & \cdots & Q_{k+\alpha-1} \\ Q_{k-1} & Q_k & \cdots & Q_{k+\alpha-2} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k-\alpha+1} & Q_{k-\alpha+2} & \cdots & Q_k \end{pmatrix}_{\alpha\times\alpha}, \quad Q_k := Q_k^{(1)}.$$

Then $Q_k^{(\alpha)} = \mathrm{Det}\ W_k^{(\alpha)}$ and the Q-system is the Desnanot-Jacobi relation for $W = W^{(\alpha)}$:

$$|W||W_{1,\alpha}^{1,\alpha}| = |W_1^1||W_\alpha^\alpha| - |W_1^\alpha||W_\alpha^1|,$$

with $W^{(0)} = 1$ and additional condition $|W^{(n)}| = 1$.

Discrete integrability

Corollary
$\{Q_k^{(1)}, k \in \mathbb{Z}\}$ *satisfy linear recursion relations*

$$Q_k - C_1 Q_{k+1} + C_2 Q_{k+2} \cdots \pm C_{n-1} Q_{k+n-1} \mp Q_{k+n} = 0$$

Proof.
The boundary condition $Q_k^{(n)} = 1$ implies $Q_k^{(n+1)} = 0$. Expand

$$0 = Q_k^{(n+1)} = \mathrm{Det} \begin{pmatrix} Q_k & Q_{k+1} & \cdots & Q_{k+n} \\ Q_{k-1} & Q_k & \cdots & Q_{k+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k-n} & Q_{k-n+1} & \cdots & Q_k \end{pmatrix}_{(n+1)\times(n+1)}$$

along any row or column. $\square$

Constants of the motion

Lemma
The coefficients of the linear recursion relation for $Q_k^{(1)}$ are independent of k.

Proof.
Subtract $Q_{k+1}^{(n)} - Q_k^{(n)} = 1 - 1 = 0$:

$$0 = \left|\begin{pmatrix} Q_{k+1} & Q_{k+2} & \cdots & Q_{k+n} \\ Q_k & Q_{k+1} & \cdots & Q_{k+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k-n+2} & Q_{k-n+1} & \cdots & Q_{k+1} \end{pmatrix}\right| - \left|\begin{pmatrix} Q_k & Q_{k+1} & \cdots & Q_{k+n-1} \\ Q_{k-1} & Q_k & \cdots & Q_{k+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k-n+1} & Q_{k-n+2} & \cdots & Q_k \end{pmatrix}\right|$$

$$0 = \left|\begin{pmatrix} Q_{k+n} - (-1)^n Q_k & Q_{k+1} & \cdots & Q_{k+n-1} \\ Q_{k+n-1} - (-1)^n Q_{k-1} & Q_k & \cdots & Q_{k+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ Q_{k+1} - (-1)^n Q_{k-n+1} & Q_{k-n-2} & \cdots & Q_k \end{pmatrix}\right|$$

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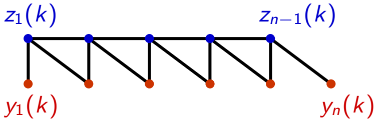
The columns are linearly dependent, whereas the rows are an index shift. The coefficients of the linear equation are the constants of the motion. □

Combinatorial formula for the constants of motion

The integrals of motion are C_i have a combinatorial description.

Theorem (DFK10)

C_j = partition function of j hard particles on the weighted graph



$$\textit{Weights}\quad y_\alpha(k) = \frac{Q_{k+1}^{(\alpha)} Q_k^{(\alpha-1)}}{Q_{k+1}^{(\alpha-1)} Q_k^{(\alpha)}}, \quad z_\alpha(k) = -\frac{Q_{k+1}^{(\alpha+1)} Q_k^{(\alpha-1)}}{Q_{k+1}^{(\alpha)} Q_k^{(\alpha)}}.$$

Quantization of the Q-system

Consider the non-commuting, invertible elements $\{\mathcal{Q}_k^{(\alpha)}\}$, satisfying the quantum Q-system (evolution)

$$q^{C^{-1}_{\alpha\alpha}}\mathcal{Q}_{k+1}^{(\alpha)}\mathcal{Q}_{k-1}^{(\alpha)} = (\mathcal{Q}_k^{(\alpha)})^2 - \mathcal{Q}_k^{(\alpha+1)}\mathcal{Q}_n^{(\alpha-1)},$$

with commutation relations

$$\mathcal{Q}_k^{(\alpha)}\mathcal{Q}_{k+1}^{(\beta)} = q^{C^{-1}_{\alpha,\beta}}\mathcal{Q}_{k+1}^{(\beta)}\mathcal{Q}_k^{(\alpha)}, \qquad C = \text{the Cartan matrix of } \mathfrak{sl}_n.$$

Theorem

1. *The commutation relations are compatible with the evolution (independent of k).*
2. *$\mathcal{Q}_k^{(\alpha)}$ is a Laurent polynomial in the initial data $\{\mathcal{Q}_i^{(\alpha)}\}_{i=0,1}$ over $\mathbb{Z}_t$ ($t = q^{1/n}$).*

Proof: The quantum Q-system is a mutation in a quantum cluster algebra.

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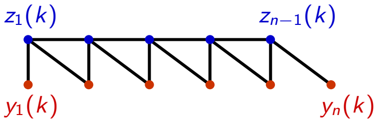
Discrete integrability of the quantum Q-system

The quantum Q-system has $n-1$ integrals of motion in involution:

Theorem (DFK10)

The partition functions $\mathcal{C}_j$ of j hard particles on the graph



with weights

$$y_\alpha(k) = \mathcal{Q}_{k+1}^{(\alpha)}(\mathcal{Q}_{k+1}^{(\alpha-1)})^{-1}(\mathcal{Q}_{k}^{(\alpha)})^{-1}\mathcal{Q}_{k}^{(\alpha-1)},$$

$$z_\alpha(k) = -\mathcal{Q}_{k+1}^{(\alpha+1)}(\mathcal{Q}_{k+1}^{(\alpha)})^{-1}(\mathcal{Q}_{k}^{(\alpha)})^{-1}\mathcal{Q}_{k}^{(\alpha-1)}.$$

$\mathcal{C}_j[k]$ *independent of k, for $j = 1, \ldots, n-1$, commute with each other.*

Note: [DF11] The commutation relations between the weights are encoded by the graph edges.

Example of $\mathfrak{sl}_2$:

The quantum Q-system is

$$t\mathcal{Q}_{n+1}\mathcal{Q}_{n-1} = \mathcal{Q}_n^2 - 1$$

with $t = q^{1/2}$, commutation relations given by

$$\mathcal{Q}_k\mathcal{Q}_{k+1} = t\mathcal{Q}_{k+1}\mathcal{Q}_k$$

and one integral of motion

$$\mathcal{C}_1 = \mathcal{C} = \mathcal{Q}_1\mathcal{Q}_0^{-1} - \mathcal{Q}_1^{-1}\mathcal{Q}_0^{-1} + \mathcal{Q}_1^{-1}\mathcal{Q}_0$$

$$= \mathcal{Q}_{k+1}\mathcal{Q}_k^{-1} - \mathcal{Q}_{k+1}^{-1}\mathcal{Q}_k^{-1} + \mathcal{Q}_{k+1}^{-1}\mathcal{Q}_k$$

The series τ

Lemma: For each α, considered as a function of initial data $\{\mathcal{Q}_i^{(\alpha)} : i = 0, 1\}$, the limit

$$\xi_\alpha = q^{C^{-1}_{\alpha,\alpha}/2} \lim_{k\to\infty} \mathcal{Q}_k^{(\alpha)} (\mathcal{Q}_{k+1}^{(\alpha)})^{-1}$$

exists, and can be expanded as a power series in $\{(\mathcal{Q}_1^{(\alpha)})^{-1}\}$ with **no constant term.**

Define the "tail" function $\tau(q, \mathbf{z})$:

$$\tau(\mathbf{z}) = \sum_{\lambda\in P^+} \prod_{\alpha=1}^{n-1} (\xi_\alpha)^{\ell_\alpha+1} S_\lambda(\mathbf{z}), \quad \lambda := \sum_\alpha \ell_\alpha \omega_\alpha.$$

Then

$$\tau(\mathbf{z}) \in \left(\prod_\alpha (\mathcal{Q}_1^{(\alpha)})^{-1}\right) \mathbb{Z}_t[\{(\mathcal{Q}_0^{(\alpha)})^{\pm 1}\}][\{(\mathcal{Q}_1^{(\alpha)})^{-1}\}].$$

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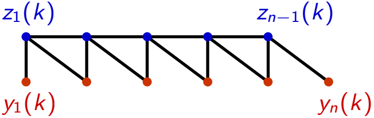
Action of conserved quantities on $\tau(\mathbf{z})$

The conserved quantities are partition functions on the graph



are independent of k.

Lemma: In the limit $k \to \infty$, $z_\alpha(k) \to 0$, $y_\alpha(k) \to y_\alpha := t^{(n-1)/2}\xi_{\alpha-1}\xi_\alpha^{-1}$. The $y_\alpha(k)$ commute, and $\mathcal{C}_i(\mathbf{y}) = e_i(\mathbf{y})$, the elementary symmetric functions.

Corollary: The conserved quantities $\mathcal{C}_i$ acting on $\tau(\mathbf{z})$ give:

$$\mathcal{C}_i\tau(\mathbf{z}) = e_i(\mathbf{z})\tau(\mathbf{z}) + \text{ lower terms}$$

where "lower terms" means terms independent of ξ_α for some α (will evaluate to 0 in next slide).

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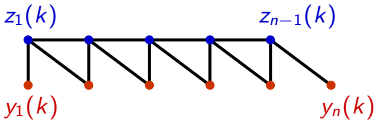

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The evaluation ϕ

For any polynomial $f(\{\mathcal{Q}_i^{(\alpha)}\})$, define the evaluation $\phi : \mathbb{Z}_t[\mathcal{Q}_k^{(\alpha)}, \mathbf{z}] \to \mathbb{Z}_t[\mathbf{z}]$ as follows:

1. Multiply from the left and the right
$$f \mapsto \left(\prod_{\alpha} \mathcal{Q}_1^{(\alpha)}\right) \ f \ \tau(\mathbf{z}).$$
2. Express as a **function of the initial** data $\{\mathcal{Q}_0^{(\alpha)}, \mathcal{Q}_1^{(\alpha)}\}$.
3. **Normal order** this expression: Move (by q-commuting) all $\mathcal{Q}_0^{(\alpha)}$s to the left of all the $\mathcal{Q}_1^{(\alpha)}$s.
4. **Evaluate** the result at $\mathcal{Q}_0^{(\alpha)} = 1$ for all α.
5. Extract the **constant term** in $\mathcal{Q}_1^{(\alpha)}$ in the resulting expression. Call the result $\phi(f)$.

Example: $\phi(\mathcal{C}_i) \sim e_i(\mathbf{z})$.

Constant term identity: $\mathfrak{sl}_2$

Theorem
(Up to an overall power of q) the character of the graded tensor product

$$\mathcal{F}^*_{\mathbf{n}} = V(\omega_1)^{*n_1} * \cdots * V(k\omega_1)^{*n_k}$$

is

$$\overline{\chi}_{\mathbf{n}}(q;z) = \phi\left(\overrightarrow{\prod_{1\leq j\leq k}} \mathcal{Q}_j^{n_j}\right)$$

Proof: Induction on the explicit character formula involving q-binomial coefficients (fermionic formula), using the quantum Q-system. (See DFK-fusion `arXiv:1109.6261`).

Constant term representation of characters: $\mathfrak{g} = \mathfrak{sl}_n$

Theorem

The normalized character of the graded tensor product of representations

$$\mathcal{F}_{\mathbf{n}}^* = V(\omega_1)^{*n_1^{(1)}} * \cdots * V(j\omega_\alpha)^{*n_j^{(\alpha)}} * \cdots * V(k\omega_{n-1})^{n_k^{(n-1)}}$$

is

$$\overline{\chi}_{\mathbf{n}}(q;\mathbf{z}) = \phi\left(\prod_{i=k}^{1}\prod_{\alpha=1}^{N-1}(\mathcal{Q}_i^{(\alpha)})^{n_i^{(\alpha)}}\right).$$

Right action on $\phi(f)$

Let g, f be Laurent polynomials in the initial data $\{\mathcal{Q}_i^{(\alpha)} : i = 0, 1\}$. Define the action of g on $\phi(f)$ as:

$$g \circ \phi(f) = \phi(fg).$$

Theorem: The conserved quantities of the quantum Q-system act on $\phi(f)$ as multiplication by the fundamental characters of $\mathfrak{sl}_n$:

$$\mathcal{C}_j(k) \circ \phi(f) = \phi(f\ \mathcal{C}_j(k)) = \phi(f\ \mathcal{C}_j(\infty)) = e_i(\mathbf{z})\phi(f).$$

Proof: We showed

$$\mathcal{C}_i \tau(\mathbf{z}) = e_i(\mathbf{z})\tau(\mathbf{z}) + \text{ lower terms.}$$

The "lower terms" which are missing a factor of ξ_α for some α contribute 0 to the evaluation ϕ, because they do not have negative powers of $\mathcal{Q}_1^{(\alpha)}$.

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Difference equations for $\mathfrak{sl}_2$

For $\mathcal{F}_{\mathbf{n}}^* = V(\omega_1)^{*n_1} * \cdots * V(k\omega_1)^{*n_k}$, we have the graded character $\chi_{\mathbf{n}}(q,\mathbf{z}) = \phi(\mathcal{Q}_1^{n_1} \cdots \mathcal{Q}_k^{n_k})$.

Reminder: For $\mathfrak{sl}_2$,
$C_1[\infty] = C_1[k-1] = \mathcal{Q}_k \mathcal{Q}_{k-1}^{-1} - \mathcal{Q}_k^{-1}\mathcal{Q}_{k-1}^{-1} + \mathcal{Q}_k^{-1}\mathcal{Q}_{k-1}$

The equation $C_1[k-1] \circ \chi_{\mathbf{n}}(q,\mathbf{z}) = C_1[\infty] \circ \chi_{\mathbf{n}}(q,\mathbf{z})$ becomes:
Lemma:

$$\chi_{\ldots,n_{k-1}-1,n_k+1} + \chi_{\ldots,n_{k-1}+1,n_k-1} - q^{|\mathbf{n}|-k+1}\chi_{\ldots,n_{k-1}-1,n_k-1} = (z_1+z_2)\chi_{\mathbf{n}}.$$

At $k=1$ this specializes to an equation for $\chi_n(q,z) = \chi_{n_1}(q,(z,z^{-1}))$:

$$\chi_{n+1} + (1-q^n)\chi_{n-1} = (z+z^{-1})\chi_n$$ specialized Difference Toda equation.

(Relativistic Toda operator for $U_{q'}(\mathfrak{sl}_2)$ on the discrete variable $\mathbf{n}$).

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Lemma:

$$\chi_{\dots,n_{k-1}-1,n_k+1} + \chi_{\dots,n_{k-1}+1,n_k-1} - q^{|\mathbf{n}|-k+1}\chi_{\dots,n_{k-1}-1,n_k-1} = (z_1+z_2)\chi_{\mathbf{n}}.$$

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(Relativistic Toda operator for $U_{q'}(\mathfrak{sl}_2)$ on the discrete variable $\mathbf{n}$).

Level-1 difference equations for $\mathfrak{sl}_n$:

For $\mathfrak{g} = \mathfrak{sl}_n$, acting with $\mathcal{C}_j$ on $\chi^{(1)}_{\mathbf{n}}(q, \mathbf{z}) := \mathrm{ch}\, V(\omega_1)^{*n_1^{(1)}} * \cdots * V(\omega_{n-1})^{*n_1^{(N-1)}}$ gives $n-1$ difference equations. Insertion of $\mathcal{C}_1$ gives Toda:

Theorem: The character of the tensor product of fundamental representations satisfies difference Toda equations for $U_q(\mathfrak{sl}_n)$.

$$\sum_{\alpha=1}^{n} \chi_{\mathbf{n}+\epsilon_\alpha-\epsilon_{\alpha-1}}(q, \mathbf{z}) - q^{|\mathbf{n}|} \sum_{\alpha=1}^{n-1} \chi_{\mathbf{n}+\epsilon_{\alpha+1}-\epsilon_\alpha}(q, \mathbf{z}) = e_1(\mathbf{z})\chi_{\mathbf{n}}(q, \mathbf{z}).$$

and $n-2$ higher order equations.

Corollary: In this simple case (level-1), $\chi_{\mathbf{n}}(q, \mathbf{z})$ are specialized q-Whittaker functions of $U_q(\mathfrak{sl}_n)$, aka degenerate Macdonald polynomials (at $t \to 0$).

Macdonald operators

Theorem: If $k \geq \max\{i : n_i^{(\alpha)} > 0\} - 1$ then

$$\mathcal{Q}_k^{(\alpha)} \circ \chi_{\mathbf{n}}(q;\mathbf{z}) = q^{\sharp} \mathcal{D}_k^{(\alpha)} \chi_{\mathbf{n}}(;\mathbf{z})$$

where $\mathcal{D}_k^{(\alpha)}$ is the difference operator:

$$\mathcal{D}_k^{(\alpha)} = \sum_{\substack{I \subset \{1,\ldots,n-1\} \\ |I|=\alpha}} \prod_{i \in I} z_i^k \left(\prod_{j \notin I} \frac{z_i}{z_i - z_j} \right) \prod_{i \in I} D_i, \quad D_i z_j = q^{-\delta_{ij}} z_j.$$

- When $k = 0$, $\mathcal{D}_0^{(\alpha)}$ is the $t \to \infty$ degeneration of the Macdonald operator.
- When $k = 1$, $\mathcal{D}_1^{(\alpha)}$ are the Kirillov-Noumi Macdonald creation operators at $t \to \infty \Rightarrow$ The characters are q-Whittaker functions or degenerate Macdonald polynomials; (also Demazure characters.)

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Realization of the quantum Q-system by difference operators

Theorem: The operators $\mathcal{D}_k^{(\alpha)}$ acting on the space of functions in $\mathbf{z}$ give a presentation of the dual quantum Q-system:

$$q^{C^{-1}_{\alpha,\alpha}}\mathcal{D}^{(\alpha)}_{k-1}\mathcal{D}^{(\alpha)}_{k+1} = (\mathcal{D}^{(\alpha)}_k)^2 - \prod_{\beta\sim\alpha}\mathcal{D}^{(\beta)}_k, \quad \mathcal{D}^{(\alpha)}_{k+1}\mathcal{D}^{(\beta)}_k = q^{C^{-1}_{\alpha,\beta}}\mathcal{D}^{(\beta)}_k\mathcal{D}^{(\alpha)}_{k+1}.$$

Recalling $\chi_{\mathbf{n}}(q;\mathbf{z}) = \phi(\prod(\mathcal{Q}_i^{(\alpha)})^{n_i^{(\alpha)}})$, we have **Corollary:**

$$\chi_{\mathbf{n}}(q,\mathbf{z}) = q^{\sharp}\prod_{i=k}^{1}\prod_{\alpha=1}^{n-1}(\mathcal{D}_i^{(\alpha)})^{n_i^{(\alpha)}}1$$

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Example

If $\mathfrak{g} = \mathfrak{sl}_2$ and $k = 1$, we have

$$\chi_n(q; z) = q^{n(n-1)/2} (\mathcal{D}_1^{(1)})^n \cdot 1$$

where

$$\mathcal{D}_1^{(1)} = \frac{1}{z_1 - z_2} (z_1^2 D_1 - z_2^2 D_2), \quad z_1 z_2 = 1.$$

Conclusion

- The graded tensor product characters can be realized in terms of an action of quantum Q-system solutions on $\phi(1)$.
- Integrability of quantum Q-system implies difference equations satisfied by the graded characters – specialize to quantum Toda when all reps are fundamental.
- Solutions in simplest case (fundamental modules) are q-Whittaker functions at integral values of the parameters: polynomial solutions.
- We have obtained expression for characters in terms of (generalized degenerate Macdonald) difference operators.
- Yet to be completed: use this to construct characters in stabilized limits to obtain CFT characters.
- Reference: `arXiv:1109.6261; arXiv:1505.01657`.