Ground state scalar product for twisted XXZ spin 1/2 at $\Delta = -1/2$

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Contents

XXZ model

- The model is defined by the Hamiltonian

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- *Can we approach the correlation functions using the Slavnov determinant?*

ABA

The central object of the Algebraic Bethe Ansatz for the spin 1/2 XXZ model is the R-matrix:

$$R(\mu, t) = \begin{pmatrix} a(\mu,t) & 0 & 0 & 0 \\ 0 & b(\mu,t) & c(\mu,t) & 0 \\ 0 & c(\mu,t) & b(\mu,t) & 0 \\ 0 & 0 & 0 & a(\mu,t) \end{pmatrix}.$$

$$a(\mu, t) = \frac{q^2\mu - t}{q(\mu - t)}, \quad b(\mu, t) = 1,$$
$$c(\mu, t) = \frac{(q^2 - 1)\sqrt{\mu\, t}}{q(\mu - t)}.$$

This R-matrix satisfies the Yang-Baxter equation:

$$R_{i,j}(\mu_i, \mu_j) R_{i,k}(\mu_i, \mu_k) R_{j,k}(\mu_j, \mu_k) = R_{j,k}(\mu_j, \mu_k) R_{i,k}(\mu_i, \mu_k) R_{i,j}(\mu_i, \mu_j).$$

ABA

Introduce the L-matrix:

$$L_j(\mu, z_j) = R_{0,j}(\mu, z_j),$$

which we use to construct the monodromy matrix:

$$T(\mu) = L_N(\mu, z_N)..L_1(\mu, z_1).$$

The monodromy matrix can be written as a 2×2 matrix:

$$T(\mu) = \begin{pmatrix} A(\mu) & B(\mu) \\ \kappa C(\mu) & \kappa D(\mu) \end{pmatrix},$$

The operators A, B, C and D form the Yang-Baxter algebra with the commutation relations given by the RTT relations

$$R_{1,2}(\mu_1, \mu_2)T_1(\mu_1)T_2(\mu_2) = T_2(\mu_2)T_1(\mu_1)R_{1,2}(\mu_1, \mu_2),$$

where we used the notation $T_1(\mu) = T(\mu) \otimes Id$ and $T_2(\mu) = Id \otimes T(\mu)$.
The Hamiltonian H can be written in terms if the transfer matrix:

$$\mathcal{T}(\mu) = A(\mu) + \kappa D(\mu)$$

ABA

The *n*-particle eigenstate of the transfer matrix is:

$$\psi_n = \prod_{i=1}^{n} B(\zeta_i | z_1, .., z_N)|0\rangle,$$

where the parameters $\zeta_1, .., \zeta_n$ satisfy the Bethe equations:

$$\prod_{i=1}^{N} a(\zeta_k, z_i) \prod_{\substack{i=1 \\ i\neq k}}^{n} a(\zeta_i, \zeta_k) - \kappa \prod_{i=1}^{N} b(\zeta_k, z_i) \prod_{\substack{i=1 \\ i\neq k}}^{n} a(\zeta_k, \zeta_i) = 0, \quad k = 1, 2, ..n.$$

The eigenvalues of the transfer matrix are:

$$\tau_n(\mu) = \prod_{i=1}^{N} a(\mu, z_i) \prod_{i=1}^{n} a(\zeta_i, \mu) + \kappa \prod_{i=1}^{N} b(\mu, z_i) \prod_{i=1}^{n} a(\mu, \zeta_i).$$

The scalar products of states are defined as

$$S_n(\mu_1, .., \mu_n; \zeta_1, .., \zeta_n) = \langle 0| \prod_{i=1}^{n} C(\mu_i) \prod_{i=1}^{n} B(\zeta_i)|0\rangle.$$

Where $\zeta_1, .., \zeta_n$ are the Bethe roots, $\mu_1, .., \mu_n$ are free parameters.

Form factors

We are interested in computing the expectation values of local spin operators:

$$\langle \mathcal{O} \rangle = \langle 0 | \prod_{i=1}^{n_0} C(\mu_i) \mathcal{O} \prod_{i=1}^{n} B(\zeta_i) | 0 \rangle,$$

where $\mathcal{O}$ stands for σ_m^+, σ_m^- or σ_m^z, and the vector

$$\langle 0 | \prod_{i=1}^{n_0} C(\mu_i),$$

is the dual Bethe state when μ_i are the Bethe roots. If $\mathcal{O}$ can be written in terms of A, B, C and D we can use the Yang-Baxter algebra to obtain:

$$\langle 0 | \prod_{i=1}^{n_0} C(\mu_i) \mathcal{O} = \sum_k \theta_k \langle 0 | \prod_{i=1}^{n} C(\nu_i^{(k)}).$$

We get

$$\langle \mathcal{O} \rangle = \sum_k \theta_k S_n(\nu_1^{(k)}, .., \nu_n^{(k)}; \zeta_1, .., \zeta_n).$$

σ^z expectation value

Take σ^z_m and write it in the F-basis (Kitanine, Maillet, Terras):

$$\sigma^z_m = \prod_{i<m} \mathcal{T}(z_i)(A(z_m) - \kappa D(z_m)) \prod_{i>m} \mathcal{T}(z_i),$$

where z's are the inhomogeneities. This expression boils down to

$$\langle \sigma^z_m \rangle = 2 \prod_{i=1}^{m-1} \tau_n(z_i) \prod_{i=m+1}^{n} \tau_n(z_i) \langle 0| \prod_{i=1}^{n} C(\mu_i) A(z_m) \prod_{i=1}^{n} B(\zeta_i)|0\rangle - S_n(\{\mu\};\{\zeta\}).$$

Use the commutation relations to commute A through the B's:

$$\langle \sigma^z_m \rangle = S_n(\{\mu\};\{\zeta\}) - \sum_{a=1}^{n} f_a S_n(\{\mu\};\zeta_1,..,\hat{\zeta}_a,..,\zeta_n,z_m),$$

where f_a are some coefficients depending on the Bethe roots (μ_j and ζ_j are the Bethe roots).

We need to compute the scalar product $S_n(\{\mu\},\{\zeta\})$, with Bethe roots ζ and free parameters μ.

Slavnov determinant

Introduce the matrix:

$$\Omega_{j,k} = \frac{\partial \tau_n(\mu_k|\zeta_1, .., \zeta_n)}{\partial \zeta_j} \prod_{i=1}^{n} c^{-1}(\zeta_i, \mu_k).$$

Then the Slavnov determinant reads:

$$S_n(\mu_1, .., \mu_n; \zeta_1, .., \zeta_n) = \frac{(q^2-1)^n}{2^n} \prod_{i<j} c(\mu_j, \mu_i) c(\zeta_i, \zeta_j) \det_{1 \le j,k \le n} \Omega_{j,k}.$$

Introduce two functions:

$$F_N(x) = \prod_{i=1}^{N} (x - q^2 z_i), \qquad Q_n(x) = \prod_{i=1}^{n} (x - \zeta_i).$$

The matrix elements become $\Omega_{j,k} \propto c_{j,k}$:

$$c_{j,k} = \frac{1}{\mu_k - \zeta_j} \left(\frac{Q(q^{-2}\mu_k) F(q^4 \mu_k)}{\mu_k - q^2 \zeta_j} + \kappa \frac{Q(q^2 \mu_k) F(q^2 \mu_k)}{\zeta_j - q^2 \mu_k} \right).$$

We need to compute the determinant:

$$S_n \propto \frac{1}{\prod_{i \le j} (\zeta_j - \zeta_i)(\mu_j - \mu_i)} \det_{1 \le j,k \le n} c_{j,k}.$$

Symmetrization of the Slavnov determinant

- Our goal is to compute the Slavnov determinant at $\Delta = -1/2$ for the ground state ($N = 2n$) and twist $\kappa = q^2$.

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- At this point we know the Q-function (Di Francesco, Zinn-Justin)

$$Q(t) = q^{2n} \frac{s_{Y_{n+1}}(z_1, .., z_{2n}; t)}{s_{\tilde{Y}_n}(z_1, .., z_{2n})}.$$

s_{Y_n} and $s_{\tilde{Y}_n}$ are the Schur functions of the partitions $Y_n = \{n, n, n-1, n-1, .., 1, 1, 0\}$ and $\tilde{Y}_n = \{n, n-1, n-1, .., 1, 1, 0\}$.

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- We also know the eigenvalue of the transfer matrix:

$$\tau_n(\mu) = -q^{2n+1} \frac{F(\mu)}{F(q^2\mu)},$$

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- A symmetric function of the Bethe roots $f(\zeta_1(z_1, .., z_{2n}), .., \zeta_n(z_1, .., z_{2n}))$ can be written as a function of z's $\tilde{f}(z_1, .., z_{2n})$ explicitly using the Q-function.

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- We need to symmetrize the matrix elements $c_{j,k}$ w.r.t. the Bethe roots ζ_i.

Symmetrization of the Slavnov determinant

- Introduce the Izergin matrices:

$$a_{j,k} = \frac{1}{(\mu_k - \zeta_j)(\mu_k - q^2\zeta_j)},$$
$$b_{j,k} = \frac{1}{(\mu_k - \zeta_j)(\zeta_j - q^2\mu_k)},$$

so that

$$c_{j,k} = Q(q^{-2}\mu_k)F(q^4\mu_k)a_{j,k} + \kappa Q(q^2\mu_k)F(q^2\mu_k)b_{j,k}.$$

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- Take the matrix $a_{j,k}$ and apply the symmetrization transform ρ:

$$\rho_{i,j} = \frac{\zeta_j^{i-1}}{\prod_{k\neq j}(\zeta_j - \zeta_k)},$$

$$\mu_k^{2-i}Q(\mu_k)Q(q^{-2}\mu_k) \times \rho_{i,j}a_{j,k} = \sum_{j=1}^{n} \frac{\zeta_j^{i-1}}{\prod_{l\neq j}(\zeta_j - \zeta_l)} \frac{\mu_k^{2-i}Q(\mu_k)Q(q^{-2}\mu_k)}{(\mu_k - \zeta_j)(\mu_k - q^2\zeta_j)}$$
$$= \sum_{j=1}^{n} q^{-2}\zeta_j^{i-1}\mu_k^{2-i}\prod_{l\neq j}\frac{(\mu_k - \zeta_l)(\mu_k q^{-2} - \zeta_l)}{(\zeta_j - \zeta_l)} = f(\mu_k).$$

- The Lagrange polynomial $f(\mu_k)$ is known at $\mu_k = \zeta_i$:

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- The transform ρ brings the matrices $a_{j,k}$ and $b_{j,k}$ to the form:

$$\rho_{i,j} a_{j,k} = \mu_k^{i-2} \frac{Q(\mu_k)q^{2-2i} - Q(q^{-2}\mu_k)}{(q^2-1)Q(\mu_k)Q(q^{-2}\mu_k)},$$

$$\rho_{i,j} b_{j,k} = \mu_k^{i-2} \frac{Q(\mu_k)q^{2i-2} - Q(q^2\mu_k)}{(q^2-1)Q(\mu_k)Q(q^2\mu_k)}.$$

- After the symmetrization we obtain:

$$\tilde{c}_{i,k} = \frac{\mu_k^{i-2}}{(q^2-1)}\Big(q^{2-2i}F(q^4\mu_k) + \kappa q^{2i-2}F(q^2\mu_k)$$
$$-\big(\frac{F(q^4\mu_k)}{Q(\mu_k)}Q(q^{-2}\mu_k) + \kappa\frac{F(q^2\mu_k)}{Q(\mu_k)}Q(q^2\mu_k)\big)\Big).$$

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- Recall the T-Q relation:

$$q^n\tau(\mu)F(q^2\mu) = \frac{F(q^4\mu)}{Q(\mu)}Q(q^{-2}\mu) + \kappa\frac{Q(q^2\mu)}{Q(\mu)}F(q^2\mu).$$

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- The scalar product becomes:

$$\tilde{S}_n = \frac{1}{\prod_{i<j}(\mu_j-\mu_i)}\det_{1\le i,k\le n}\mu_k^{i-2}\left(q^{2-2i}\frac{F(q^4\mu_k)}{F(q^2\mu_k)} + q^{-2+2i}\kappa - q^n\tau(\mu_k)\right).$$

Note, this expression is valid for generic q.

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- Matrix elements $\tilde{c}_{i,k}$ become

$$\tilde{c}_{j,k} = \frac{\mu_k^{j-2}}{q^2 - 1} \left(q^{2-2j} F(q^4 \mu_k) + q^{2j} F(q^2 \mu_k) + q F(\mu_k) \right).$$

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- The functions F are the generating functions of the elementary symmetric polynomials:

$$F_n(x) = \sum_{i=0}^{2n}(-q^2)^{2n-i}x^i e_{2n-i}(z_1,..,z_{2n}),$$

$$e_k(z_1,..,z_m) = \sum_{1\le i_1<i_2<\cdots<i_k\le m} z_{i_1}z_{i_2}\dots z_{i_k},$$

$$e_k(z_1,..,z_m) = 0, \quad \text{for } k<0 \text{ or } k>m.$$

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- Substituting this into the expression $\tilde{c}_{j,k}$:

$$\tilde{c}_{j,k} = \frac{\mu_k^{j-2}q^{4n}}{q^2-1}\sum_{s=0}^{2n}(-1)^s\mu_k^s e_{2n-s}\left(q^{1-2s}+q^{2j}+q^{2+2s+2j}\right),$$

- The matrix elements $\tilde{c}$ can be rewritten as:

$$\tilde{c}_{j,k} = \frac{3q^{n-j}}{q^2 - 1} \sum_{m=1}^{n} (-1)^{j+m+1} \mu_k^{3m-3} e_{2n-3m+j+1}.$$

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- The last expression is nothing but the product of two $n \times n$ matrices

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- The determinant S_n becomes:

$$S_n = \frac{3^n q^n \prod_{i=1}^{n} \mu_i^{1/2} \zeta_i^{1/2}}{\prod_{i=1}^{n} q^{2n} F(q^2 \mu_i)} s_{Y_n}(\mu_1, .., \mu_n) s_{\tilde{Y}_n}(z_1, .., z_{2n}).$$

Normalization of the ground state

- When the parameters μ are also the Bethe roots in:

$$S_n(\mu_1, .., \mu_n; \zeta_1, .., \zeta_n) = \langle 0| \prod_{i=1}^{n} C(\mu_i) \prod_{i=1}^{n} B(\zeta_i)|0\rangle.$$

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$$\prod_{i=1}^{n} Q(q\zeta_i) = \frac{s_{\tilde{Y}_n}(z_1, .., z_{2n}) s^2_{Y'_n}(z_1, .., z_{2n})}{s_{Y_n^0}(z_1, .., z_{2n})} \prod_{i=1}^{2n} \frac{Q(z_i)}{F(z_i)}$$

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- Hence the norm of the ground state is equal to:

$$\mathcal{N}_n = \frac{s^2_{\tilde{Y}_n}(z_1, .., z_{2n}) s^2_{Y'_n}(z_1, .., z_{2n})}{s^2_{Y^0_n}(z_1, .., z_{2n}) s_{Y_{2n}}(z_1, .., z_{2n})}$$

$\langle \sigma_m^z \rangle$

Recall the equation:

$$\langle \sigma_m^z \rangle = S_n(\zeta_1, ..) - 2 \prod_{i=1}^{n} \frac{b(\zeta_i, z_m)}{a(\zeta_i, z_m)} \sum_{i=1}^{n} \frac{c(\zeta_i, z_m)}{b(\zeta_i, z_m)} \prod_{j \neq i} \frac{a(\zeta_j, \zeta_i)}{b(\zeta_j, \zeta_i)} S_n(z_m, \zeta_1, .., \hat{\zeta}_i, ..).$$

- In terms of the symmetrized scalar product $\tilde{S}_n$:

$$\langle \sigma_m^z \rangle = \tilde{S}_n \left(1 + 6q \frac{Q(z_m)Q(q^2 z_m)}{F(qz_m)} G_n \right),$$

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$$\langle \sigma_m^z \rangle \propto \det_{1 \leq j,k \leq n} \left(\zeta_k^{3j-2} F(qz_m)Q(qz_m) - 2z_m^{3j-2} F(q\zeta_k)Q(q\zeta_k) \right). \qquad (1)$$

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- Now the problem is to symmetrize this w.r.t. the Bethe roots ζ.

$\langle \sigma_m^z \rangle$

- Use the transformation ρ to symmetrize the $\langle \sigma_m^z \rangle$ determinant:

$$\langle \sigma_m^z \rangle \propto \det_{1 \leq j,k \leq n} \left(h_{j+3k-2-n} F(qz_m) Q(qz_m) - 2 z_m^{3k-2} \sum_{i=0}^{3n} (-q)^i h_{2n+j-i} \gamma_i \right).$$

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- This is an explicit formula since we know $h_i(\zeta_1, .., \zeta_n)$.
- This formula is not computationally efficient.
  We need a different symmetrization procedure instead of ρ, or we need to perform other manipulations to simplify this determinant.

Bethe roots

- Since we know the Q-function we can derive the correspondence between the pair $(e^B,\ h^B)$ and $(e^s,\ h^s)$

$$F(q^2 t)Q(q^2 t) = \sum_{k=0}^{3n}(-1)^k t^{3n-k} \sum_{j=0}^{k} e_j^B e_{k-j}^s = \sum_{k=0}^{3n}(-1)^k t^{3n-k}\gamma_k.$$

where γ's are Schur functions:

$$\gamma_{3j} = (-1)^j \frac{s_{\pi_{2j}}}{s_{\tilde{Y}}}, \quad j = 0, .., n$$

$$\gamma_{3j+1} = (-1)^j \frac{s_{\pi_{2j+1}}}{s_{\tilde{Y}}}, \quad \gamma_{3j+2} = 0, \quad j = 0, .., n-1.$$

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- Define $\delta_k = \det_{1\le i,j\le k}\gamma_{1-i+j}$, then

$$e_k^s = \sum_{i=0}^{k}\gamma_{k-i}h_i^B, \qquad h_k^s = \sum_{i=0}^{k}(-1)^i\delta_{k-i}e_i^B,$$

$$e_k^B = \sum_{i=0}^{k}\gamma_{k-i}h_i^s, \qquad h_k^B = \sum_{i=0}^{k}(-1)^{i+k}\delta_{k-i}e_i^s,$$

Conclusion

- Root of unity

  A similar computation for the systems with odd length.

  Computation of more complicated correlation functions.

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- Generic q

  Symmetrization of Slavnov determinants for other boundary conditions?

  Can we relate the symmetrized scalar product determinant and the determinants appearing in the separation of variables (SoV) approach?