Combinatorial aspects of correlation functions of integrable models

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The Heisenberg $XX0$ model on the chain is defined by the Hamiltonian

$$\mathcal{H} \equiv -\frac{1}{2}\sum(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-}) \,.$$

The local spin operators $\sigma_k^{\pm} = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$ and σ_k^z obey the commutation rules: $[\sigma_k^+, \sigma_l^-] = \delta_{kl}\,\sigma_l^z$, $[\sigma_k^z, \sigma_l^{\pm}] = \pm 2\,\delta_{kl}\,\sigma_l^{\pm}$ (δ_{kl} is the Kronecker symbol). The spin operators act in the space $\mathfrak{H}_{M+1}$ spanned over the states $\bigotimes_{k=0}^{M} |s\rangle_k$, where $|s\rangle_k$ implies either spin "up", $|\uparrow\rangle$, or spin "down", $|\downarrow\rangle$, state at k$^{\rm th}$ site. The states $|\uparrow\rangle \equiv \begin{pmatrix}1\\0\end{pmatrix}$ and $|\downarrow\rangle \equiv \begin{pmatrix}0\\1\end{pmatrix}$ provide a natural basis of the linear space $\mathbb{C}^2$. The state $|\Uparrow\rangle$ with all spins "up": $|\Uparrow\rangle \equiv \bigotimes_{n=0}^{M} |\uparrow\rangle_n$ is annihilated by the Hamiltonian (2):

$$\hat{H}\,|\Uparrow\rangle = 0$$

The N-particle state-vectors, the states with N spins "down", is convenient to express by means of the Schur functions:

$$|\,\Psi(\mathbf{u}_N)\rangle = \sum_{\boldsymbol{\lambda}\subseteq\{\mathcal{M}^N\}} S_{\boldsymbol{\lambda}}(\mathbf{u}_N^2)\left(\prod_{k=1}^{N}\sigma_{\mu_k}^{-}\right)|\Uparrow\rangle\,.$$

The summation is over all partitions $\boldsymbol{\lambda}$ satisfying $\mathcal{M}\equiv M+1-N\geq\lambda_1\geq\lambda_2\geq\dots\geq\lambda_N\geq 0$. The sites with spin "down" states are labeled by the coordinates μ_i, $1\leq i\leq N$. These coordinates constitute a strictly decreasing partition $M\geq\mu_1>\mu_2>\ \dots\ >\mu_N\geq 0$. The relation $\lambda_j=\mu_j-N+j$, where $1\leq j\leq N$, connects the parts of $\boldsymbol{\lambda}$ to those of $\boldsymbol{\mu}$. Therefore, we can write: $\boldsymbol{\lambda}=\boldsymbol{\mu}-\boldsymbol{\delta}_N$, where $\boldsymbol{\delta}_N$ is the strict partition $(N-1,\dots,1,0)$. The parameters $\mathbf{u}_N^2\equiv(u_1^2,\dots,u_N^2)$ are arbitrary complex numbers. The *Schur functions* $S_{\boldsymbol{\lambda}}$ are defined by the Jacobi-Trudi relation:

$$S_{\boldsymbol{\lambda}}(\mathbf{x}_N)\ \equiv\ S_{\boldsymbol{\lambda}}(x_1,x_2,\dots,x_N)\ \equiv\ \frac{\det(x_j^{\lambda_k+N-k})_{1\leq j,k\leq N}}{\mathcal{V}(\mathbf{x}_N)}\,,$$

in which $\mathcal{V}(\mathbf{x}_N)$ is the Vandermonde determinant

$$\mathcal{V}(\mathbf{x}_N)\equiv\det(x_j^{N-k})_{1\leq j,k\leq N}\ =\prod_{1\leq m<l\leq N}(x_l-x_m)\,.$$

The conjugated state-vectors are given by

$$\langle \Psi(\mathbf{v}_N) \,| = \sum_{\boldsymbol{\lambda} \subseteq \{\mathcal{M}^N\}} \langle \Uparrow | \left(\prod_{k=1}^{N} \sigma_{\mu_k}^{+} \right) S_{\boldsymbol{\lambda}}(\mathbf{v}_N^{-2}) \,.$$

There is a natural correspondence between the coordinates of the spin "down" states $\boldsymbol{\mu}$ and the partition $\boldsymbol{\lambda}$ expressed by the Young diagram

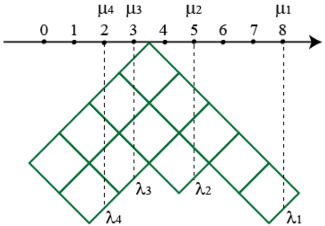

Рис.: Relation of the spin "down" coordinates $\boldsymbol{\mu} = (8, 5, 3, 2)$ and partition $\boldsymbol{\lambda} = (5, 3, 2, 2)$ for $M = 8$, $N = 4$.

For the periodic boundary conditions $\sigma^{\#}_{k+(M+1)} = \sigma^{\#}_{k}$ if parameters $u_j^2 \equiv e^{i\theta_j}$ $(1 \le j \le N)$ satisfy the Bethe equations,

$$e^{i(M+1)\theta_j} = (-1)^{N-1}\,, \quad 1 \le j \le N\,,$$

then the state-vectors become the eigen vectors of the Hamiltonian:

$$\mathcal{H}\,|\Psi(\boldsymbol{\theta}_N)\rangle \;=\; E_N(\boldsymbol{\theta}_N)\,|\Psi(\boldsymbol{\theta}_N)\rangle\,.$$

The solutions θ_j to the Bethe equations can be parametrized such that

$$\theta_j = \frac{2\pi}{M+1}\left(I_j - \frac{N-1}{2}\right)\,, \quad 1 \le j \le N\,,$$

where I_j are integers or half-integers depending on whether N is odd or even.

The eigen energies of the model are equal to

$$E_N(\boldsymbol{\theta}_N) = -\sum_{j=1}^{N} \cos\theta_j = -\sum_{j=1}^{N} \cos\left(\frac{2\pi}{M+1}\left(I_j - \frac{N-1}{2}\right)\right).$$

The *ground state* of the model is the eigen-state that corresponds to the lowest eigen energy $E_N(\boldsymbol{\theta}_N^{\mathrm{g}})$. It is determined by the solution to the Bethe equations at $I_j = N - j$:

$$\theta_j^{\mathrm{g}} \equiv \frac{2\pi}{M+1}\left(\frac{N+1}{2} - j\right), \quad 1 \le j \le N,$$

and is equal to

$$E_N(\boldsymbol{\theta}_N^{\mathrm{g}}) = -\frac{\sin\frac{\pi N}{M+1}}{\sin\frac{\pi}{M+1}}.$$

We shall consider the two types of the the thermal correlation functions in a system of finite size will be considered. We call them the *persistence of ferromagnetic string* and the *persistence of domain wall*. The persistence of ferromagnetic string is related to the projection operator $\bar{\Pi}_n$ that forbids spin "down" states on the first n sites of the chain:

$$\mathcal{T}(\boldsymbol{\theta}_N^{\mathrm{g}}, n, t) \equiv \frac{\langle \Psi(\boldsymbol{\theta}_N^{\mathrm{g}}) \mid \bar{\Pi}_n \, e^{-t\mathcal{H}} \, \bar{\Pi}_n \mid \Psi(\boldsymbol{\theta}_N^{\mathrm{g}}) \rangle}{\langle \Psi(\boldsymbol{\theta}_N^{\mathrm{g}}) \mid e^{-t\mathcal{H}} \mid \Psi(\boldsymbol{\theta}_N^{\mathrm{g}}) \rangle}, \qquad \bar{\Pi}_n \equiv \prod_{j=0}^{n-1} \frac{\sigma_j^0 + \sigma_j^z}{2},$$

where t is the inverse temperature $t = 1/T$.
The persistence of domain wall is related to the operator $\bar{\mathsf{F}}_n$ that creates a sequence of spin "down" states on the first n sites of the chain:

$$\mathcal{F}(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}, n, t) \equiv \frac{\langle \Psi(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}) \mid \bar{\mathsf{F}}_n^{+} \, e^{-t\mathcal{H}} \, \bar{\mathsf{F}}_n \mid \Psi(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}) \rangle}{\langle \Psi(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}) \mid e^{-t\mathcal{H}} \mid \Psi(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}) \rangle}, \qquad \bar{\mathsf{F}}_n \equiv \prod_{j=0}^{n-1} \sigma_j^- .$$

Here $\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}$ is the set of ground state solutions to the Bethe equations for the system of $N - n$ particles.

The calculation of introduced correlation functions will be based on the Binet–Cauchy formula adapted for the Schur functions:

$$\mathcal{P}_{L/n}(\mathbf{y}_N, \mathbf{x}_N) \equiv \sum_{\boldsymbol{\lambda}\subseteq\{(L/n)^N\}} S_{\boldsymbol{\lambda}}(\mathbf{y}_N) S_{\boldsymbol{\lambda}}(\mathbf{x}_N)$$

$$= \left(\prod_{l=1}^{N} y_l^n x_l^n\right) \frac{\det(T_{kj})_{1\le k,j\le N}}{\mathcal{V}(\mathbf{y}_N)\mathcal{V}(\mathbf{x}_N)},$$

where the summation is over all partitions $\boldsymbol{\lambda}$ satisfying: $L \ge \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_N \ge n$, and the entries T_{kj} are given by:

$$T_{kj} = \frac{1 - (x_k y_j)^{N+L-n}}{1 - x_k y_j}.$$

Consider the simplest one-particle correlation function

$$G(j,m|t) \equiv \langle\Uparrow|\, \sigma_j^+ e^{t\mathcal{H}} \sigma_m^- \,|\Uparrow\rangle.$$

The Hamiltonian may be expressed in the form

$$\mathcal{H} \equiv -\frac{1}{2}\sum(\sigma_{k+1}^- \sigma_k^+ + \sigma_{k+1}^+ \sigma_k^-) = -\sum_{n,m} \Delta_{nm} \sigma_n^- \sigma_m^+ ,$$

where Δ_{nm} is the hopping matrix with the entries equal to

$$\Delta_{nm} = \delta_{n+1,m} + \delta_{n-1,m}.$$

Differentiating $G(j,m|t)$ with respect to t and applying the commutation relation

$$[\mathcal{H}, \sigma_m^-] = \sum_n \Delta_{nm} \sigma_n^- \sigma_m^z ,$$

we obtain the equality

$$\frac{d}{dt} G(j,m|t) = \langle\Uparrow|\, \sigma_j^+ e^{t\mathcal{H}} \mathcal{H} \sigma_m^- \,|\Uparrow\rangle$$

$$= \sum_n \Delta_{nm} \langle\Uparrow|\, \sigma_j^+ e^{t\mathcal{H}} \sigma_n^- \,|\Uparrow\rangle = \sum_n \Delta_{nm} G(j,n|t) .$$

The correlation function satisfies the difference equation:

$$\frac{d}{dt}G\left(j,m|t\right)=G\left(j,m-1|t\right)+G\left(j,m+1|t\right),$$

for the fixed subindex j, and the same equation for the subindex j with the fixed m. The initial condition is defined by the equality $G\left(j,m|0\right)=\delta_{jm}$. The correlator $G\left(j,l|t\right)$ may be considered as the exponential generating function of random walks. Expanding correlation functions in powers of t one has

$$G\left(j,m|t\right)=\sum_{K}\frac{t^K}{K!}\langle\Uparrow|\,\sigma_j^+(\mathcal{H})^K\sigma_m^-\,|\Uparrow\rangle.$$

Applying then the commutation relations, one obtains

$$\begin{aligned}\mathcal{H}^K\sigma_m^-\,|\Uparrow\rangle=\mathcal{H}^{K-1}[\hat{H},\sigma_m^-]\,|\Uparrow\rangle&=\hat{H}^{K-1}\sum_{n_1}\Delta_{n_1 m}\sigma_{n_1}^-\,|\Uparrow\rangle\\&=\sum_{n_1,\ldots,n_K}\Delta_{n_K n_{K-1}}\ldots\Delta_{n_2 n_1}\Delta_{n_1 m}\sigma_{n_K}^-\,|\Uparrow\rangle.\end{aligned}$$

The multiplication on $\langle\Uparrow|\, \sigma_j^+$ will fix the ending point of the trajectory:

$$\langle\Uparrow|\, \sigma_j^+ (\mathcal{H})^K \sigma_m^- \,|\Uparrow\rangle = \mathfrak{G}(j, m|K) = \sum_{n_1,\dots,n_{K-1}} \Delta_{jn_{K-1}} \dots \Delta_{n_2 n_1} \Delta_{n_1 m}\,.$$

The position of the walker on a lattice is labelled by the spin down state, while the spin up states correspond to the empty sites, Δ_{ps} is an elementary step. The obtained equality enumerates all admissible trajectories of the walker starting from the site j and terminating at m. The function $\mathfrak{G}(j, m|K)$ satisfies equation:

$$\mathfrak{G}(j, m|K+1) = \mathfrak{G}(j, m-1|K) + \mathfrak{G}(j, m+1|K)\,.$$

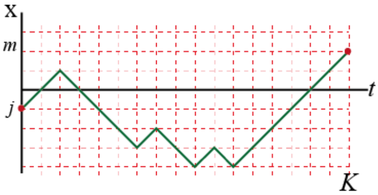

Рис.: A random walk in $K = 17$ steps.

The multi-particle correlation function

$$G(j_1,j_2,...,j_N;l_1,l_2,...,l_N|t) = \langle \Uparrow | \, \sigma^+_{j_1}\sigma^+_{j_2}...\sigma^+_{j_N} e^{-t\mathcal{H}} \sigma^-_{l_1}\sigma^-_{l_2}...\sigma^-_{l_N} \, | \Uparrow \rangle \, .$$

Applying the of commutation relation

$$[\mathcal{H}, \sigma^-_{l_1}\sigma^-_{l_2}...\sigma^-_{l_N}] = \sum_{k=1}^{N} \sigma^-_{l_1}...\sigma^-_{l_{k-1}}[\mathcal{H}, \sigma^-_{l_k}]\sigma^-_{l_{k+1}}...\sigma^-_{l_N}$$

we obtain the equation

$$\frac{d}{dt}G(j_1,...,j_N;l_1,...,l_N|t) = \sum_{k=1}^{N}(G(j_1,...,j_N;l_1,l_2,...,l_k+1,...,l_N|t) + G(j_1,...,j_N;l_1,l_2,...,l_k-1,...,l_N|t))$$

The condition $G(j_1,...,j_N;l_1,l_2,...,l_N|t) = 0$ if $l_k = l_p, j_k = j_p$ for any $1 \leq j,l \leq N$ is guaranteed by the property of the Pauli matrices $(\sigma^{\pm}_k)^2 = 0$.

The solution of the equation is represented in the determinant form

$$G(j_1, ..., j_N; l_1, ..., l_N|t) = \det \{G(j_r, l_s|t)\}_{r,s=1,...,N} \ .$$

where $G(j, l|t)$ are the one-particle correlation functions.
The average $\langle \Uparrow | \, \sigma^+_{j_1} \sigma^+_{j_2} ... \sigma^+_{j_N} (-\mathcal{H})^K \sigma^-_{l_1} \sigma^-_{l_2} ... \sigma^-_{l_N} \, | \Uparrow \rangle$ is equal to the number of configurations that have the N *random turns walkers* being initially located on the lattice sites $l_1 > l_2 > ... > l_N$ and after K steps arrived at the positions $j_1 > j_2 > ... > j_N$.

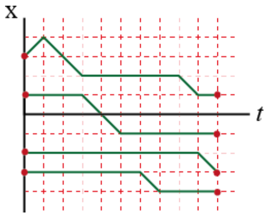

Рис.: Random turns walkers.

For the infinite in the both sides lattice the one-particle correlation function is

$$G(j,m|t) = I_{m-j}(2t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2t\cos\theta} e^{i(m-j)\theta} d\theta.$$

Expanding modified Bessel function in powers of t

$$I_{m-j}(2t) = \sum_{k\geq|l-j|}^{\infty} \frac{t^k}{\left(\frac{k-j+m}{2}\right)!\left(\frac{k+j-m}{2}\right)!}$$

where the sum is taken over k satisfying the condition $k+|j-m| = 0(mod2)$, we obtain

$$\mathfrak{G}(j,m|K) = \frac{K!}{\left(\frac{K-j+m}{2}\right)!\left(\frac{K+j-m}{2}\right)!}.$$

It is a well known binomial formula for a number of all lattice paths from m to j of length K on the infinite lattice.

The generating function of N random turns walkers being initially located on the lattice sites $l_1 > l_2 > ... > l_N \geq 0$ and arrived at the positions $j_1 > j_2 > ... > j_N \geq 0$ is

$$G(j_1, ..., j_N; l_1, ..., l_N|t) = \left(\frac{1}{2\pi}\right)^N \int_{-\pi}^{\pi} d\theta_1 ... \int_{-\pi}^{\pi} d\theta_N e^{2t\sum_{m=1}^N \cos\theta_m} \det\left\{e^{i(l_s - j_r)\theta_r}\right\}_{r,s=1,...,N} .$$

Making use of the symmetry of the integrand with respect to permutations of the variables $\theta_1, \ldots, \theta_N$

$$G(j_1, ..., j_N; l_1, ..., l_N|t) = \frac{1}{(2\pi)^N N!} \int_{-\pi}^{\pi} d\theta_1 ... \int_{-\pi}^{\pi} d\theta_N e^{2t\sum_{m=1}^N \cos\theta_m}$$
$$\times s_\lambda(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_N}) s_\mu(e^{-i\theta_1}, e^{-i\theta_2}, ..., e^{-i\theta_N}) \prod_{1\leq j<k\leq N} |e^{i\theta_j} - e^{i\theta_k}|^2 ,$$

where $\lambda_k = j_k - N + k$ and $\mu_k = l_k - N + k$.

The one-particle correlation function on the semiaxis $(0 \le j, m < \infty)$ satisfies the boundary condition $G(j, m|t) = 0$ for $j, m = -1$:

$$G(j, m|t) = \frac{1}{\pi} \int_{-\pi}^{\pi} e^{2t\cos\theta} \sin[(j+1)\theta] \sin[(m+1)\theta]\, d\theta$$
$$= I_{j-m}(2t) - I_{j+m+2}(2t)\,.$$

This function is the exponential generating function of Catalan paths. The Catalan path is a lattice path that starts at $(0, j)$, ends in (K, m), and only contains upsteps $(1, 1)$ and downsteps $(1 - 1)$, in such a way that it never goes below the t-axis.

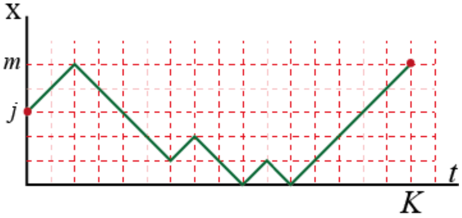

Рис.: A Catalan path in $K = 16$ steps starting from a point $(0, 3)$ and terminating at $(16, 5)$.

The number of Catalan paths in K steps from j to m is equal to

$$\mathfrak{G}(j,m|K) = \binom{K}{\frac{K-j+m}{2}} - \binom{K}{\frac{K+j+m+2}{2}} .$$

The Dyck path is the Catalan path that starts at the origin and ends at $(2K,0)$. The exponential generating function of such paths is

$$G(0,0|t) = \frac{1}{t} I_1(2t) = \sum_{k=0}^{\infty} \frac{t^{2k}}{2k!} C_k ,$$

where C_k are Catalan numbers

$$C_k = \frac{1}{k+1}\binom{2k}{k} ,$$

and the number of Dyck paths is

$$\mathfrak{G}(0,0|2K) = C_K .$$

The multi-particle correlation function on a semi-infinite lattice is an exponential generating function of the random turn paths starting at $l_1 > l_2 > ... > l_N \geq 0$ and terminating at $j_1 > j_2 > ... > j_N \geq 0$ that do not pass below t axis is equal to

$$G(j_1, ..., j_N; l_1, ..., l_N|t) = \frac{1}{(\pi)^N N!} \int_{-\pi}^{\pi} d\theta_1 ... \int_{-\pi}^{\pi} d\theta_N e^{2t\sum_{m=1}^{N} \cos\theta_m}$$

$$\times sp_\lambda(e^{i\theta_1}, e^{i\theta_2}, ..., e^{i\theta_N}) sp_\mu(e^{-i\theta_1}, e^{-i\theta_2}, ..., e^{-i\theta_N})$$

$$\times \prod_{1\leq j<k\leq N} \left(\det_{1\leq r,s\leq N} \{\sin s\theta_r\} \right)^2 ,$$

where $\lambda_k = j_k - N + k$, $\mu_k = l_k - N + k$ and

$$sp_\lambda(x_1, x_2, ..., x_K) = \frac{\det_{1\leq j,k\leq K}(x_j^{\lambda_k+K-k+1} - x_j^{-(\lambda_k+K-k+1)})}{\det_{1\leq j,k\leq K}(x_j^{K-k+1} - x_j^{-(K-k+1)})}$$

is the character of the irreducible representation of the symplectic Lie algebra corresponding to a partition λ.

The calculation of correlation functions of the model is based on Binet-Cauchy formula:

$$\sum_{\boldsymbol{\lambda}\subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{x})S_{\boldsymbol{\lambda}}(\mathbf{y}) = \frac{\det(T_{kj})_{1\leq k,j\leq N}}{\mathcal{V}_N(\mathbf{x})\mathcal{V}_N(\mathbf{y})}\,,$$

where the summation is over all partitions $\boldsymbol{\lambda}$ satisfying: $M \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$. The entries

$$T_{kj} = \frac{1-(x_k y_j)^{M+N}}{1-x_k y_j}\,.$$

To study the asymptotical behaviour of the introduced correlation functions we need to calculate the the q-parameterized Binet-Cauchy relation. We put $\mathbf{y}=\mathbf{q}\equiv(q,q^2,\ldots,q^N)$, $\mathbf{x}=\mathbf{q}/q\equiv(1,q,\ldots,q^{N-1})$ and obtain

$$\sum_{\boldsymbol{\lambda} \subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q) = \mathcal{V}_N^{-1}(\mathbf{q}) \mathcal{V}_N^{-1}(\mathbf{q}/q) \det \left(\frac{1 - q^{(M+N)(j+k-1)}}{1 - q^{j+k-1}} \right)_{1 \le j,k \le N}$$

$$= q^{\frac{NM}{2}(1-M)} \det \left(\begin{bmatrix} 2N+i-1 \\ N+j-1 \end{bmatrix} \right)_{1 \le i,j \le M} = Z(N,N,M)\,.$$

The entries of the last determinant are the q-*binomial coefficients*:

$$\begin{bmatrix} R \\ r \end{bmatrix} \equiv \frac{[R]!}{[r]!\,[R-r]!}\,, \quad [n] \equiv \frac{1-q^n}{1-q}\,,$$

and

$$Z(N,N,M) = \prod_{k=1}^{N} \prod_{j=1}^{N} \frac{1-q^{M+j+k-1}}{1-q^{j+k-1}}$$

is the MacMahon generating function of plane partitions in the $N \times N \times M$ box.

A combinatorial description of the Schur functions may be given in terms of *semistandard Young tableaux*. A filling of the cells of the Young diagram of $\boldsymbol{\lambda}$ with positive integers $n \in \mathbb{N}^+$ is called a *semistandard tableau of shape* $\boldsymbol{\lambda}$ provided it is weakly increasing along rows and strictly increasing along columns. The weight $\mathbf{x}^T$ of a tableau T is defined as

$$\mathbf{x}^T \equiv \prod_{i,j} x_{T_{ij}} \, ,$$

where the product is over all entries T_{ij} of the tableau T. An equivalent definition of the Schur function is given by

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \ldots, x_m) = \sum_T \mathbf{x}^T \, ,$$

where $m \geq N$, and the sum is over all tableaux T of shape $\boldsymbol{\lambda}$ with the entries being numbers from the set $\{1, 2, \ldots, m\}$.

There is a natural way of representing each semistandard tableau of shape $\boldsymbol{\lambda}$ with entries not exceeding N as a nest of self-avoiding lattice paths with prescribed start and end points.

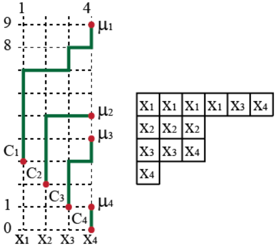

Рис.: A semistandard tableau of shape $\boldsymbol{\lambda} = (6, 3, 3, 1)$.

An equivalent representation of the Schur function

$$S_{\boldsymbol{\lambda}}(x_1, x_2, \dots, x_N) = \sum_{C} \prod_{j=1}^{N} x_j^{l_j},$$

where summation is over all admissible nests C, the power l_j of x_j is the number of steps to north taken along the vertical line x_j.

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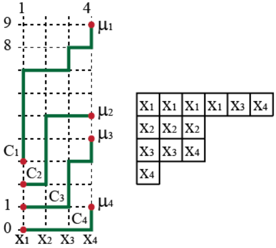

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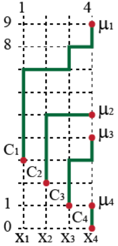


x_1	x_1	x_1	x_1	x_3	x_4
x_2	x_2	x_2			
x_3	x_3	x_4			
x_4					

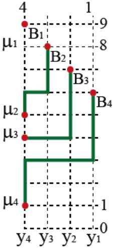
The k^{th} lattice path is contained in a rectangle of the size $\lambda_k \times (N-k)$. The starting point of each path is the lower left vertex. The volume of the path is the number of squares below it in the corresponding rectangle. The volume of the nest of lattice paths C is:

$$| \zeta |_C = \sum_{j=1}^{N} (N-j) l_j = \sum_{j=1}^{N} (j-1) l_{N-j+1} .$$

The q-parametrized Schur function is a partition function of the nest:

$$S_{\boldsymbol{\lambda}}(\mathbf{q}) = \sum_{C} q^{|\xi|_C} = q^{|\lambda|} \sum_{C} q^{|\zeta|_C} \, , |\boldsymbol{\lambda}| = \sum_{k=1}^{N} \lambda_k \, .$$

The representation of the Schur function corresponding to the conjugate nest of self-avoiding lattice paths



$$S_{\boldsymbol{\lambda}}(y_1, y_2, \ldots, y_N) = \sum_B \prod_{j=1}^{N} y_j^{(M-l_j)},$$

where summation is over all admissible nests B of N self-avoiding lattice paths. The volume of the nest B of lattice paths is given by

$$|\ \zeta\ |_B = \sum_{j=1}^{N} (j-1)(M-l_j).$$

The scalar product may be graphically expressed as a nest of N self-avoiding lattice paths starting at the equidistant points C_i and terminating at the equidistant points B_i $(i = 1, \ldots, N)$. This configuration is known as *watermelon*

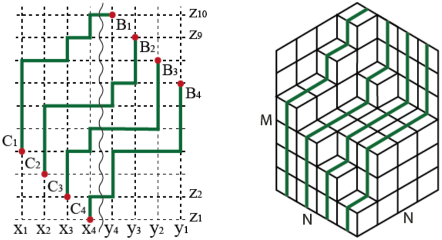


Рис.: Watermelon configuration and correspondent plane partition.

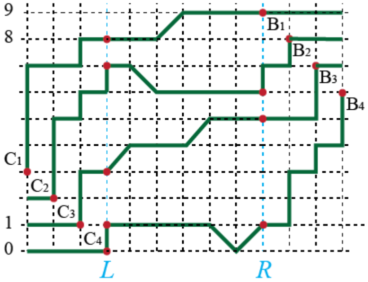
The partition function of watermelons (the generating function of watermelons) is equal to the q-parameterized scalar product:

$$Z(N, N, M) = \sum_{W} q^{|\xi|_C + |\zeta|_B} = \sum_{\lambda \subseteq M^N} S_{\boldsymbol{\lambda}}(\mathbf{q}) S_{\boldsymbol{\lambda}}(\mathbf{q}/q)$$

$$= \langle \Psi_N(\mathbf{q}^{-\frac{1}{2}}) | \Psi_N(\mathbf{q}/q)^{\frac{1}{2}} \rangle \,,$$

The matrix element of the projection operator $\mathit{\Pi}_n \equiv \prod_{j=0}^{n-1} \frac{\sigma_j^0+\sigma_j^z}{2}$ is equal to

$$\langle \Psi(\mathbf{v}_N) \mid \bar{\mathit{\Pi}}_n \, e^{-t\mathcal{H}} \, \bar{\mathit{\Pi}}_n \mid \Psi(\mathbf{u}_N) \rangle = \sum_{\boldsymbol{\lambda}^L, \boldsymbol{\lambda}^R \subseteq \{(\mathcal{M}/n)^N\}} S_{\boldsymbol{\lambda}^L}(\mathbf{v}_N^{-2}) S_{\boldsymbol{\lambda}^R}(\mathbf{u}_N^2)$$
$$\times \langle \Uparrow | \left(\prod_{l=1}^{N} \sigma_{\mu_l^L}^{+} \right) e^{-t\mathcal{H}} \left(\prod_{p=1}^{N} \sigma_{\mu_p^R}^{-} \right) | \Uparrow \rangle$$



$$\langle \Psi(\mathbf{v}_N) \mid \bar{\Pi}_0 \, e^{-t\mathcal{H}} \, \bar{\Pi}_0 \mid \Psi(\mathbf{u}_N) \rangle = \sum_{k=0}^{\infty} \frac{t^k}{k!} \langle \Psi(\mathbf{v}_N) \mid \bar{\Pi}_0 \, \mathcal{H}^k \, \bar{\Pi}_0 \mid \Psi(\mathbf{u}_N) \rangle$$

The answer for the persistence of the ferromagnetic string is

$$\mathcal{T}(\boldsymbol{\theta}_N^{\mathrm{g}}, n, t) \equiv \frac{\langle \Psi(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}) \mid \bar{\mathsf{F}}_n^{+} \, e^{-t\mathcal{H}} \, \bar{\mathsf{F}}_n \mid \Psi(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}) \rangle}{\langle \Psi(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}) \mid e^{-t\mathcal{H}} \mid \Psi(\widetilde{\boldsymbol{\theta}}_{N-n}^{\mathrm{g}}) \rangle}$$

$$= \frac{e^{tE_N(\boldsymbol{\theta}_N^{\mathrm{g}})}}{(M+1)^N} \det \left(\sum_{k,l=n}^{M} G(k,l|t) \, e^{i(l\theta_i^{\mathrm{g}} - k\theta_j^{\mathrm{g}})} \right)_{1 \le i,j \le N} .$$

An alternative expression:

$$\mathcal{T}(\boldsymbol{\theta}_N^{\mathrm{g}}, n, t) = \frac{|\mathcal{V}(e^{i\boldsymbol{\theta}_N^{g}})|^2}{(M+1)^{2N}} \sum_{\{\boldsymbol{\theta}_N\}} e^{-t(E_N(\boldsymbol{\theta}_N) - E_N(\boldsymbol{\theta}_N^{\mathrm{g}}))}$$

$$\times \left| \mathcal{V}(e^{i\boldsymbol{\theta}_N}) \, \mathcal{P}_{\mathcal{M}/n}(e^{-i\boldsymbol{\theta}_N}, e^{i\boldsymbol{\theta}_N^{\mathrm{g}}}) \right|^2 ,$$

where $\mathcal{M} = M + 1 - N$ and

$$\mathcal{P}_{\mathcal{M}/n}(e^{-i\boldsymbol{\theta}_N}, e^{i\boldsymbol{\theta}_N^{\mathrm{g}}}) = \sum_{\boldsymbol{\lambda} \subseteq \{(\mathcal{M}/n)^N\}} S_{\boldsymbol{\lambda}}(e^{-i\boldsymbol{\theta}_N}) S_{\boldsymbol{\lambda}}(e^{i\boldsymbol{\theta}_N^{\mathrm{g}}}) .$$

For a long enough chain, $M \gg 1$, and N moderate: $1 \ll N \ll M$, the correlation function may be expressed as

$$\mathcal{T}(\boldsymbol{\theta}_N^{\mathrm{g}}, n, t) \simeq \frac{|\mathcal{V}(e^{i\boldsymbol{\theta}_N^g})|^2}{(M+1)^N N!} \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} \ldots \int\limits_{-\pi}^{\pi} e^{t \sum\limits_{l=1}^{N} (\cos\theta_l - \cos\theta_l^{\mathrm{g}})}$$
$$\times \left|\mathcal{P}_{\mathcal{M}/n}(e^{-i\boldsymbol{\theta}_N}, e^{i\boldsymbol{\theta}_N^{\mathrm{g}}})\right|^2 \prod_{1 \le k < l \le N} \left|e^{i\theta_k} - e^{i\theta_l}\right|^2 \frac{d\theta_1 d\theta_2 \ldots d\theta_N}{(2\pi)^N}\,.$$

In the large t limit (small temperature limit $t = 1/T$)) in the leading order in t^{-1}:

$$\mathcal{T}(\boldsymbol{\theta}^{\mathrm{g}} \approx \mathbf{0}, n, t) \simeq A^2(N, N, M - N + 1 - n)\, e^{\Phi(N,M,t)}\,,$$
$$\Phi(N, M, t) = N^2 \log \frac{2\pi}{M+1} - \frac{N^2}{2} \log t + 3\phi_N\,,$$

where $A(N, N, M - N + 1 - n)$ is the number of plane partitions in a box $N \times N \times (M - N + 1 - n)$ and

$$\phi_N = \log G(N+1) - \frac{N}{2} \log 2\pi\,,$$

Barnes function

$$\phi_N \;=\; \log G(N+1) - \frac{N}{2}\log 2\pi\,.$$

In terms of Barnes function, the number of boxed plane partition

$$A(N,N,M-N+1-n) \;=\; \frac{G^2(N+1)\,G(M+2-n+N)\,G(M+2-n-N)}{G(2N+1)\,G^2(M+2-n)}\,.$$

The asymptotic behavior of the approximate expression of the persistence of ferromagnetic string is:

$$\log\mathcal{T}(\boldsymbol{\theta}^{\,\mathrm{g}}\approx\mathbf{0},n,t) \;\simeq\; N^2\log\left(\mathsf{C}\,\frac{(M-n)^2}{M(Nt)^{1/2}}\right)\,.$$

For the persistence of domain wall correlation function the asymptotic we have:

$$\log\mathcal{F}(\boldsymbol{\theta}^{\,\mathrm{g}}\approx\mathbf{0},n,t) \;\simeq\; N^2\log\left(\mathsf{B}\,\frac{N^{3/2}}{Mt^{1/2}}\right)+2N(N-n)\log\left(\mathsf{D}\,\frac{M-n}{2N-n}\right)\,.$$

$$\hat{H}_{XXZ} = -\frac{1}{2}\sum_{k=0}^{M}\left(\sigma_{k+1}^{-}\sigma_{k}^{+} + \sigma_{k+1}^{+}\sigma_{k}^{-} + \frac{\Delta}{2}\left(\sigma_{k+1}^{z}\sigma_{k}^{z} - 1\right) + h\sigma_{k}^{z}\right),$$

- $\Delta \to 0$
- $\Delta \to -\infty$

Izing limit:

$$\lim_{\Delta\to-\infty}\frac{1}{\Delta}\,\hat{H}_{\mathrm{XXZ}} \;=\; \hat{H}_{\mathrm{IZ}} \;\equiv\; \frac{-1}{4}\sum_{k=0}^{M}(\sigma_{k+1}^{z}\sigma_{k}^{z} - 1)\,.$$

Strong anisotropy limit:

$$\hat{H}_{\mathrm{SA}} \;=\; -\frac{1}{2}\sum_{k=0}^{M}\mathcal{P}\left(\sigma_{k+1}^{-}\sigma_{k}^{+}+\sigma_{k+1}^{+}\sigma_{k}^{-}+h\,\sigma_{k}^{z}\right)\mathcal{P}\,, \qquad \mathcal{P}\equiv\prod_{k=0}^{M}(\mathbb{I}-\hat{q}_{k+1}\hat{q}_{k})\,,$$

where $\hat{q}_k \;\equiv\; \frac{1}{2}\,(\mathbb{I} - \sigma_k^z)$, and $[\hat{H}_{\mathrm{SA}}, \hat{H}_{\mathrm{IZ}}] = 0$. The nearest neighbours with spins "down"are not allowed.

The state vector:

$$|\Psi_N(\mathbf{u})\rangle = \sum_{\widetilde{\boldsymbol{\lambda}}\subseteq\{(M-2(N-1))^N\}} S_{\widetilde{\boldsymbol{\lambda}}}(\mathbf{u}^2)\left(\prod_{k=1}^{N}\sigma^-_{\widetilde{\mu}_k}\right)|\Uparrow\rangle\,.$$

The summation is over all partitions
$\widetilde{\boldsymbol{\lambda}} = \widetilde{\boldsymbol{\mu}} - 2\boldsymbol{\delta}_N$ $(M+2(1-N)\geq\widetilde{\lambda}_1\geq\widetilde{\lambda}_2\geq\cdots\geq\widetilde{\lambda}_N\geq 0)$, and
$\widetilde{\mu}_i > \widetilde{\mu}_{i+1}+1$.

Four vertex model