Airy diffusions and $N^{1/3}$ fluctuations in the 2D and 3D Ising models

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joint work with

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The content

1. 3D and 2D Ising models.
2. Random walks and Airy diffusion.
3. Airy diffusion as a limit of the transfer matrix semigroup.

3D Ising model with $(\pm)$ boundary condition $\bar{\sigma}_{\pm} \equiv \pm 1$:

$$-H_V(\sigma_V | \bar{\sigma}_{\pm}) = \sum_{x \sim y \ \in V} \sigma_x \sigma_y \pm \sum_{x \in \partial V} \sigma_x .$$

The Gibbs state in V at the temperature β^{-1} is given by

$$\mu_{\pm}(\sigma_V) = \frac{1}{Z(V,\beta)} \exp\left\{-\beta H_V(\sigma_V | \bar{\sigma}_{\pm})\right\}.$$

We take $\beta > \beta_{cr}$.

We want to make the two phases to coexist in the same box. So we introduce the magnetization

$$M(\sigma_V) = \sum_{x \in V} \sigma_x$$

and consider the conditional distribution

$$\mu_- (\cdot | M(\cdot) = m |V|),$$

called ‘canonical ensemble’. If $m = -\frac{1}{2}$, say, then the volume of the $(+)$-droplet is $\approx \frac{1}{4} |V|$.

We want to study the shape of the giant component of the $(+)$-phase.
2D case – Wulff construction: a global shape from local interaction, R. Dobrushin, R. Kotecký, S. S. 1992.
3D case – The Wulff construction in three and more dimensions, T. Bodineau, 1999; On the Wulff Crystal in the Ising Model, R. Cerf, A. Pisztora, 2000.

We want to study the evolution of the droplet as m increases. To see it better we change the setting:

$$-H_V(\sigma_V|\bar{\sigma}_{pm}) = \sum_{x\sim y\ \in V} \sigma_x\sigma_y + \sum_{x\in\partial_\downarrow V} \sigma_x - \sum_{x\in\partial_\uparrow V} \sigma_x.$$

We consider

$$\mu_{pm}(\sigma_V) = \frac{1}{Z(V,\beta)} \exp\{-\beta H_V(\sigma_V|\bar{\sigma}_{pm})\},$$

and we study

$$\mu_{pm}\left(\cdot|M(\cdot) = aN^2\right)$$

as a function of $a \geq 0$; $V = N \times N \times N$.

Dima Ioffe, S. S.: Ising model fog drip: the first two droplets, In: "In and Out of Equilibrium 2", Progress in Probability 60, 2008.
Dima Ioffe, S. S.: Ising model fog drip: the shallow puddle, $o(N)$ deep. Actes des rencontres du CIRM, (2010)
Dima Ioffe, S. S.: Formation of Facets for an Effective Model of Crystal Growth

Look on the blackboard.

2D Ising model

2D Ising model with $(-)$ boundary condition $\bar{\sigma}_- \equiv -1$ and competing magnetic field $h > 0$:

$$-H_V\left(\sigma_V|\bar{\sigma}_-\right) = \sum_{x\sim y\ \in V} \sigma_x\sigma_y + h\sum_{x\in V}\sigma_x - \sum_{x\in\partial V}\sigma_x.$$

The Gibbs state in V at the temperature β^{-1} is given by

$$\mu\left(\sigma_V\right) = \frac{1}{Z\left(V,\beta\right)}\exp\left\{-\beta H_V\left(\sigma_V|\bar{\sigma}_-\right)\right\}.$$

We take $\beta > \beta_{cr}$.

2D Ising model

In order that the magnetic field h and the boundary condition $\bar{\sigma}_-$ have the same influence in a box $V_N = N \times N$ it has to be that $hN^2 \sim N$, i.e. $h \sim 1/N$.

In R.H. Schonmann and S.S:

Constrained variational problem with applications to the Ising model, J. Stat. Phys. (1996)

we have shown that there exists a function $B_c(\beta)$, such that the following happens:

if $h = B/N$ with $B < B_c(\beta)$, then the boundary condition wins, and we see in V_N the 'minus-phase';

if $h = B/N$ with $B > B_c(\beta)$, then the magnetic field wins, and we see in V_N a droplet W_N of 'plus-phase'. This droplet has its asymptotic shape.

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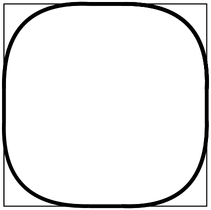
2D Ising model



The droplet in the box.

2D Ising model

The fluctuations of the droplet boundary along the wall are of the order of $N^{1/3}$. This was established in
Pietro Caputo, Eyal Lubetzky, Fabio Martinelli, Allan Sly and Fabio Lucio Toninelli: The shape of the $(2+1)$D SOS surface above a wall, http://arxiv.org/pdf/1207.3580.pdf
for SOS model, and the same methods apply for the Ising model at low temperatures.
They were able to show that for every $\varepsilon > 0$ the contour stays in the strip $N^{1/3+\varepsilon}$, and does not fit the strip $N^{1/3-\varepsilon}$, as $N \to \infty$.
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The scaling limit

We show that after the vertical scaling by $\frac{N^{1/3}}{\left(\beta e^{\beta}\right)^{1/3}}$ and horizontal scaling by $\frac{N^{2/3}e^{\beta/3}}{(\beta)^{2/3}}$ we will see in the limit $N \to \infty$ the stationary diffusion process

$$dX(t) = a(X(t))dt + db_t$$

with the drift

$$a(x) = [\ln A(x)]' = \frac{A'(x)}{A(x)}.$$

The function $A(x)$, $x > 0$ is given by

$$A(x) = \frac{Ai(-\omega_1 + x)}{Ai'(-\omega_1)},$$

where $Ai(\cdot)$ is the Airy function, and $-\omega_1$ is its first zero.
The generator is given by

$$L\varphi = \frac{1}{2}\frac{1}{A^2}\frac{d}{dx}\left(A^2\frac{d}{dx}\varphi\right).$$

This diffusion process stays positive and has the unique stationary measure with density $[A(x)]^2$.

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The scaling limit

The function $A(x)$ is the leading eigenfunction of the operator $-\frac{d^2}{dx^2}+x$ on $\mathbb{R}^+$ with zero Dirichlet b.c. at $x=0$.

This process first appeared in the paper by P. Ferrari and H. Spohn: Constrained Brownian motion: fluctuations away from circular and parabolic barriers, The Annals of Probability, 2005.

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The scaling limit

In case of $n > 1$ interfaces the operator

$$-\frac{d^2}{dx^2} + x$$

is replaced by

$$-\frac{d^2}{dx_1^2} - \ldots - \frac{d^2}{dx_n^2} + x_1 + \ldots + x_n$$

on $0 \leq x_1 \leq \ldots \leq x_n$ with zero b.c. on the boundary of the chamber.

The scaling limit

Let $\varphi_1 = A, \varphi_2, ..., \varphi_n$ are the first eigenfunctions of the Sturm–Liouville operator $-\frac{d^2}{dx^2} + x$ with zero boundary condition. Then the function

$$\det ||\varphi_i (x_j)||$$

is its principal eigenfunction, with the eigenvalue given by the sum of the first n eigenvalues of $-\frac{d^2}{dx^2} + x$. The square of this function,

$$(\det ||\varphi_i (x_j)||)^2$$

is proportional to the stationary distribution of the limiting n-dimensional diffusion process.

Consider a random walk $\mathbb{X} = (X_0 = 0, X_1, X_2, ..., X_N = 0)$ and a convex function $V \geq 0$ on $\mathbb{R}^1$, $V(0) = 0$. Let $\mathbb{V}(\mathbb{X}) = \sum V(X_j)$.
We study the asymptotic properties of $\mathbb{X}$ under the distribution

$$\mathbb{P}_N \{\mathbb{X}\} \sim \exp\{-\lambda_N \mathbb{V}(\mathbb{X})\} \prod_{j=0}^{N-1} p(X_{j+1} - X_j).$$

Let $V(x) \sim x^{\alpha}$ as $x \to \infty$, $V(x) \sim |x|^{\gamma}$ as $x \to -\infty$, with $\alpha \leq \gamma$. (Ising: $\alpha = 1$, $\gamma = +\infty$.) Define the height $H_N = H_N(\lambda_N) > 0$ as the unique positive solution of the equation:

$$\lambda_N V(H_N) H_N^2 = 1.$$

This is the condition of the survival of the excursion of the size (H_N^2, H_N). (We assume that $H_N^2(\lambda_N) \ll N$.) Then under height scaling by H_N and time scaling by H_N^2 the process converges weakly to:

The diffusion with the generator

$$\mathcal{L} = \frac{1}{2}\frac{1}{A^2}\frac{d}{dx}\left(A^2\frac{d}{dx}\varphi\right),$$

where A is the ground state of the Schrodinger operator

$$-\frac{d^2}{dx^2} + |x|^\alpha$$

on $\mathbb{R}^1$, if $\gamma = \alpha$, or the ground state of the Schrodinger operator

$$-\frac{d^2}{dx^2} + x^\alpha$$

on $\mathbb{R}_+$ with zero boundary condition at $x = 0$ if $\gamma > \alpha$. The stationary distribution is $\sim A^2(x)$, and the drift is $[\ln A(x)]'$.

For example, if $\alpha = 1$, $\gamma = +\infty$, $\lambda_N = \frac{1}{N}$ we get Ferrari-Spohn diffusion, after height scaling $H_N = N^{1/3}$ and time scaling $N^{2/3}$. Here $A(x) \sim Ai(x - \omega_1)$, $x \geq 0$, and $-\omega_1$ is the maximal root of $Ai(\cdot)$.

If $\alpha = \gamma = 1$, $\lambda_N = \frac{1}{N}$ we get after the same height scaling $H_N = N^{1/3}$ and time scaling $N^{2/3}$ the diffusion with the function

$$A(x) = Ai(\varpi_1 + |x|),$$

where ϖ_1 is the location of the rightmost maximum of $Ai(\cdot)$.

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If $\alpha = \gamma = 2$, we have after height scaling $H_N(\lambda_N)$ and time scaling $H_N^2(\lambda_N)$ the OU diffusion, with

$$A(x) \sim \exp\left\{-x^2\right\},$$

$x \in \mathbb{R}^1$,
while if $\alpha = 2$, $\gamma = +\infty$, we have

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1D Gibbs fields

Let $U(u,v) = U(u-v)$ be a n.n. interaction, $u, v \in \mathbb{Z}^1$. Consider the Gibbs field $\mathbb{X}_0$, corresponding to the Hamiltonian $\mathbf{H}(X) = \sum_s U(X_s, X_{s+1})$. If the interaction is balanced:

$$\sum_v v e^{-U(v)} = 0,$$

then its scaling limit is the 1D Brownian motion. Assume $\|U\| = \sum_v e^{-U(v)} = 1$. The variance:

$$\sigma^2(U) = \sum_v v^2 e^{-U(v)}.$$

We need to add to the Hamiltonian the additional stabilizing self-interaction, $V(s)$. So we change to $\mathbf{H}(X) = \sum_s U(X_s, X_{s+1}) + \sum_s V(X_s)$. We suppose that

$$V(u) = +\infty \text{ for } u < 0,\ V(0) = 0,\ \lim_{u\to\infty} V(u) = +\infty.$$

When we weaken the self-interaction V, by passing to λV, with λ small and then take the limit $\lambda \to 0$, the corresponding Gibbs field starts to diverge. Such a divergence has a universal character, and depends on very few details of the stabilizing self-interaction V.

Define the value H_λ by

$$H_\lambda^2 \lambda V(H_\lambda) = 1,$$

and suppose that $H_\lambda \to \infty$ as $\lambda \to 0$, and that the limiting function

$$q(r) = \lim_{\lambda \to 0} H_\lambda^2 \lambda V(rH_\lambda)$$

exists. Let $\mathbb{X}_\lambda = \{X_s\}$ be the (infinite-volume) 1D Gibbs field, corresponding to the Hamiltonian

$$\mathbf{H}(X) = \sum_s U(X_s, X_{s+1}) + \sum_s \lambda V(X_s).$$

1D Gibbs fields

The Gibbs field $\mathbb{X}_\lambda$ exists and is unique. Let $\mathbb{P}_\lambda\{\cdot\}$ denote the corresponding state; it is a Markov chain. It diverges as $\lambda \to 0$. But its scaling limit exists, as $\lambda \to 0$. Namely, let $\mathbf{x}_\lambda$ be the result of scaling of the random field $\mathbb{X}_\lambda$ by a factor H_λ vertically and by H_λ^2 horizontally. Then as $\lambda \to 0$, the $\left(H_\lambda, H_\lambda^2\right)$-rescaled process $\mathbf{x}_\lambda$ converges weakly to a certain diffusion process $\mathbf{x}_{\sigma,q}$. It is defined by some diffusion operator $G_{\sigma,q}$, which in turn is a generator of the corresponding diffusion semigroup $S_{\sigma,q}^t$.

Our 1D Gibbs field is naturally associated with the transfer matrix T_λ with matrix elements

$$T_\lambda(u,v) = \exp\left\{-\frac{1}{2}\left(\lambda V(u) + \lambda V(v)\right) - U(u-v)\right\}.$$

The corresponding (discrete time t) transfer matrix semigroup T_λ^t is not stochastic, of course. The relation between our Markov chain and the semigroup T_λ^t is the following:

Let $\phi_\lambda > 0$ be the unique positive right eigenfunction of T_λ, it corresponds to the principal eigenvalue E_λ of T_λ. (The free energy then is $\ln E_\lambda$.) The transition probabilities P of our Markov chain (which corresponds to the semigroup S_λ^t):

$$P(u,v) = \exp\left\{-\frac{1}{2}(\lambda V(u) + \lambda V(v)) - U(u-v)\right\} \frac{\phi_\lambda(v)}{E_\lambda \phi_\lambda(u)}.$$

The n-step transition probabilities are given by

$$P^{(n)}(u,v) = \frac{\phi_\lambda(v)}{E_\lambda^n \phi_\lambda(u)} T_\lambda^n(u,v).$$

One can check that the (H_λ, H_λ^2)-rescaling of the operator $T_\lambda - I$ converges, as $\lambda \to 0$ to the operator

$$L = \frac{\sigma^2}{2}\frac{d^2}{dx^2} - q(x).$$

The operator L generates the semigroup

$$T^t = \exp\{-tL\}.$$

By Trotter-Kurtz, the rescaled discrete semigroup T_λ^t converges to the continuous time semigroup $T^t = \exp\{-tL\}$.
The operator L on $x \geq 0$, with zero boundary condition has all eigenvalues simple. Let φ_0 be its ground state, and $-e_0$ be the corresponding eigenvalue. Note that the function φ_0 is positive.

Doob transforms

The ground-state transform of L is the diffusion operator $G_{\sigma,q}$:

$$G_{\sigma,q}\psi = \frac{1}{\varphi_0}\left(L + e_0\right)\left(\psi\varphi_0\right) \equiv \frac{\sigma^2}{2}\frac{d^2}{dr^2}\psi + \sigma^2\frac{\varphi_0'}{\varphi_0}\frac{d}{dr}\psi.$$

Doob transforms

It generates the diffusion semigroup $S^t_{\sigma,q}$, which can be written as

$$S^t_{\sigma,q}\psi = \frac{e^{e_0 t}}{\varphi_0} T^t\left(\psi\varphi_0\right).$$

Denote by $\mathbf{x}\left(t\right) = \mathbf{x}_{\sigma,q}\left(t\right)$ the corresponding diffusion process. Since discrete semigroup T^t_λ converges to the continuous time semigroup T^t, and since

$$P^{(n)}\left(u,v\right) = \frac{\phi_\lambda\left(v\right)}{E^n_\lambda \phi_\lambda\left(u\right)} T^n_\lambda\left(u,v\right),$$

in order to conclude the convergence of S^t_λ to $S^t_{\sigma,q}$ we just need to know that the eigenfunctions ϕ_λ and the eigenvalues E_λ of T_λ converge to φ_0 and e_0. To see that, it is sufficient to prove in advance the compactness of the family $\left\{\phi_\lambda\right\}$. That implies the convergence $\phi_\lambda \to \varphi_0$ and $E_\lambda \to e_0$.

The End