Sum rule for a mixed boundary qKZ equation

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In collaboration with Jan de Gier

Outline

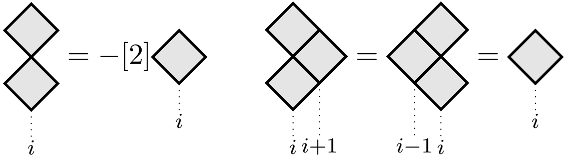
1. Mixed boundary qKZ equation
2. Sum rule
3. Bases of the Hecke algebra

Section 1

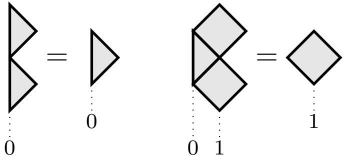
Mixed boundary qKZ equation

One boundary Temperley–Lieb algebra

- Generators $e_0, \dots e_{N-1}$
- Bulk relations $e_i^2 = -[2]e_i$, $e_i e_{i\pm1} e_i = e_i$:



- Boundary relations $e_0^2 = e_0$, $e_1 e_0 e_1 = e_1$:

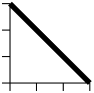
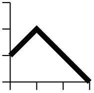
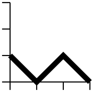


- With t-number

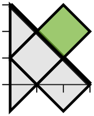
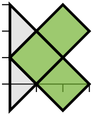
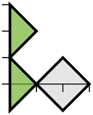
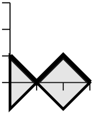
$$[u] = \frac{t^u - t^{-u}}{t - t^{-1}}.$$

Action on Ballot paths

- Ballot path: $(\alpha_0, \ldots, \alpha_N)$, with $\alpha_i \geq 0$, $\alpha_{i+1} - \alpha_i = \pm 1$, and $\alpha_N = 0$.
- Ballot paths of length $N = 3$

$\Omega =$  $\alpha_1 =$  $\alpha_2 =$ 

- Example

$e_2|\Omega\rangle =$  $=$  $=$  $=$  $= |\alpha_2\rangle$

- Will take general vector of the form

$$|\Psi(z_1, \ldots, z_N)\rangle = \sum_{\alpha} \psi_\alpha(z_1, \ldots, z_N)|\alpha\rangle$$

Mixed boundary qKZ equation

- Write qKZ equation in component form.
- For $0 \leq i \leq N-1$ (bulk and left boundary)

$$\sum_\alpha \psi_\alpha(z_1,\ldots,z_N)\Big(e_i|\alpha\rangle\Big) = \sum_\alpha \Big(T_i(-1)\psi_\alpha(z_1,\ldots,z_N)\Big)|\alpha\rangle,$$

  where $T_i(u)$ are generators of a Baxterized Hecke algebra (will be defined)
- Reflection at the right boundary

$$\psi_\alpha(\ldots,z_{N-1},z_N) = \psi_\alpha(\ldots,z_{N-1},t^3 z_N^{-1})$$

- Relates Temperley–Lieb action on Ballot paths (LHS) to Hecke algebra action on coefficient functions (RHS).

Baxterized Hecke algebra

- Bulk generators ($1 \leq i \leq N-1$):

$$T_i(u) = (tz_i - t^{-1}z_{i+1})\frac{1-\pi_i}{z_i - z_{i+1}} - \frac{[u-1]}{[u]}, \qquad \pi_i : z_i \leftrightarrow z_{i+1}$$

- Boundary generator

$$T_0(u) = k(z_1, \zeta_1)\frac{1-\pi_0}{z_1 - z_1^{-1}} - B_0(u), \qquad \pi_0 : z_1 \leftrightarrow z_1^{-1},$$

  and $k(z_1, \zeta_1)$, $B_0(u)$ are simple functions.

- These generators obey Yang–Baxter (bulk) and reflection equations (boundary).

Solution of the qKZ equation

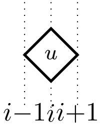
Theorem (de Gier, Pyatov, 2010)

The solutions of the qKZ equation have a factorised form

$$\psi_\alpha(z_1, \ldots, z_N) = \prod_{i,j}^{\nearrow u_{i,j}} T_i(u_{i,j})\psi_\Omega$$

The product is constructed using a graphical representation of the Hecke generators

$T_0(u) =$ 

, $T_i(u) =$ 

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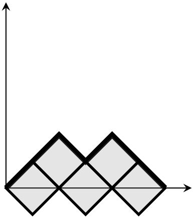
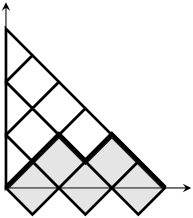
Factorised solutions

- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$



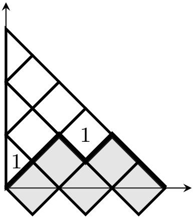
Factorised solutions

- Factorised solution for $\psi_\alpha(z_1,\ldots,z_N)$
- Fill to maximal Ballot path $\Omega = (N, N-1, \ldots, 0)$



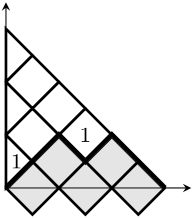
Factorised solutions

- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$
- Fill to maximal Ballot path $\Omega = (N, N-1, \ldots, 0)$
- Label corners with 1



Factorised solutions

- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$
- Fill to maximal Ballot path $\Omega = (N, N-1, \ldots, 0)$
- Label corners with 1
- Label remaining tiles by rule
  $u_{i,j} = \max\{u_{i+1,j-1}, u_{i-1,j-1}\} + 1$



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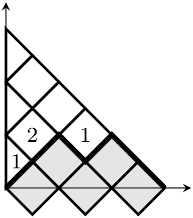

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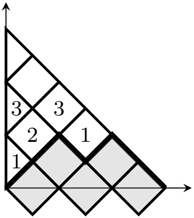

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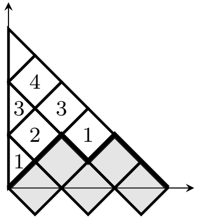

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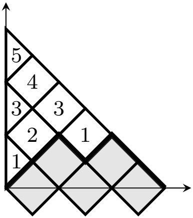

Factorised solutions

- Factorised solution for $\psi_\alpha(z_1, \ldots, z_N)$
- Fill to maximal Ballot path $\Omega = (N, N-1, \ldots, 0)$
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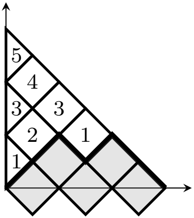


$$\psi_\alpha = T_0(1).T_1(2)T_0(3).T_3(1)T_2(3)T_1(4)T_0(5)\psi_\Omega$$

and

$$\psi_\Omega = \Delta_t^-(z_1, \ldots, z_N)\Delta_t^+(z_1, \ldots, z_N)$$

Section 2

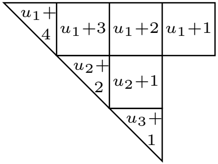
Sum rule

Consecutive integer filling

- Fill with consecutive integers along rows, e.g. for previous shape tilted by 45 °

$$\psi_{4,2,1}(u_1+1, u_2+1, u_3+1)$$

$=$



- In terms of Hecke generators

$$\psi_{a_1,\ldots,a_n}(u_1+1,\ldots,u_n+1) = \mathcal{T}_{a_n}(u_n+1)\ldots\mathcal{T}_{a_1}(u_1+1)\psi_\Omega$$

where

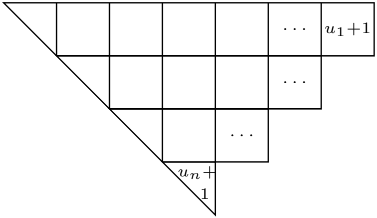
$$\mathcal{T}_a(u+1) = T_{a-1}(u+1)\ldots T_1(u+a-1)T_0(u+a)$$

- $\mathcal{T}_a(u+1)$ gives a row of length a, numbered from $u+1$.

Staircase diagram

- Call the largest such element the *staircase diagram*:

$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1) =$$



,

  where $n = \lfloor N/2 \rfloor$, $\bar{a}_i = N - 2i + 1$.

- In terms of Hecke generators

$$\begin{aligned} &\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1) \\ &= \mathcal{T}_{N-2n+1}(u_n+1)\ldots\mathcal{T}_{N-3}(u_2+1)\mathcal{T}_{N-1}(u_1+1)\psi_{\Omega}. \end{aligned}$$

Generalised sum rule

Theorem

The staircase diagram has the expansion

$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1)=\sum_{\alpha} c_\alpha \psi_\alpha(z_1,\ldots,z_N),$$

where the coefficients c_α are non-zero and are monomials in

$$y_i=-\frac{[u_i]}{[u_i+1]},\qquad \tilde{y}_i=-B_0(u_i+1).$$

- At specialization $u_i=1$, $t=\mathrm{e}^{\pm 2\pi \mathrm{i}/3}$, all coefficients $c_\alpha=1$. The sum gives the normalization of Temperley-Lieb loop model ground state vector. The sum has been computed at this point [Zinn-Justin 2007].
- Proof of the sum rule requires expanding staircase diagram in two stages.

First expansion

The first stage of the expansion gives the form of the coefficients.

Lemma (First expansion)

$$\begin{aligned}
&\psi_{a_1,\ldots,a_n}(u_1+1,\ldots,u_n+1)\\
&=\mathcal{T}_{a_n}(u_n+1)\ldots\mathcal{T}_{a_1}(u_1+1)\psi_\Omega\\
&=\prod_{i=n,n-1,\ldots,1}\left(\mathcal{T}_{a_i}(1)+y_i\mathcal{T}_{a_i-1}(1)+\tilde{y}_i\right)\psi_\Omega.
\end{aligned}$$

where

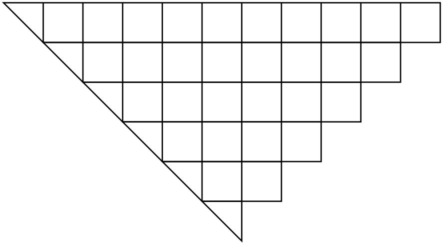
$$y_i=-\frac{[u_i]}{[u_i+1]},\qquad \tilde{y}_i=-B_0(u_i+1).$$

First expansion terms

Procedure to expand

$$(\mathcal{T}_{a_n}(1) + y_n \mathcal{T}_{a_n - 1}(1) + \tilde{y}_n) \ldots (\mathcal{T}_{a_1}(1) + y_1 \mathcal{T}_{a_1 - 1}(1) + \tilde{y}_1)\, \psi_\Omega$$

- Start from the empty outline.
- Working from top down, a row may be left empty (factor $\tilde{y}_i$), filled one short (factor y_i), or filled completely (no additional factor).
- Delete empty rows and boxes.



First expansion terms

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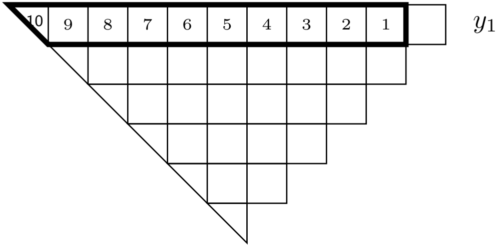

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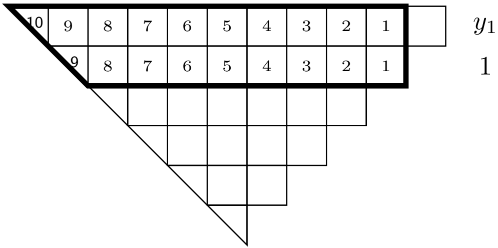

First expansion terms

Procedure to expand

$$(\mathcal{T}_{a_n}(1) + y_n\mathcal{T}_{a_n-1}(1) + \tilde{y}_n)\ldots(\mathcal{T}_{a_1}(1) + y_1\mathcal{T}_{a_1-1}(1) + \tilde{y}_1)\,\psi_\Omega$$

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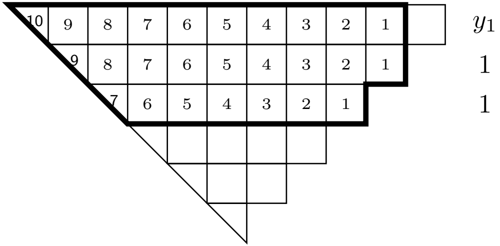

Procedure to expand

$$(\mathcal{T}_{a_n}(1) + y_n \mathcal{T}_{a_n - 1}(1) + \tilde{y}_n) \ldots (\mathcal{T}_{a_1}(1) + y_1 \mathcal{T}_{a_1 - 1}(1) + \tilde{y}_1)\, \psi_\Omega$$

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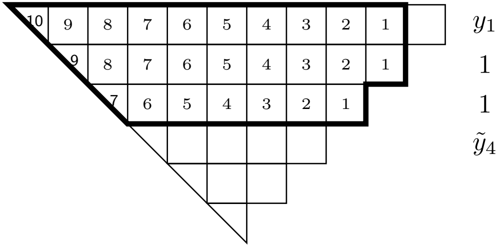

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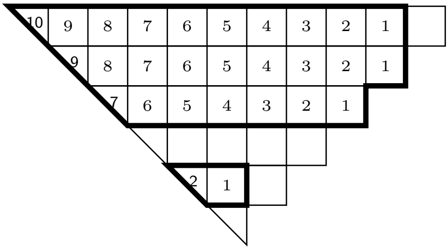

y_1
1
1
$\tilde{y}_4$
y_5

First expansion terms

Procedure to expand

$$(\mathcal{T}_{a_n}(1) + y_n \mathcal{T}_{a_n - 1}(1) + \tilde{y}_n) \ldots (\mathcal{T}_{a_1}(1) + y_1 \mathcal{T}_{a_1 - 1}(1) + \tilde{y}_1)\, \psi_\Omega$$

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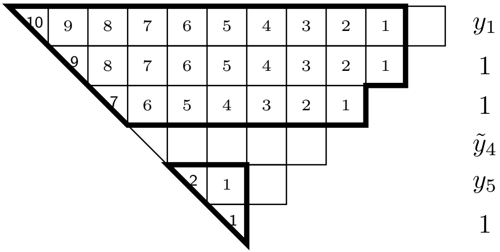

First expansion terms

Procedure to expand

$$(\mathcal{T}_{a_n}(1) + y_n \mathcal{T}_{a_n-1}(1) + \tilde{y}_n) \ldots (\mathcal{T}_{a_1}(1) + y_1 \mathcal{T}_{a_1-1}(1) + \tilde{y}_1)\, \psi_\Omega$$

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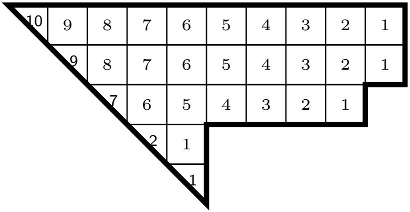


- Coefficient $y_1 \tilde{y}_4 y_5$

Second expansion

- When the resulting term is not a proper component ψ_α, a second expansion is required.

Lemma (Second expansion)

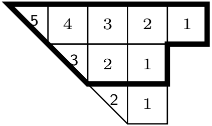
Let $\psi_\alpha(z_1, \ldots, z_N)$ be a component of the qKZ solution, with last row of length $a+1$, then

$$T_{a-1}(1) \ldots T_1(a-1) T_0(a) \psi_\alpha(z_1, \ldots, z_N) = \sum_{\alpha'} \psi_{\alpha'}(z_1, \ldots, z_N)$$

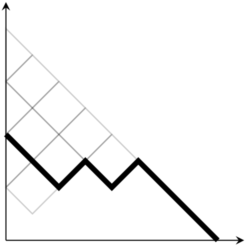
- The terms in the sum are found through a graphical rule, and all have coefficient 1.

Second expansion example

$$T_1(1)T_0(2)\psi_\alpha(z_1,\ldots,z_N) =$$



Ballot path



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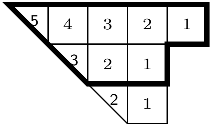

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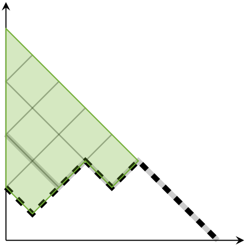
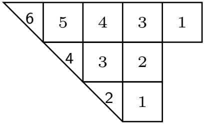


Terms



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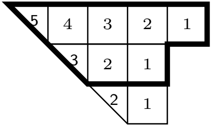

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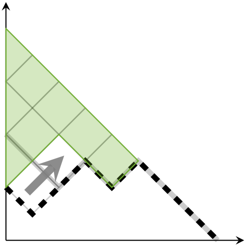

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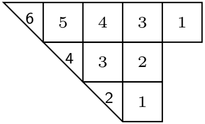

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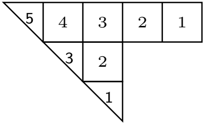
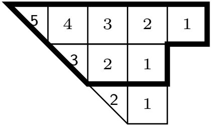
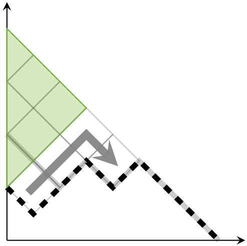


Second expansion example

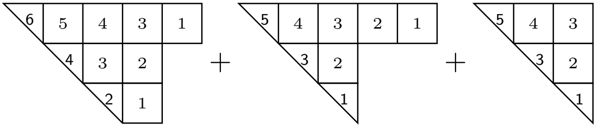
$$T_1(1)T_0(2)\psi_\alpha(z_1,\ldots,z_N) =$$



Ballot path



Terms



Proof of the sum rule

- Recall the sum rule

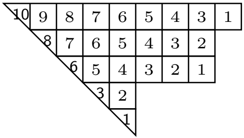
$$\psi_{\bar{a}_1,\ldots,\bar{a}_n}(u_1+1,\ldots,u_n+1) = \sum_{\alpha} c_\alpha \psi_\alpha(z_1,\ldots,z_N),$$

  where the coefficients c_α are non-zero and are monomials in y_i, $\tilde{y}_i$.
- We have shown via the two expansions that the staircase diagram can be expanded in terms of components ψ_α.
- To show that the coefficients are non-zero and monomials, we must show that each component ψ_α arises from a single term in the first expansion.

Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



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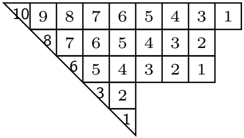
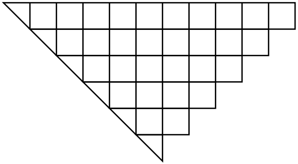


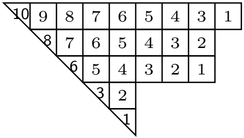
- Draw empty maximal staircase



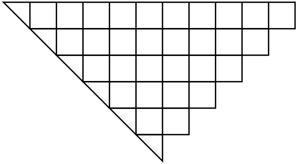
Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



- Draw empty maximal staircase
- Add rows to staircase, bottom up, in lowest place each fits



Example of the algorithm

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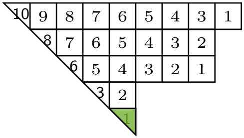
$$\psi_\alpha(z_1, \ldots, z_N) =$$



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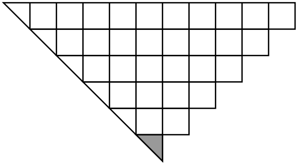

Example of the algorithm

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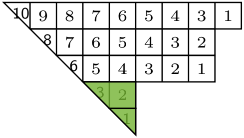

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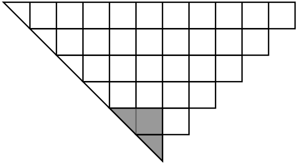

Example of the algorithm

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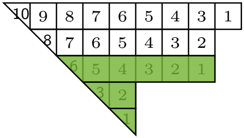

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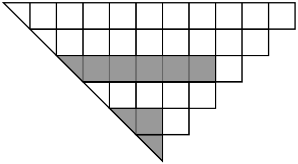

Example of the algorithm

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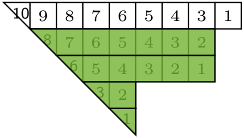

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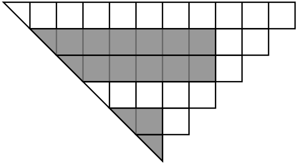

Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

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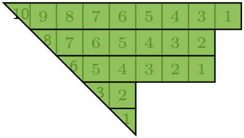

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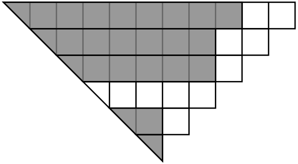
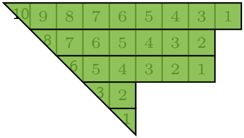


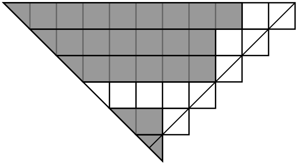
Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



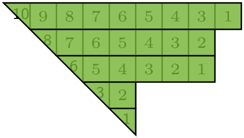
- Draw empty maximal staircase
- Add rows to staircase, bottom up, in lowest place each fits
- Draw in ribbons, starting from outer diagonal



Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



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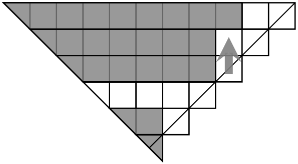
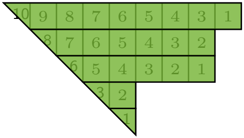


Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1, \ldots, z_N) =$$



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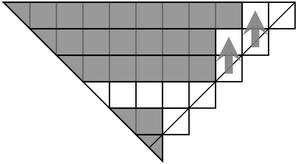
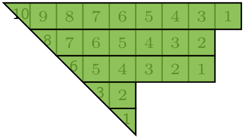


Example of the algorithm

- Work backwards from ψ_α to term from staircase expansion.

$$\psi_\alpha(z_1,\dots,z_N) =$$



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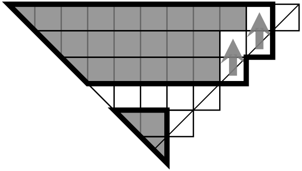


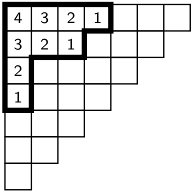
- Coefficient $c_\alpha = y_1 \tilde{y}_4 y_5$.

Section 3

Bases of the Hecke algebra

qKZ equation for type A

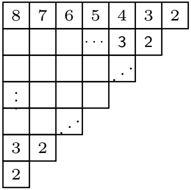
- For type A solutions given by partitions labelled with the same rule as for type B [Kirilov, Lascoux 2000, de Gier, Pyatov 2010], e.g.



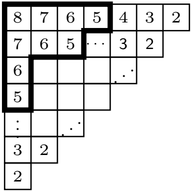
- The set of all such elements corresponds to a parabolic Kazhdan–Lusztig basis of the type A Hecke algebra.

Sum rule for type A

- Sum rule given by consecutive integer labelling [de Gier, Lascoux, Sorrell 2012]



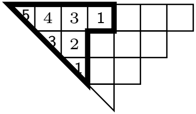
- Set of all subpartitions gives the Young basis, e.g.



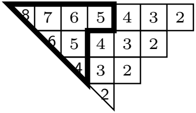
- Elements of the Young basis are specialised Macdonald polynomials.

Hecke bases for type B

- The elements of the qKZ solution correspond to the parabolic Kazhdan–Lusztig basis for the type B Hecke algebra [Shigechi 2014], e.g.



- The consecutive integer numbering corresponds* to the Young basis, e.g.



Conclusion and future work

- We have found a factorised form for a sum rule for the type B qKZ equation.
- Our construction also gives the change of basis from the Kazhdan–Lusztig to the Young basis.
- We still need to determine if the type B Young basis corresponds to a specialization of the Macdonald polynomials.
- Our main goal now is to find a way to evaluate the type B sum.