Aalto University
School of Science
and Technology

# Boundary correlation functions with a hidden quantum group

Kalle Kytölä kalle.kytola@aalto.fi

Department of Mathematics and Systems Analysis, Aalto University

joint work with Eveliina Peltola (Univ. Helsinki)

Niko Jokela (Univ. Helsinki) & Matti Järvinen (ENS, Paris)

May 22, 2015 — Galileo Galilei Institute

Lattice models: exact methods and combinatorics

# This talk

## Quantum group construction of boundary correlation fns:

(K. & Peltola [arXiv:1408.1384])

## Applications to random conformally invariant curves:

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- Extremal multiple SLEs (K. & Peltola [arXiv:????.????])
  - $\rightsquigarrow$ classification of random curves w/ non-trivial conformal moduli
  - $\rightsquigarrow$ effect of bdry conditions on interfaces in lattice models
  - $\rightsquigarrow$ crossing probabilities

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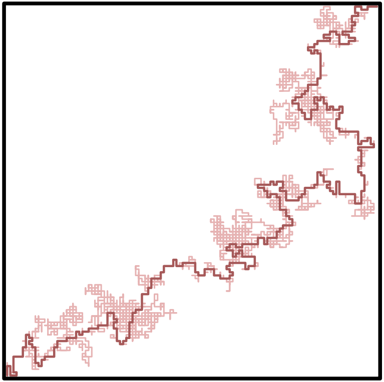
  - $\rightsquigarrow$ effect of bdry conditions on interfaces in lattice models
  - $\rightsquigarrow$ crossing probabilities
- Chordal SLE bdry visiting probabilities (Jokela & Järvinen & K. [arXiv:1311.2297])
  - $\rightsquigarrow$ multi-point boundary Green's function for SLE [Lawler & ...]
  - $\rightsquigarrow$ correlation fn of SLE covariant measure on bdry [Alberts & Sheffield]
  - $\rightsquigarrow$ lattice model probas, e.g. Potts model bdry spin correlation

# 1. Introduction: conformally invariant random curves

# Lattice model interfaces: loop-erased random walk



SRW: simple random walk

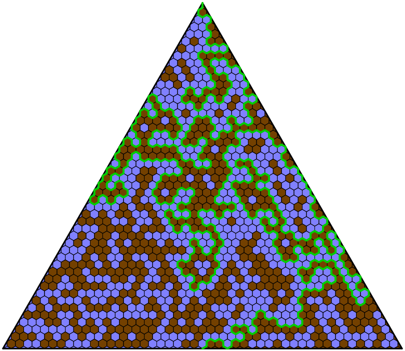
- started at bottom left corner
- conditioned to exit the box through top right corner

LERW: loop-erased random walk

- chronologically erasing the loops of the SRW $\rightsquigarrow$ simple path from bottom left to top right

Scaling limit result [Lawler & Schramm & Werner 2004, Zhan 2004]

# Lattice model interfaces: critical percolation



Critical site percolation

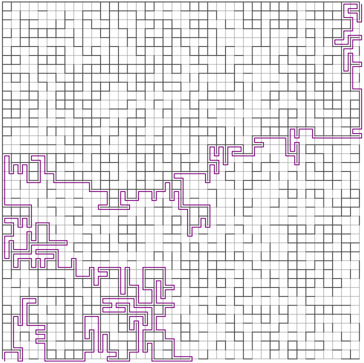
- color the hexagons black or white according to independent fair coin tosses

The exploration path

- path from midpoint of the bottom side to the top corner of the triangle, leaving white hexagons on its immediate left and black hexagons on its immediate right

Scaling limit result [Smirnov 2001, Camia & Newman 2007]

# Lattice model interfaces: random cluster model



FK-model [Fortuin-Kasteleyn 1972]

- random subset $\omega$ of edges
  $\mathsf{P}[\{\omega\}] \propto \left(\frac{p}{1-p}\right)^{|\omega|} Q^{\#\text{conn. comp.}(\omega)}$
- parameters: $Q > 0$, $p \in (0,1)$
  $$(p_{\text{critical}} = \frac{\sqrt{Q}}{1+\sqrt{Q}})$$
- all left and top side edges conditioned to be in $\omega$

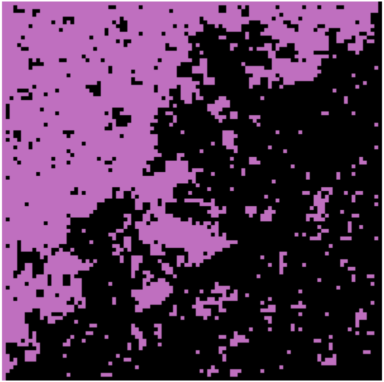
The exploration path

- path from bottom left to top right closely surrounding the connected component of left and top sides

Scaling limit result for $Q = 2$

[Smirnov 2010, Chelkak & Duminil-Copin & Hongler & Kemppainen & Smirnov 2013]

# Lattice model interfaces: loop-erased random walk



Critical Ising model with Dobrushin boundary conditions

- spins $\sigma_z \in \{\pm 1\}$ at each pixel $z$, $+1$ on top and left bdry, $-1$ on bottom and right bdry
- probability $\propto x^{\#\text{ disagreeing neighbors}}$ with $x = x_{\text{critical}} = \sqrt{2} - 1$

interface / domain wall

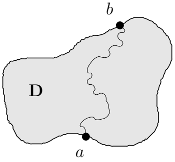
- curve between the macroscopic components of $+1$, $-1$ spins

Scaling limit result [Chelkak & Duminil-Copin & Hongler & Kemppainen & Smirnov 2013]

# Schramm's classification result

## The chordal Schramm-Loewner evolution (SLE$_\kappa$):

[Schramm 2000, Lawler & Schramm & Werner, Rohde & Schramm, ...]




Random curve $\gamma_{D;a,b}$ in domain $D$ from boundary point $a$ to $b$

$\boxed{\text{law } \mathsf{P}_{D;a,b}}$

Classification:

- Conformal invariance:
- Domain Markov property:

$\Longrightarrow \exists \kappa > 0 : \mathsf{P}_{D;a,b} =$ "chordal SLE$_\kappa$ in $D$ from $a$ to $b$"

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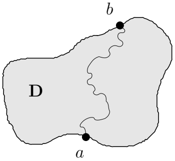


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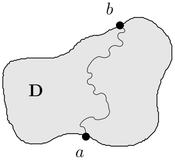


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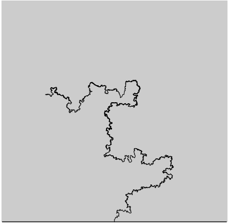
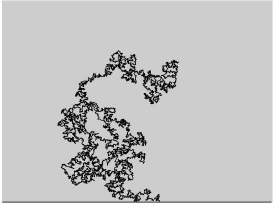
- Domain Markov property: given an initial piece $\gamma'$ of $\gamma_{D;a,b}$ starting from $a$, the rest has the law of $\gamma_{D\setminus\gamma';\mathrm{tip},b}$.

$\Longrightarrow$ $\exists\kappa > 0 : \mathsf{P}_{D;a,b} =$ "chordal SLE$_\kappa$ in $D$ from $a$ to $b$"

# Properties of SLE

random fractal curve, $\dim_{\text{Hausdorff}}(\gamma) = 1 + \frac{\kappa}{8}$ for $\kappa \leq 8$ [Beffara 2008]





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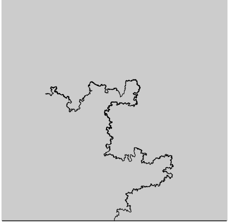
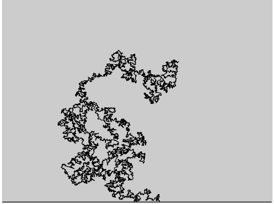


$\boxed{0 < \kappa \leq 4}$

simple curve

doesn't touch boundary



$\boxed{4 < \kappa < 8}$

non-self-crossing curve

touches boundary on a random Cantor set

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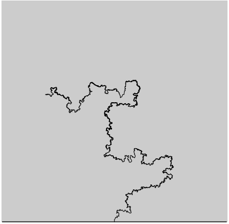

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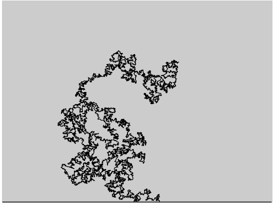

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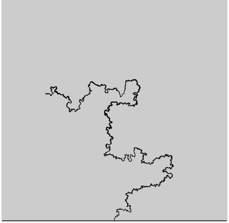

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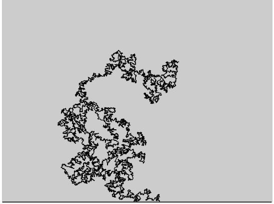

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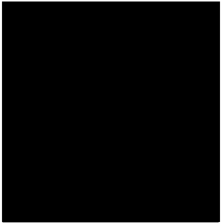

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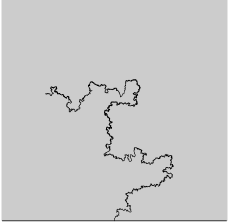

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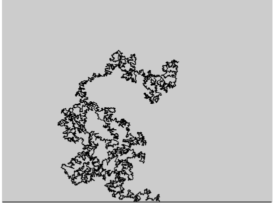

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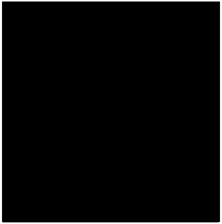

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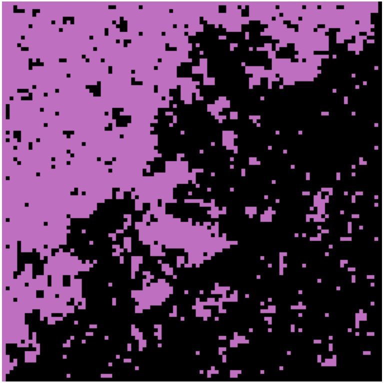
in this talk $\kappa < 8$

# Generalizing the Dobrushin boundary conditions

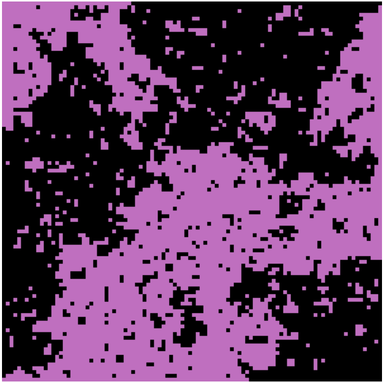
Critical Ising model with Dobrushin boundary conditions



[simulation and picture by Eveliina Peltola]

# Generalizing the Dobrushin boundary conditions

Critical Ising model with alternating boundary conditions



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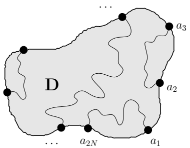
# Classification problem of multiple SLEs




[Dubédat 2007]

[Bauer & Bernard & K. 2005]

Random curves $(\gamma^{(i)})_{i=1}^{N}$ in domain $D$ connecting boundary points $a_1, a_2, \ldots, a_{2N}$

$\boxed{\text{law } \mathsf{P}_{D; a_1, \ldots, a_{2N}}}$

Can we give a classification?

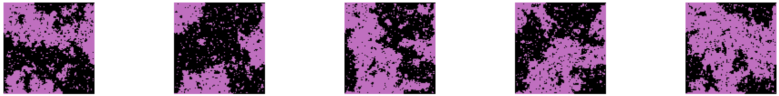
- Conformal invariance
- Domain Markov property (w.r.t. all initial segments)
- \- initial segments absolutely continuous w.r.t. chordal $\mathrm{SLE}_\kappa$

! Convex set of multiple-$\mathrm{SLE}_\kappa$'s $(\mathsf{P}_{D; a_1, \ldots, a_{2N}})$

# 2. Multiple Schramm-Loewner Evolutions growth processes

# Overview of classification of multiple SLEs

| local multiple SLEs with curves starting from $2N$ boundary points | $\longleftrightarrow$ | Möbius covariant positive solutions $\mathcal{Z}$ to a system of PDEs |
|---|---|---|



[pictures by Eveliina Peltola]

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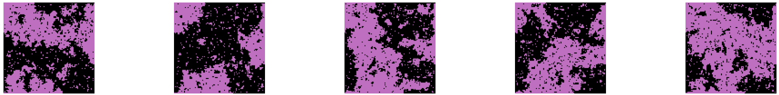
convex set of probability measures
(finite dimensional)

vector space of solutions
(finite dimensional)



[pictures by Eveliina Peltola]

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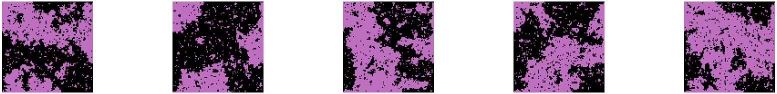
extremal points: deterministic connectivity pattern (a planar pair partition $\alpha \in \mathrm{PPP}_N$)



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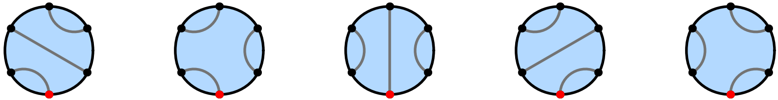

$$\mathrm{PPP} = \bigsqcup_{N \in \mathbb{N}} \mathrm{PPP}_N, \qquad \#\mathrm{PPP}_N = \mathcal{C}_N = \frac{1}{N+1}\binom{2N}{N}$$

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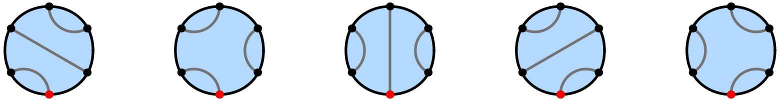

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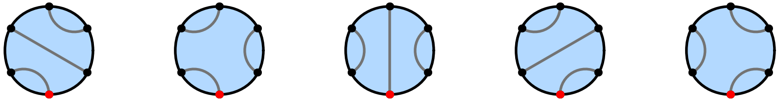
solutions $\mathcal{Z}_\alpha$ with particular asymptotic behavior



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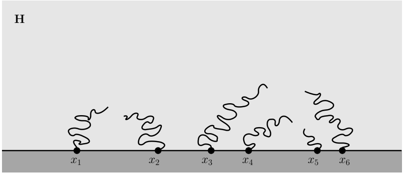
# Role of the partition function




Local multiple SLE$_\kappa$ classification:

$\mathcal{Z}$ "partition function" defined on

$$\mathfrak{X}_{2N} = \{x_1 < x_2 < \cdots < x_{2N}\}$$

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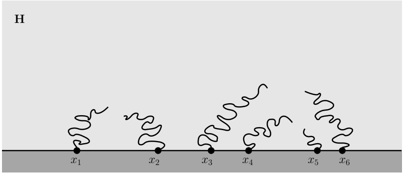


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$$(h = h_{1,2} = \tfrac{6-\kappa}{2\kappa})$$

$\mathcal{Z}$ specifies Girsanov transforms w.r.t. chordal SLE$_\kappa$:

$$\frac{\mathrm{d}(j\text{:th curve})}{\mathrm{d}(\text{chordal SLE}_\kappa)} \propto \prod_{k \neq j} g'(x_k)^h \times \mathcal{Z}\big(g(x_1), \ldots, g(\text{tip}), \ldots, g(x_{2N})\big).$$

where $g \colon \mathbb{H} \setminus (j\text{:th curve}) \to \mathbb{H}$ is conformal s.t. $g(z) = z + o(1)$.

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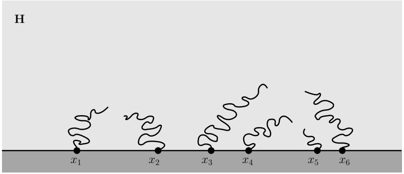


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(PDE) $\mathcal{D}_j \mathcal{Z} = 0$ for all $j = 1, \ldots, 2N$, where

$$\mathcal{D}_j = \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2} \right).$$

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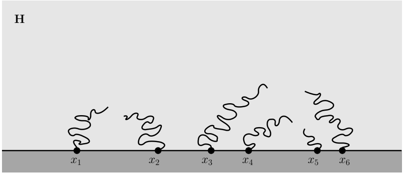


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$\mathcal{D}_j = \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2} \right)$.

(COV) For $\mu \colon \mathbb{H} \to \mathbb{H}$ Möbius s.t. $\mu(x_1) < \cdots < \mu(x_{2N})$ we have
$\mathcal{Z}(x_1, \ldots, x_{2N}) = \prod_{j=1}^{2N} \mu'(x_j)^h \times \mathcal{Z}(\mu(x_1), \ldots, \mu(x_{2N}))$.

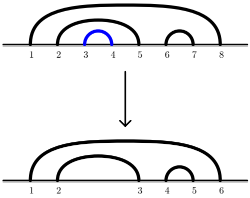
$\mathcal{Z}$ specifies Girsanov transforms w.r.t. chordal $\text{SLE}_\kappa$:

$$\frac{\mathrm{d}(j\text{:th curve})}{\mathrm{d}(\text{chordal SLE}_\kappa)} \propto \prod_{k \neq j} g'(x_k)^h \times \mathcal{Z}(g(x_1), \ldots, g(\text{tip}), \ldots, g(x_{2N})).$$

where $g \colon \mathbb{H} \setminus (j\text{:th curve}) \to \mathbb{H}$ is conformal s.t. $g(z) = z + o(1)$.

# Collapsing marked points

Suppose
$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}(x_1, \ldots, x_{2N})}{(x_{j+1} - x_j)^{-2h}}$
$= \hat{\mathcal{Z}}(x_1, \ldots, x_{j-1}, x_{j+2}, \ldots, x_{2N})$.
Then the law of the curves other than $j, j+1$ under local $2N$-SLE$_\kappa$ defined by $\mathcal{Z}$ tends to the local $(2N-2)$-SLE$_\kappa$ defined by $\hat{\mathcal{Z}}$ as $x_j, x_{j+1} \to \xi$.




For pure partition functions $\mathcal{Z}_\alpha$, $\alpha \in$ PPP, thus require

(ASY) $$\lim_{x_j, x_{j+1} \to \xi} \frac{\mathcal{Z}_\alpha}{(x_{j+1} - x_j)^{-2h}} = \begin{cases} \mathcal{Z}_{\alpha / \{j, j+1\}} & \text{if } \{j, j+1\} \in \alpha \\ 0 & \text{if } \{j, j+1\} \notin \alpha \end{cases}$$

# Multiple SLEs pure partition function problem

$$\left(\mathcal{Z}_\alpha\right)_{\alpha \in \mathrm{PPP}}$$

for $\alpha \in \mathrm{PPP}_N$, function $\mathcal{Z}_\alpha$ on $\mathfrak{X}_{2N} = \left\{x_1 < x_2 < \cdots < x_{2N}\right\}$ s.t.

(PDE) $\mathcal{D}_j \mathcal{Z}_\alpha = 0$ where $\mathcal{D}_j = \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2} \right)$

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# 3. Solution of pure partition functions by a hidden quantum group

# Overview of the quantum group method

**Correspondence:**

| | | |
|---|---|---|
| vectors in an $n$-fold tensor product representation of a quantum group | $\longleftrightarrow$ | functions of $n$ variables |
| highest weight vectors of subrepresentations | $\longleftrightarrow$ | solutions to partial differential equations |
| vectors in the trivial subrepresentation | $\longleftrightarrow$ | Möbius covariant functions |
| prescribed projections to subrepresentations | $\longleftrightarrow$ | prescribed asymptotic behavior |

# Idea: Integral solutions to PDEs

**How to solve the PDEs?** $(\mathcal{D}_j \mathcal{Z})(x_1, \ldots, x_{2N}) = 0$,
where $\mathcal{D}_j = \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2} \right)$.

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$$\mathcal{Z}(x_1, \ldots, x_{2N}) = \int_\Gamma f(x_1, \ldots, x_{2N}; w_1, \ldots, w_\ell) \, \mathrm{d}w_1 \cdots \mathrm{d}w_\ell \quad ?$$

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quantum group $\mathcal{U}_q(\mathfrak{sl}_2)$ acts on $\Gamma$

[Felder & Wieczerkowski 1991, Peltola & K. 2014]

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# The correspondence theorem (special case)

$\kappa \in (0,8) \setminus \mathbb{Q}$, $q = e^{\mathrm{i}\pi 4/\kappa}$

Theorem (K. & Peltola)

$$\mathcal{F}^{(x_0)} \colon \; M_2^{\otimes 2N} \longrightarrow \{\text{functions on } \mathfrak{X}_{2N}^{(x_0)}\}$$

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# The correspondence theorem (special case)

- $M_2 \otimes M_2 \cong M_1 \oplus M_3$, proj. to $M_1 \cong \mathbb{C}$ is $\hat{\pi} \colon M_2 \otimes M_2 \to \mathbb{C}$.
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$\kappa \in (0,8) \setminus \mathbb{Q}$, $q = e^{\mathrm{i}\pi 4/\kappa}$

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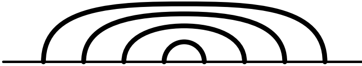
**Uniqueness of solutions:** The only solution of the homogeneous problem, $\hat{\pi}_j(v) = 0 \; \forall j$ & (SING), is $v = 0$.

# Explicit solution for the maximally nested case

**Rainbow configuration:**

$$\Cap_N = \{\{1, 2N\}, \{2, 2N-1\}, \ldots, \{N, N+1\}\}$$



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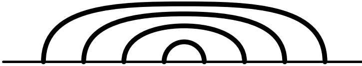

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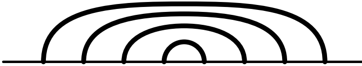

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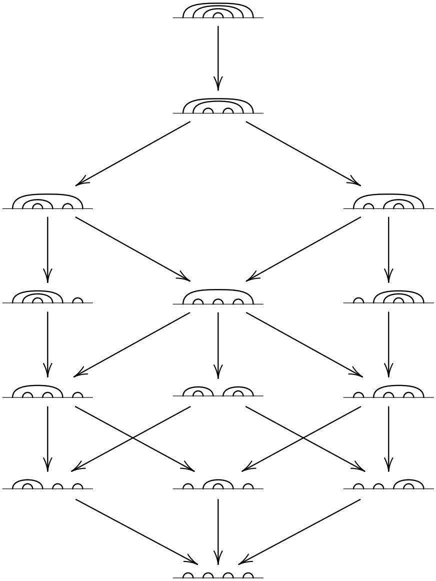
**Explicit formula**: The solution for rainbow configurations is

$$v_{\underline{\Cap}_N} = \mathrm{const.} \times \sum_{k=0}^{N} (-1)^k q^{k(N-k-1)} \times (F^k.(e_0^{\otimes N})) \otimes (F^{N-k}.(e_0^{\otimes N})).$$

# Recursive solution on the poset of configurations



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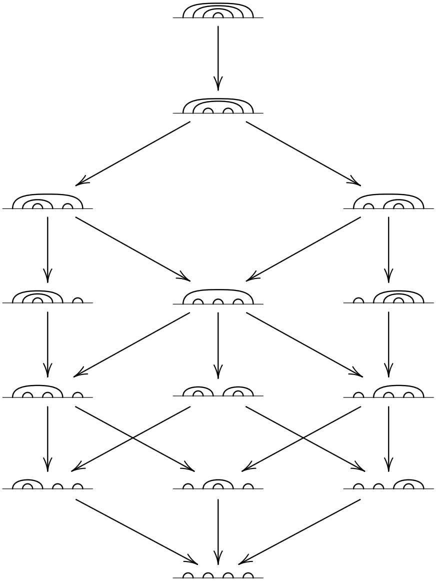

Tying operation

$\wp_j : \mathrm{PPP}_N \to \mathrm{PPP}_N$:

- connect $j$ and $j+1$
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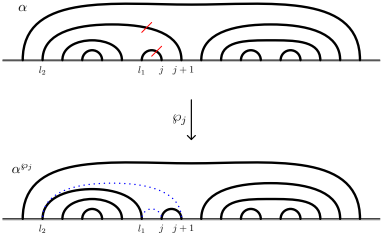

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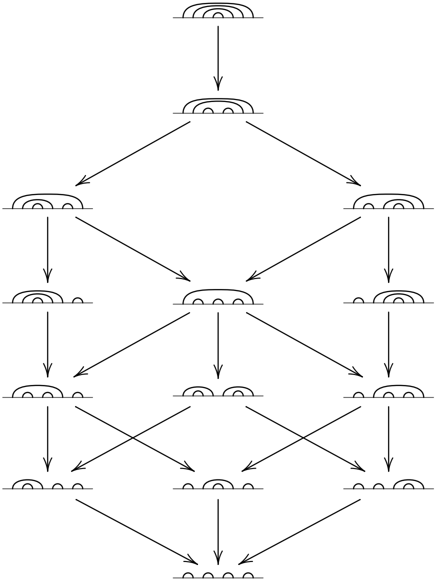

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**Recursion based on formula:**
if $\{j, j+1\} \in \varrho \in \mathrm{PPP}_N$, then

$$(\mathrm{id} - \pi_j)\,(v_\varrho) = \frac{-1}{[2]} \sum_{\beta \in \wp_j^{-1}(\varrho) \setminus \{\varrho\}} v_\beta$$

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- With $\mathcal{V}_\emptyset = 1$, there is a unique collection $(\mathcal{V}_\alpha)_{\alpha \in \mathsf{PPP}}$ solving the system (SING) & (PROJ).

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- The functions $\mathcal{Z}_\alpha = \mathcal{F}[v_\alpha]$, span $c_N$-dimensional solution spaces of the system

(PDE) $\mathcal{D}_j \mathcal{Z}_\alpha = 0$, $\mathcal{D}_j = \frac{\kappa}{2} \frac{\partial^2}{\partial x_j^2} + \sum_{i \neq j} \left( \frac{2}{x_i - x_j} \frac{\partial}{\partial x_i} - \frac{2h}{(x_i - x_j)^2} \right)$

(COV) $\mathcal{Z}(x_1, \ldots) = \prod_{j=1}^{2N} \mu'(x_j)^h \times \mathcal{Z}(\mu(x_1), \ldots)$

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# 4. GENERAL QUANTUM GROUP METHOD AND SOME DETAILS

# Overview of the quantum group method (again)

**Correspondence:**

| | | |
|---|---|---|
| vectors in an *n*-fold tensor product representation of a quantum group | $\longleftrightarrow$ | functions of *n* variables |
| highest weight vectors of subrepresentations | $\longleftrightarrow$ | solutions to partial differential equations |
| vectors in the trivial subrepresentation | $\longleftrightarrow$ | Möbius covariant functions |
| prescribed projections to subrepresentations | $\longleftrightarrow$ | prescribed asymptotic behavior |

# Integral solutions to PDEs of CFTs

# The correspondence theorem (general case)

Theorem (K. & Peltola) $\mathcal{F}^{(x_0)}_{d_1,\ldots,d_n} : \quad \bigotimes_{j=1}^{n} M_{d_j} \longrightarrow \{\text{functions on } \mathfrak{X}_n^{(x_0)}\}$

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$$\mathcal{F}^{(x_0)}_{\ldots,d_j,d_{j+1},\ldots}[v] \sim (x_{j+1} - x_j)^{\Delta_d} \times \mathcal{F}^{(x_0)}_{\ldots,d,\ldots}[v].$$

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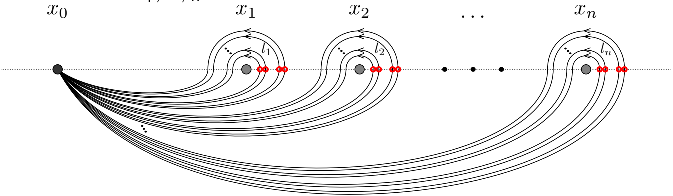
$\varphi^{(x_0)}_{l_1,\ldots,l_n}(\mathbf{x}) =$ 

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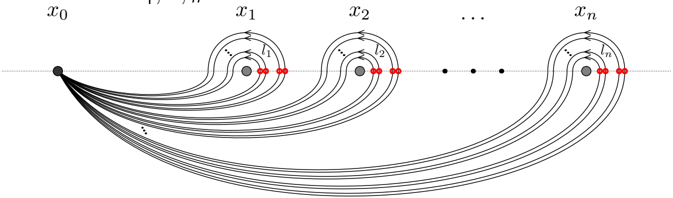


$$f \propto \prod (x_j - x_i)^{\frac{2}{\kappa}(d_i-1)(d_j-1)} \times \prod (w_s - w_r)^{\frac{8}{\kappa}} \times \prod (w_r - x_i)^{-\frac{4}{\kappa}(d_i-1)}$$

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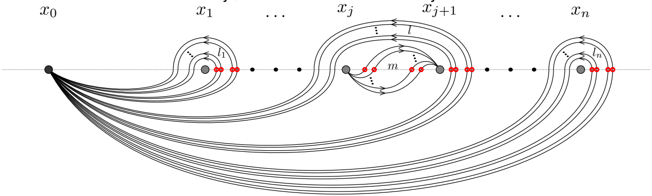

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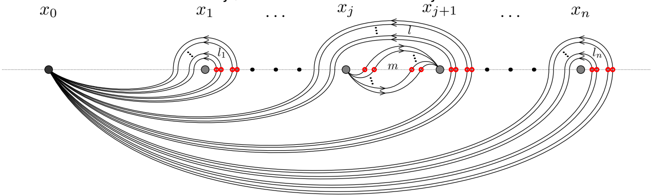


dominated convergence:

$$\frac{\mathcal{F}^{(x_0)}_{\ldots,d_j,d_{j+1},\ldots}[v](\ldots)}{(x_{j+1}-x_j)^{\Delta_d^{d_j,d_j+1}}} \underset{x_j,x_{j+1}\to\xi}{\longrightarrow} \mathcal{F}^{(x_0)}_{\ldots,d,\ldots}[v](\ldots,\xi,\ldots)$$

where $\Delta_d^{d_j,d_{j+1}} = \dfrac{2(1+d^2-d_j^2-d_{j+1}^2)+\kappa(d_j+d_{j+1}-d-1)}{2\kappa}$

# Sketch: anchor point independence

Write $\varphi^{(x_0)}_{l_1,\ldots,l_n}(\mathbf{x})$ in terms of $\alpha^{(x_0)}_{m_1,\ldots,m_n}(\mathbf{x})$

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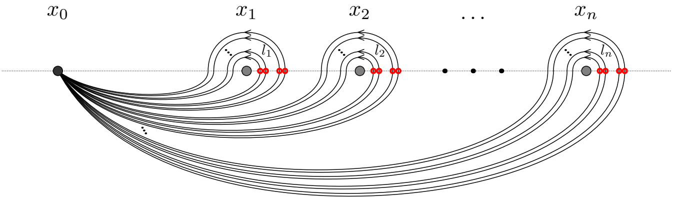


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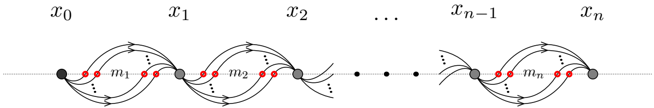

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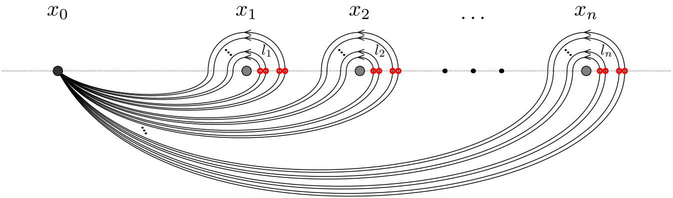


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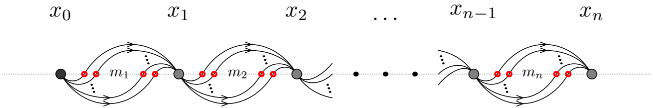


**Highest weight vectors:**

If $E.v = 0$, then in $\mathcal{F}^{(x_0)}[v](\mathbf{x})$, the coefficient of $\alpha^{(x_0)}_{m_1,\ldots,m_n}(\mathbf{x})$ vanishes whenever $m_1 \neq 0$.

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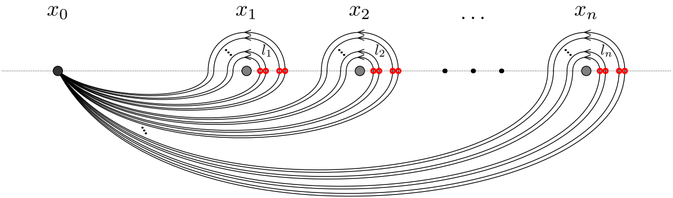


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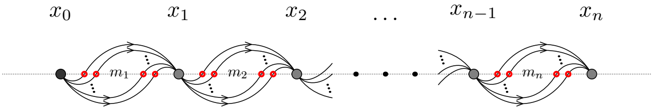


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$\rightsquigarrow$ $\mathcal{F}[v](\mathbf{x})$ well defined for $\mathbf{x} \in \mathfrak{X}_n$

# Sketch: Stokes thm and highest weight vectors

- $\Supset_{l_1,\ldots,l_n}$ the $\ell$-dimensional integration surface of $\varphi^{(x_0)}_{l_1,\ldots,l_n}$
- $g(w_1; w_2, \ldots, w_\ell)$ single valued, symmetric in last $\ell - 1$ vars

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**Stokes formula / integration by parts:**

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$$= \sum_{j=1}^{n} \Big\{ (q^{-1} - q)\, [l_j]\, [d_j - l_j]\, q^{\sum_{i<j}(d_i - 1 - 2l_i)} \times \int_{\Supset_{\ldots,l_j-1,\ldots}} \big(\gamma(w_1, \ldots, w_{\ell-1})\, f(\mathbf{x}; w_1, \ldots, w_{\ell-1})\big)\, \mathrm{d}w_1 \cdots \mathrm{d}w_{\ell-1} \Big\}$$

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where $\gamma(w_1, \ldots, w_{\ell-1})$

$$= \prod_{i=1}^{n} |x_0 - x_i|^{-\frac{4}{\kappa}(d_i - 1)} \prod_{r=1}^{\ell-1} |x_0 - w_r|^{\frac{8}{\kappa}}\, g(x_0; w_1, \ldots, w_{\ell-1}).$$

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**Highest weight vect.:** $v = \sum C_{l_1,\ldots,l_n} (e_{l_n} \otimes \cdots \otimes e_{l_1})$ s.t. $E.v = 0$

$$\sum C_{l_1,\ldots,l_n} \int_{\Supset_{l_1,\ldots,l_n}} \sum_{r=1}^{\ell} \frac{\partial}{\partial w_r} \Big( g(w_r; \ldots)\, f(\mathbf{x}; \mathbf{w}) \Big)\, \mathrm{d}\mathbf{w} = 0.$$

# Sketch: PDEs by Stokes

Benoit & Saint-Aubin differential operators:

$$\mathcal{D}^{(j)} = \sum_{k=1}^{d_j} \sum_{\substack{n_1,\ldots,n_k \geq 1 \\ n_1+\ldots+n_k = d_j}} \frac{(\kappa/4)^{d_j-k}\,(d_j-1)!^2}{\prod_{j=1}^{k-1}(\sum_{i=1}^{j} n_i)(\sum_{i=j+1}^{k} n_i)} \times \mathcal{L}^{(j)}_{-n_1} \cdots \mathcal{L}^{(j)}_{-n_k}$$

where $\mathcal{L}^{(j)}_p$ ($j = 1, \ldots, n$ and $p \in \mathbb{Z}$) are 1st order diff. operators

$$\mathcal{L}^{(j)}_p = -\sum_{i \neq j} (x_i - x_j)^p \left( (1+p)\,\frac{(d_i-1)(2(d_i+1)-\kappa)}{2\kappa} + (x_i - x_j)\frac{\partial}{\partial x_i} \right)$$

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**Highest weight vectors:**
If $E.v = 0$, Stokes formula gives $\mathcal{D}^{(j)}\,\mathcal{F}[v](\mathbf{x}) = 0$.

# Sketch: Möbius covariance

$$\varphi^{(x_0)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n) \;=\; \int_{\Supset_{l_1,\ldots,l_n}} f(x_1,\ldots,x_n;w_1,\ldots,w_\ell)\,\mathrm{d}w_1\cdots\mathrm{d}w_\ell$$

Möbius covariance: if $\nu(x_1) < \cdots < \nu(x_n)$ for $\nu(z) = \frac{az+b}{cz+d}$, want

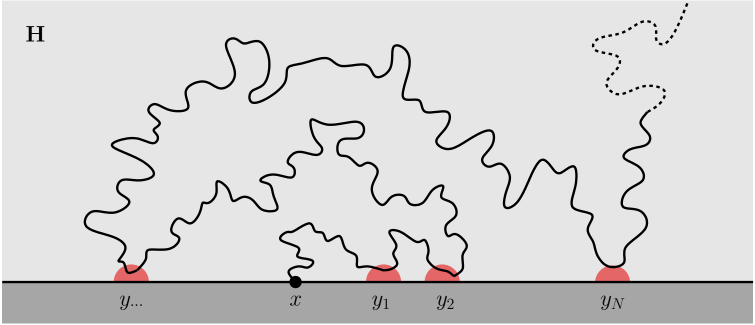
$$\mathcal{F}[v]\big(\nu(x_1),\ldots,\nu(x_n)\big) \times \prod_{j=1}^{n} \nu'(x_j)^{\frac{(d_j-1)(2(d_j+1)-\kappa)}{2\kappa}} \;=\; \mathcal{F}[v](x_1,\ldots,x_n)$$

- translation invariance, $z \mapsto z+\xi$:
  $$\varphi^{(x_0+\xi)}_{l_1,\ldots,l_n}(x_1+\xi,\ldots,x_n+\xi) \;=\; \varphi^{(x_0)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n)$$
  - make changes of variables $w'_r = w_r + \xi$
- homogeneity, $z \mapsto \lambda z$:
  $$\varphi^{(\lambda x_0)}_{l_1,\ldots,l_n}(\lambda x_1,\ldots,\lambda x_n) \;=\; \lambda^{\Delta}\,\varphi^{(x_0)}_{l_1,\ldots,l_n}(x_1,\ldots,x_n)$$
  - make changes of variables $w'_r = \lambda w_r$
- special conformal transformations, $z \mapsto \frac{z}{1+az}$:
  - vary $a$ infinitesimally
  - use a property of the integrand $f$
  - apply Stokes formula

# 5. Chordal SLE boundary visit problem

# Chordal SLE boundary visit amplitude

$$\mathsf{P}_{\mathbb{H};x,\infty}\Big[\mathrm{SLE}_\kappa \text{ visits } B_\varepsilon(y_1), \text{ then } B_\varepsilon(y_2), \text{ then } \ldots \text{ then } B_\varepsilon(y_N)\Big]$$



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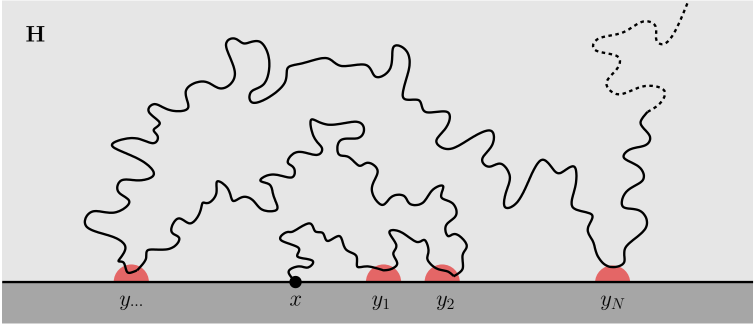
$$\sim \qquad \varepsilon^{N\frac{8-\kappa}{\kappa}}$$



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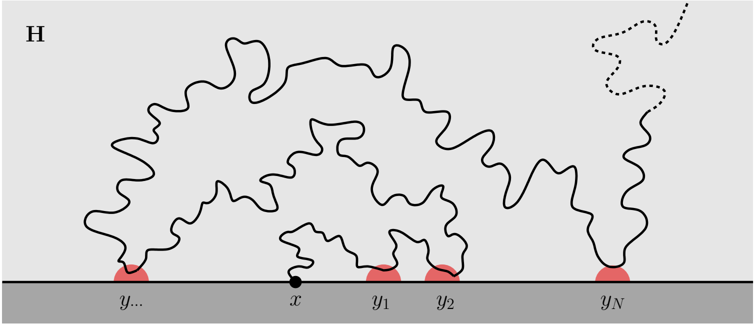
$$\sim \text{ const.} \times \varepsilon^{N\frac{8-\kappa}{\kappa}} \times \zeta_N(x; y_1, y_2, \ldots, y_N)$$



# Boundary visit amplitude problem

(COV) $\zeta_N(x; y_1, \ldots, y_N) = \lambda^{N h_{1,3}} \times \zeta_N(\lambda x + \xi; \lambda y_1 + \xi, \ldots, \lambda y_N + \xi)$

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(PDE) $\left\{ \frac{\kappa}{2} \frac{\partial^2}{\partial x^2} + \sum_{j=1}^{N} \left( \frac{2}{y_j - x} \frac{\partial}{\partial y_j} + \frac{2 \frac{\kappa - 8}{\kappa}}{(y_j - x)^2} \right) \right\} \zeta_N(x; y_1, \ldots, y_N) = 0$

* Itô for martingale $\prod_{j=1}^{N} g_t'(y_j)^{\frac{8-\kappa}{\kappa}} \times \zeta_N(X_t; g_t(y_1), \ldots, g_t(y_N))$

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---

(ASY) As $y_j \to x$, asymptotics are

$$|y_j - x|^{\frac{8-\kappa}{\kappa}} \times \zeta_N(\ldots) \to \begin{cases} \zeta_{N-1}(x; y_2, \ldots, y_N) & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}.$$

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(PDE) moreover $N$ third order linear homogeneous PDEs for $\zeta_N$

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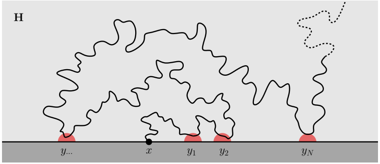
# Translation of the bdry visit amplitude problem

$L$ points on the left and $R$ on the right, $L + R = N$, visit order $\omega$:

$$v_\omega \in M_3^{\otimes R} \otimes M_2 \otimes M_3^{\otimes L}$$



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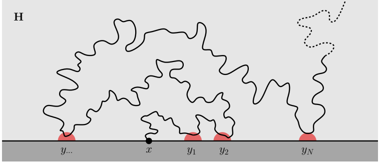

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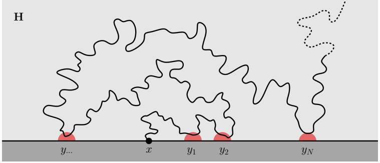

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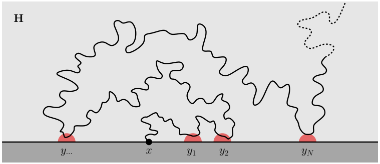
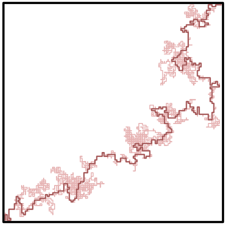



**Thm** (Jokela & Järvinen & K., K. & Peltola)

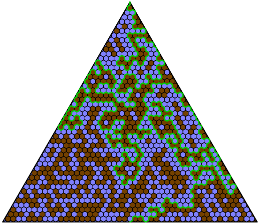
If $v_\omega$ satisfies this, then the function $\zeta_\omega = \mathcal{F}[v_\omega]$ satisfies the PDEs and asymptotics for the zig-zag problem. Solutions exist and are unique.

# Boundary visits of interfaces in lattice models

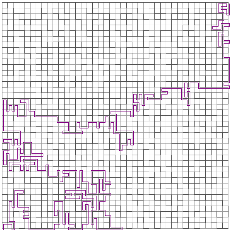
LERW



Percolation



$Q$-FK model

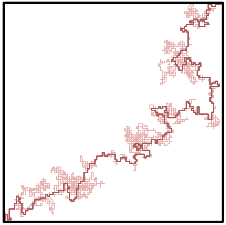


$\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=2}$ $\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=6}$ $\overset{?}{\longrightarrow}$ chordal $\mathrm{SLE}_{\kappa=\kappa(\mathrm{Q})}$

as lattice mesh $\delta \searrow 0$

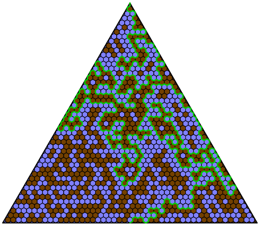
# Boundary visits of interfaces in lattice models

LERW



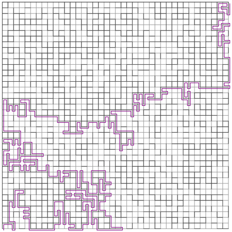
$\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=2}$

Percolation



$\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=6}$

$Q$-FK model



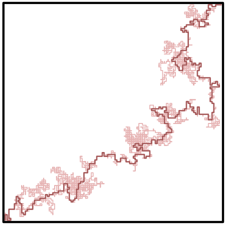
$\overset{?}{\longrightarrow}$ chordal $\mathrm{SLE}_{\kappa=\kappa(Q)}$

as lattice mesh $\delta \searrow 0$

- sample configuration and find the curve (interface)

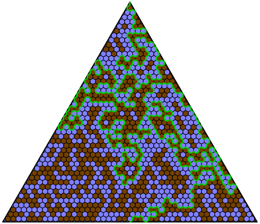
# Boundary visits of interfaces in lattice models

LERW



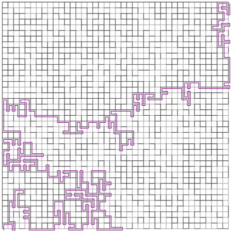
$\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=2}$

Percolation



$\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=6}$

$Q$-FK model



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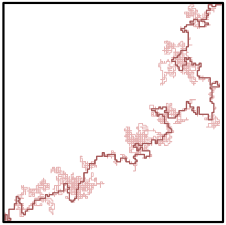
as lattice mesh $\delta \searrow 0$

- sample configuration and find the curve (interface)
- collect frequencies of boundary visits from the samples

# Boundary visits of interfaces in lattice models

LERW



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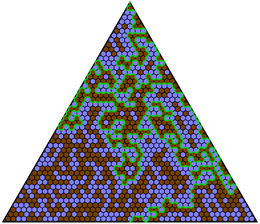
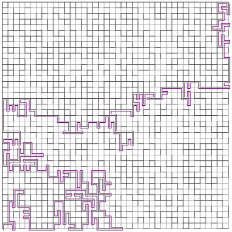
Percolation



$\longrightarrow$ chordal $\mathrm{SLE}_{\kappa=6}$

$Q$-FK model



$\stackrel{?}{\longrightarrow}$ chordal $\mathrm{SLE}_{\kappa=\kappa(Q)}$

as lattice mesh $\delta \searrow 0$

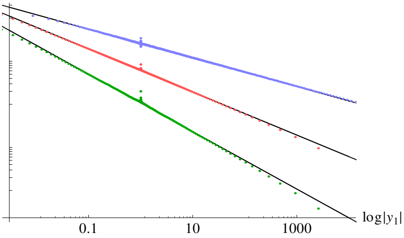
- sample configuration and find the curve (interface)
- collect frequencies of boundary visits from the samples
- $\mathsf{P}[\gamma \text{ visits } x_1, \ldots, x_N] \approx \text{const.} \times \prod_j (\delta\, f'(x_j))^{\frac{8-\kappa}{\kappa}} \zeta_N(f(x_1), \ldots)$,
  where $f =$ conformal map to $(\mathbb{H}; 0, \infty)$

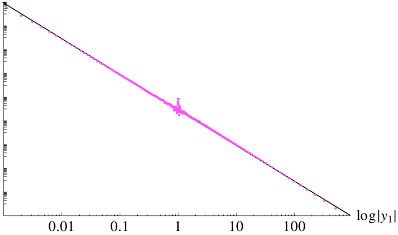
# Lattice model simulations vs. solutions

$N = 1$, one-point visit frequencies, log-log-scale

$$\zeta_1(x; y_1) \;\propto\; |y_1 - x|^{\frac{\kappa-8}{\kappa}}$$

(set $x = 0$)




blue: percolation

red: $Q = 2$ FK model

green: $Q = 3$ FK model

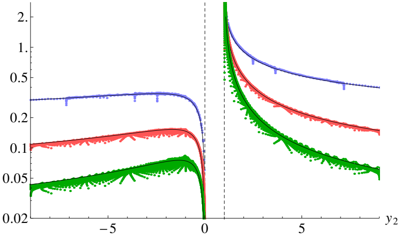
magenta: LERW

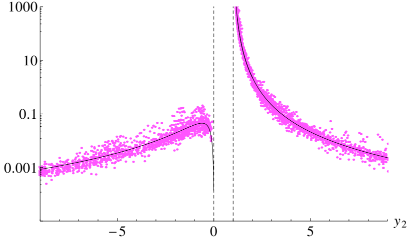
# Lattice model simulations vs. solutions

$N = 2$, two-point visit frequencies, log-scale

the 4 pieces of $\zeta_2(x; y_1, y_2)$ are hypergeometric functions

(set $x = 0$, $y_1 = 1$)




blue: percolation

red: $Q = 2$ FK model

green: $Q = 3$ FK model

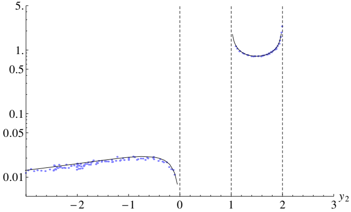
magenta: LERW

# Lattice model simulations vs. solutions

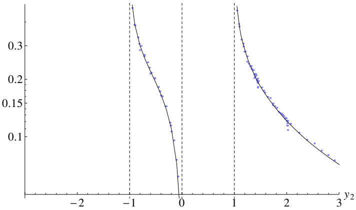
$N = 3$, three-point visit frequencies, log-scale

solving for the 8 pieces of $\zeta_3(x; y_1, y_2, y_3)$ not reducible to ODE

percolation




(set $x = 0$, $y_1 = 1$, $y_3 = 2$)



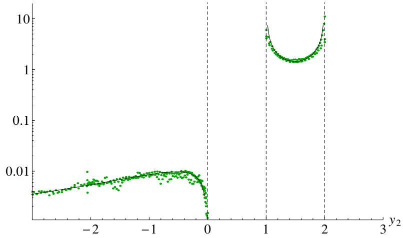

(set $x = 0$, $y_1 = 1$, $y_3 = -1$)

# Lattice model simulations vs. solutions

$N = 3$, three-point visit frequencies, log-scale

solving for the 8 pieces of $\zeta_3(x; y_1, y_2, y_3)$ not reducible to ODE

$Q = 3$ FK model



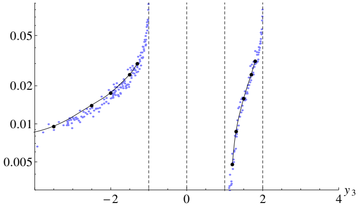

(set $x = 0$, $y_1 = 1$, $y_3 = 2$)

# Lattice model simulations vs. solutions

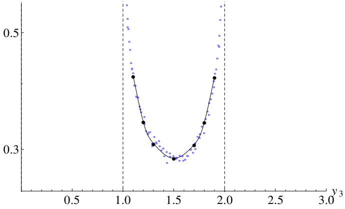
$N = 4$, four-point visit frequencies, log-scale

solving for the 16 pieces of $\zeta_4(x; y_1, y_2, y_3, y_4)$ not reducible to ODE

percolation




(set $x = 0$, $y_1 = 1$, $y_2 = -1$, $y_4 = 2$)




(set $x = 0$, $y_1 = -1$, $y_2 = 1$, $y_4 = 2$)

# THANK YOU!