From conormal varieties of Schubert varieties to loop models

A. Knutson & **P. Zinn-Justin**
LPTHE (UPMC Paris 6), CNRS

Introduction

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- My interest has been revived by the book of Maulik and Okounkov on quantum cohomology and quantum groups. Not only does it unify and formalize a lot of the work above, in the context of geometric representation theory, but it also connects to a number of hot topics, including $N = 1$ SUSY gauge theories and the AGT conjecture.
- Here we want to interpret this correspondence by means of Gröbner degenerations, which provides a more explicit and combinatorial version of them.
- This will lead us naturally to the study of exactly solvable lattice models, and more precisely loop models.

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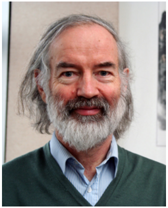
Schubert and Grothendieck polynomials



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- Lascoux and Schützenberger introduced in 1982 Schubert and Grothendieck polynomials in relation with the geometry of the flag variety (following earlier work of Bernstein, Gelfand^2; and Demazure) and Schubert calculus.
- More precisely, Schubert polynomials are identified with certain representatives of the cohomology classes of Schubert varieties.
- Here we follow Knutson and Miller (2005), who define them instead as equivariant cohomology classes of matrix Schubert varieties, and then degenerate the latter to obtain explicit formulae for these polynomials.

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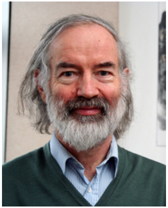

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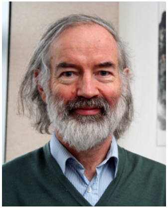

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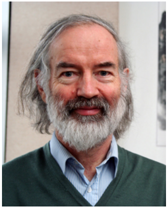

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Matrix Schubert varieties

Given an integer n and a permutation $w \in \mathcal{S}_n$, one forms a subvariety X_w of $\mathrm{Mat}(n, \mathbb{C})$ as follows:

1. (53214)

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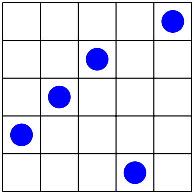
1. (53214) $\rightarrow$ 
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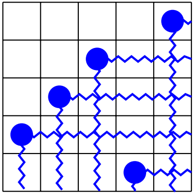
1. (53214) $\rightarrow$ 

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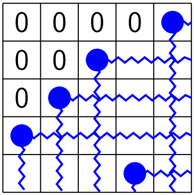
1. (53214) $\rightarrow$ 

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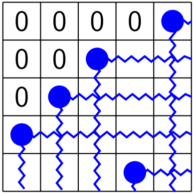
1. $(53214) \to$ 

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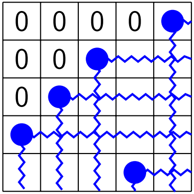

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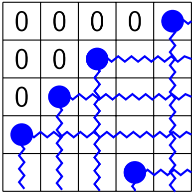
1. $(53214) \rightarrow$ 

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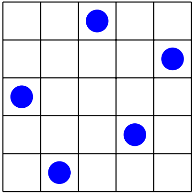
2. $(35142) \rightarrow$ 

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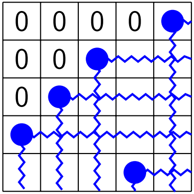
1. (53214) $\rightarrow$ 

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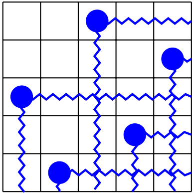
2. (35142) $\rightarrow$ 

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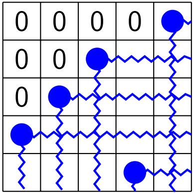
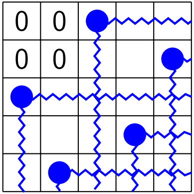
1. $(53214) \rightarrow$  $\rightarrow \left\{ (m_{ij}) : \begin{array}{l} m_{1,1}=m_{1,2}=m_{1,3}=m_{1,4} \\ =m_{2,1}=m_{2,2}=m_{3,1}=0 \end{array} \right\}$
2. $(35142) \rightarrow$ 

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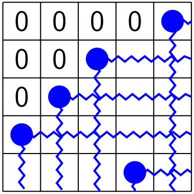
1. $(53214) \to$ 

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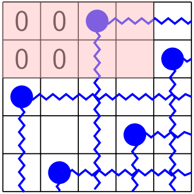
2. $(35142) \to$ 

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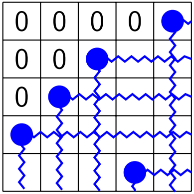
1. $(53214) \rightarrow$ 

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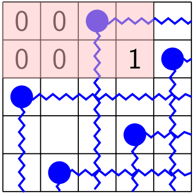
2. $(35142) \rightarrow$ 

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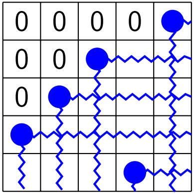
1. $(53214) \rightarrow$ 

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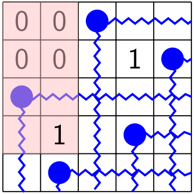
2. $(35142) \rightarrow$ 

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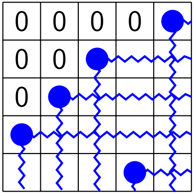
1. $(53214) \rightarrow$ 

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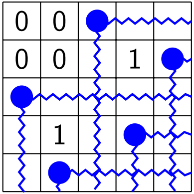
2. $(35142) \rightarrow$ 

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Multidegrees

Multidegrees are an algebraic formulation of equivariant cohomology (in the case of groups acting linearly on vector spaces).

Let V be a vector space with a linear torus action T, i.e., in practice, a basis (e_i) of V: $v = \sum v_i e_i$ with associated weights $[v_i] \in R_1$ that are degree 1 polynomials in $R = \mathbb{Z}[z_1, \ldots, z_{\dim T}]$.

To each T-invariant subscheme X of V one can associate a polynomial $\operatorname{mdeg} X \in R$ of degree the codimension of X in V. We shall not reproduce its usual definition, but only certain properties.

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Multidegrees cont'd

The following properties characterize multidegrees:

- For a coordinate subspace $W \subset V$, i.e.,

$$W = \bigoplus_{i \in I} e_i \mathbb{C} = \{v = \sum_i v_i e_i \in V : v_i = 0 \ \forall i \notin I\}$$

then

$$\text{mdeg}\, W = \prod_{i \notin I} [v_i]$$

(example: for a hyperplane, $\text{mdeg}\{v_i = 0\} = [v_i]$. 1)

- If a scheme X has top-dimensional components X_α,

$$\text{mdeg}\, X = \sum_\alpha m_\alpha \,\text{mdeg}\, X_\alpha$$

($m_\alpha \in \mathbb{Z}_{>0}$; if X is reduced, $m_\alpha = 1$)

- mdeg is invariant by flat (equivariant) degeneration...

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Gröbner degeneration

Basic idea: take the limit of the equations of X as one rescales variables.

In the "nice" case, in the limit, only one term remains in each equation $\to$ Stanley–Reisner scheme (reduced union of coordinate subspaces).

Example:

Here, degree $= 2$. (degree is a special case of multidegree)

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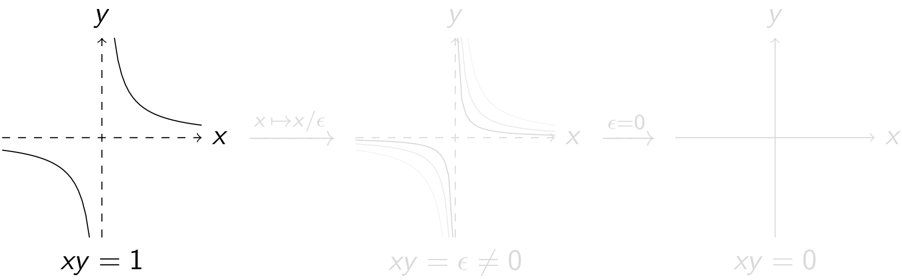

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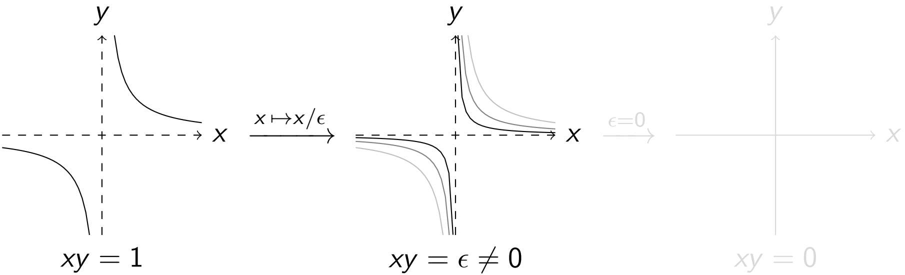

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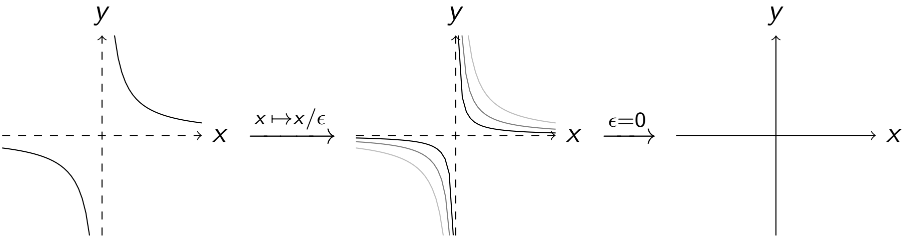

$xy = 1$ $\qquad$ $xy = \epsilon \neq 0$ $\qquad$ $xy = 0$

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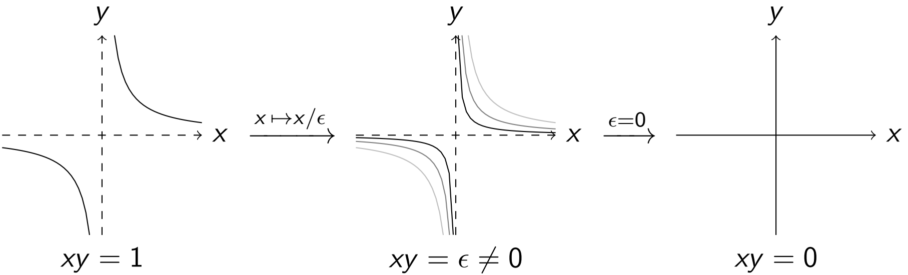

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The case of matrix Schubert varieties

The embedding space is

$$V = \mathrm{Mat}(n, \mathbb{C})$$

The torus is $2n$-dimensional, with

$$R = \mathbb{Z}[y_1, \ldots, y_n, x_1, \ldots, x_n]$$

and weights

$$[m_{ij}] = y_i - x_j \qquad i, j = 1, \ldots, n$$

We'll be computing multidegrees of matrix Schubert varieties X_w, a.k.a. (double) Schubert polynomials:

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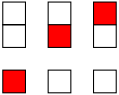
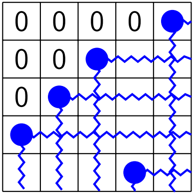
The NE/SW degeneration of matrix Schubert varieties

Theorem (Knutson, Miller)

There is a Gröbner degeneration of matrix Schubert varieties where each determinant equation is replaced with its NE/SW term.



Example 1



$$\rightarrow \left\{ (m_{ij}) : \begin{array}{l} m_{1,1}{=}m_{1,2}{=}m_{1,3}{=}m_{1,4} \\ {=}m_{2,1}{=}m_{2,2}{=}m_{3,1}{=}0 \end{array} \right\}$$

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Back

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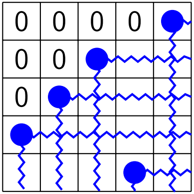

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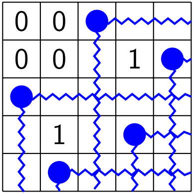

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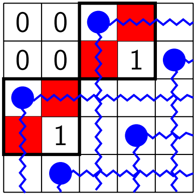

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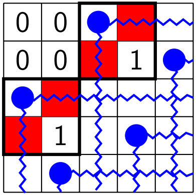

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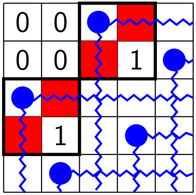

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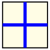
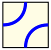
Pipedreams

Represent each coordinate subspace by a diagram in the $n \times n$ square, where each zero variable is replaced with a  and each free variable is replaced with a .

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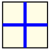
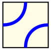
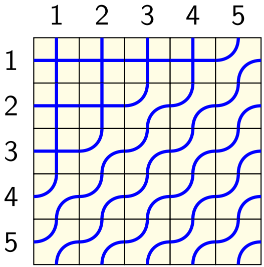
Represent each coordinate subspace by a diagram in the $n \times n$ square, where each zero variable is replaced with a  and each free variable is replaced with a .

1. $(53214) \rightarrow$ 

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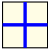
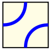
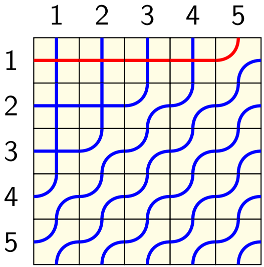
Represent each coordinate subspace by a diagram in the $n \times n$ square, where each zero variable is replaced with a  and each free variable is replaced with a .

1 (53214) $\rightarrow$ 

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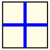
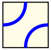
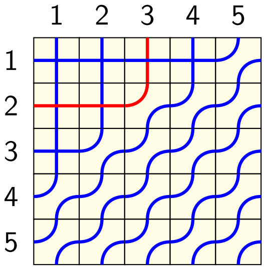
Represent each coordinate subspace by a diagram in the $n \times n$ square, where each zero variable is replaced with a  and each free variable is replaced with a .

1. $(53214) \to$ 

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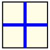
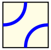
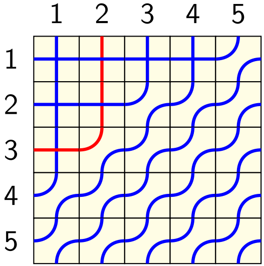
Represent each coordinate subspace by a diagram in the $n \times n$ square, where each zero variable is replaced with a  and each free variable is replaced with a .

1 $(53214) \rightarrow$ 

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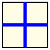
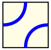
Represent each coordinate subspace by a diagram in the $n \times n$ square, where each zero variable is replaced with a  and each free variable is replaced with a .

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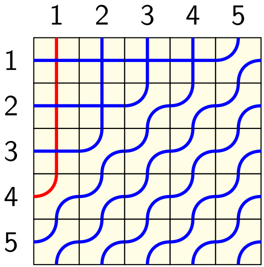

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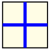
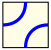
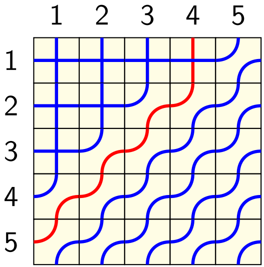
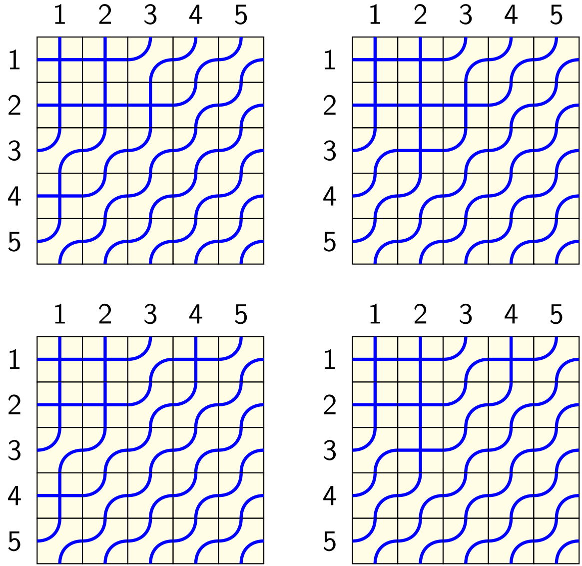
Represent each coordinate subspace by a diagram in the $n \times n$ square, where each zero variable is replaced with a  and each free variable is replaced with a .

1 (53214) $\rightarrow$ 

More pipedreams

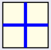
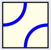
2

$(35142) \rightarrow$



Pipedreams: general case

Definition

A (reduced) pipedream is a $n \times n$ square picture made of  and  such that any two lines cross at most once.

Theorem (Knutson, Miller)

The NE/SW degeneration of a matrix Schubert variety produces a reduced union of coordinate subspaces which are in one-to-one correspondence with pipedreams representing its permutation.

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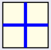
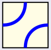
A (reduced) pipedream is a $n \times n$ square picture made of  and  such that any two lines cross at most once.

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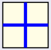
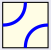
A (reduced) pipedream is a $n \times n$ square picture made of  and  such that any two lines cross at most once.

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Generalizations

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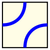
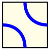
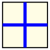
Generalizations cont'd

- It is natural to introduce three plaquettes: , ,  (and more?)
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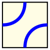
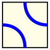
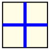
- It is natural to introduce three plaquettes: , ,  (and more?)
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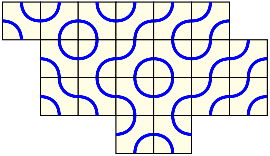

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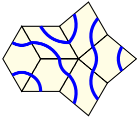

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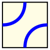
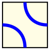
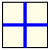
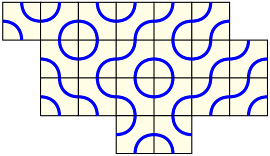
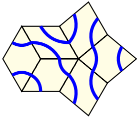
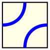
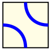
Generalizations cont'd

- It is natural to introduce three plaquettes: , ,  (and more?)
- Also, one may want more general shapes of domains:



or even



- As a first step we shall consider only  and .

From Z-lattices to crossing link patterns

Definition

A (planar) quadrangulation is a Z-lattice iff it is simply connected and its dual map, viewed as a collection of intersecting lines, has no closed loops, no two lines crossing twice and no self-intersection.

Its dual therefore defines a fixed-point-free involution of the exterior midpoints (a.k.a. chord diagram, or crossing link pattern), denoted $\mathcal{D}$. The number of boxes of the domain is also the number of crossings $|\mathcal{D}|$ of $\mathcal{D}$.

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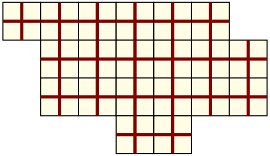

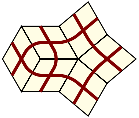

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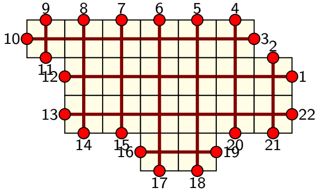
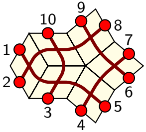
Z-lattices cont'd

One numbers all external edges from 1 to N. Then, each line connecting i to j, $i < j$ gets: (1) an orientation $i \to j$ and (2) a parameter z_i.





This allows to define unambiguously the weight of a plaquette:

$$= \begin{cases} \hbar - y + x \\ y - x \end{cases}$$

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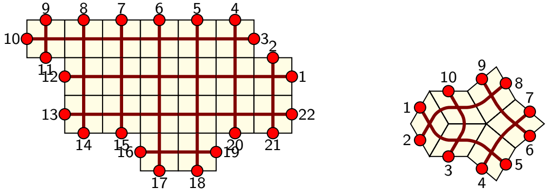

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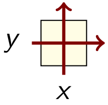

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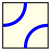
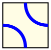
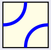
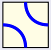




Loop configurations, link patterns

Definition

- A loop configuration of a Z-lattice D is a choice of  or  on each plaquette of D.
- A link pattern is a planar pairing inside a disk of N points on its boundary.
- A link pattern is admissible for a Z-lattice D if it can be obtained as the connectivity of boundary points of a loop configuration of D.
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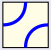
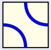
- A loop configuration of a Z-lattice D is a choice of  or  on each plaquette of D.
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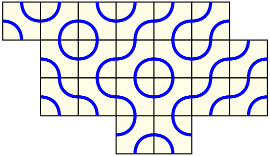

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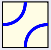
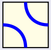
- A loop configuration of a Z-lattice D is a choice of  or  on each plaquette of D.
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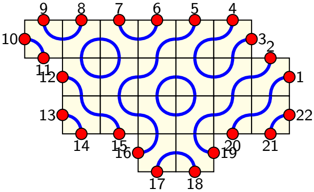

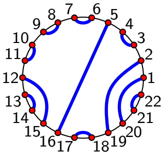

Theorem (Knutson, ZJ, '15?)

Given a crossing link pattern $\mathcal{D}$ of size $2N$, there exists an affine scheme $X_{\mathcal{D}}$ in $T^*V_{\mathcal{D}} = V_{\mathcal{D}} \times V_{\mathcal{D}}^*$ where $V_{\mathcal{D}} \cong \mathbb{C}^{|\mathcal{D}|}$, such that

1. *The irreducible components X_π of $X_{\mathcal{D}}$ are naturally indexed by link patterns $\pi \leq \mathcal{D}$.*
2. *Each X_π is Lagrangian.*

   Now let D be a Z-lattice associated to $\mathcal{D}$.
3. *There is a torus $(\mathbb{C}^\times)^{N+1} \supset (\mathbb{C}^\times)^N_{symp}$ acting on $T^*V_{\mathcal{D}}$ such that*

$$\operatorname{mdeg} X_\pi = \sum_{\substack{\text{loop configurations in } D\\ \text{boundary connectivity } \pi}} \text{(product of weights of plaquettes) } 2^{\#\bigcirc}$$

4. *There is a (symplectic, torus-equivariant, partial) Gröbner degeneration of $X_{\mathcal{D}}$ such that each term in the sum above is the multidegree of one piece of the degeneration.*

Remark: the actual theorem provides the equations of the scheme, of the torus action, of the irreducible components and of the degeneration. . .

General construction

1. Start from the orbital scheme:

$$\mathcal{O} = \{M^2 = 0,\ M \text{ upper triangular } 2N \times 2N\}$$

2. Intersect it with a certain translate of a linear subspace

$$X_{\mathcal{D}} = \mathcal{O} \cap (\mathcal{D}_< + (\mathfrak{b} \cdot \mathcal{D}_<)^\perp)$$

   where $\mathcal{D}_<$ is the upper triangle of the involution matrix of $\mathcal{D}$. (reminiscent of Slodowy or MV slice – transversality!)
3. The torus $(\mathbb{C}^\times)^{N+1}$ is a certain subtorus of $(\mathbb{C}^\times)^{2N+1}$ acting by conjugation by diagonal matrices and scaling.
4. Embed it $X_{\mathcal{D}}$ inside $T^* V_{\mathcal{D}}$ by picking $2|\mathcal{D}|$ "relevant" variables. (in particular $\mathbb{C}^\times$ acts by scaling of the fiber)

- We know defining equations for each $X_{\mathcal{D}}$ and its components X_π.
- The X_π, being Lagrangian, irreducible and conical in the fiber, are conormal varieties of certain varieties that we can describe (among which, [partial] 321-avoiding matrix Schubert varieties, and closures of certain Fomin–Zelevinsky double Bruhat cells).

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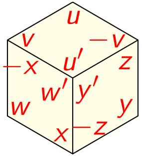
The hexagon



$-uv+yz, -uv+wx,$
$uv^2+wy, u(vz+w), u(vx+y)$
$[vz+w+w', vx+y+y', xz+u+u']$
5 components

$u\to 0,\ v\to\infty,\ uv$ fixed:
$-uv+yz, -uv+wx,$
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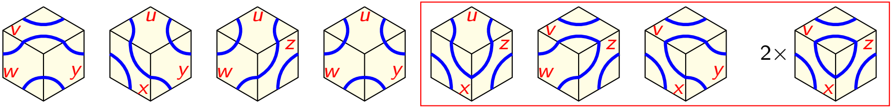
8 (linear) components:



Same equations:
$-u'v+y'z, -u'v+w'x,$
$u'v^2+w'y', u'(vz+w'), u'(vx+y')$
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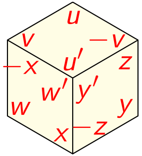

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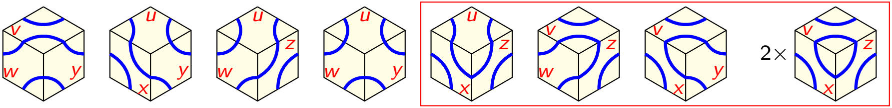

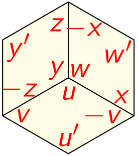

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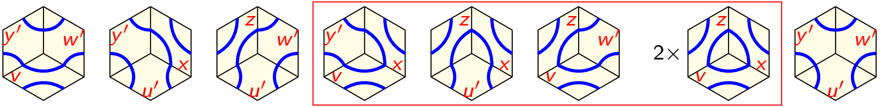

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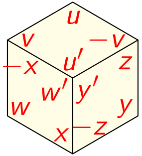

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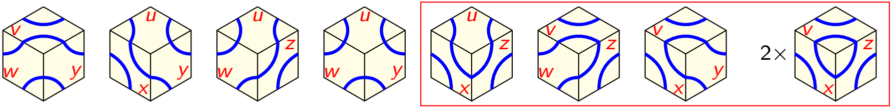

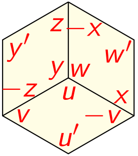

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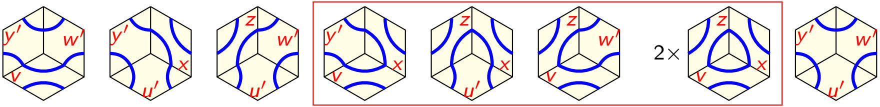

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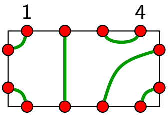
Grassmannian case

Special case: rectangular domain, link patterns of the form bottom-(left,top,right), top-(left,top,bottom):



Then the X_π are conormal varieties of (matrix) Schubert varieties of the Grassmannian $Gr(k, n)$.

See also somewhat related content in [Maulik, Okounkov, section 11.2.5] (up to loop model/link patterns $\to$ XXX/spins).

Also, in this case, the boundary conditions for the loop models are nothing but partial Domain Wall Boundary Conditions, or equivalently, define an Offshell Bethe state. (or Onshell with infinite twist).

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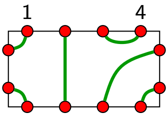

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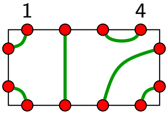

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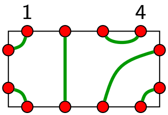

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Gröbner degeneration for the loop model

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In the rectangular case, it can also be described as:

- it is the NE/SW degeneration on the variables (m_{ij}).
- it is the NW/SE degeneration on the variables (c_{ij}).
- it preserves the symplectic structure.

Here, only partial degeneration: not all equations become monomial.

Geometrically however, all seems OK: the degeneration is a union of coordinate subspaces.

$\rightarrow$ nonreduced union of coordinate subspaces!

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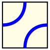
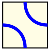
Gröbner degeneration cont'd

The reduced equations for the D-degeneration of $X_{\mathcal{D}}$ are:
$m_p c_p = 0 \ \forall p \in D$.
$\rightarrow$ for each $p \in D$ one has to make a choice:
either $c_p = 0$ , or $m_p = 0$ , i.e., each piece corresponds to a loop configuration.

At the level of multidegrees, we get

$$\operatorname{mdeg} X_{\mathcal{D}} = \sum_{\textit{pieces}} (\text{multiplicity}) \times \prod (\text{weight of eqs})$$

where weight$(m_p) = y(p) - x(p)$, weight$(c_p) = \hbar - y(p) + x(p)$.

Punch line: multiplicity $= 2^{\#\bigcirc}$.

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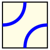
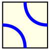
either $c_p = 0$ , or $m_p = 0$ , i.e., each piece corresponds to a loop configuration.

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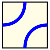
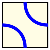
either $c_p = 0$ , or $m_p = 0$ , i.e., each piece corresponds to a loop configuration.

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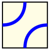
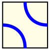
either $c_p = 0$ , or $m_p = 0$ , i.e., each piece corresponds to a loop configuration.

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Brauer loop model

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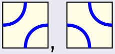
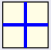
- A degenerate Brauer loop configuration of D is a choice of , or  on each plaquette of D such that no two lines cross twice and no line crosses itself.

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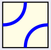
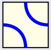
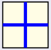
- A degenerate Brauer loop configuration of D is a choice of ,  or  on each plaquette of D such that no two lines cross twice and no line crosses itself.

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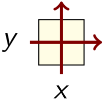

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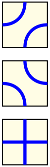

Theorem (Knutson, ZJ, '15?)

For any pairs of crossing link patterns $\pi \le \mathcal{D}$, there exists a variety $Y_\pi \subset T^* V_{\mathcal{D}}$ such that

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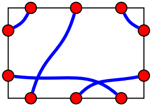
Example ($n = 3,\ k = 2$)



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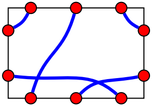

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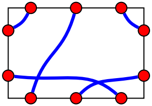

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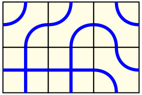

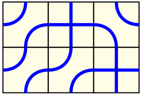

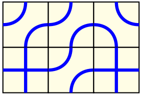

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- The “Gröbner” approach leads to a direct geometric interpretation of the partition function of exactly solvable lattice models, as well as of the Yang–Baxter equation.

- There are many possible generalizations of this work: more general loop models (including the full Brauer loop model); higher rank; other boundary conditions; trigonometric solutions of YBE (K-theory) (DONE!), and elliptic (elliptic cohomology – see Andrei’s talk!), etc.

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