# Application of the hidden fermionic structure to the CFT

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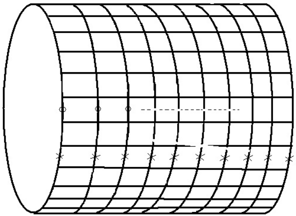
# Plan

Basic motivation: to apply fermionic basis originated from the lattice to the CFT

- Introduction
- Hidden fermionic structure on the lattice
  - Partition function on cylinder
  - Fermionic operators and fermionic basis
  - Jimbo-Miwa-Smirnov theorem
  - Functions $\rho$ and $\omega$
- Application to CFT and further perspectives
  - Conjectures on scaling limit
  - Operators: local and non-local
  - Identification of fermionic basis
  - OPE in fermionic basis and conformal blocks
  - Discussion of recursion relation for the conformal blocks
- Conclusions

# Partition function on cylinder



Cut corresponds to insertion of local operator $\mathcal{O}$

We call $q^{\alpha \sum_{j=-\infty}^{0} \sigma_j^z}\,\mathcal{O}$ quasi-local operator with tail $\alpha$

$\kappa$ – "imaginary" magnetic field
$\alpha$ – disorder parameter


Matsubara expectation values:

$$Z^{\kappa}\left\{q^{\alpha \sum_{j=-\infty}^{0} \sigma_j^z}\,\mathcal{O}\right\} = \frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}\ q^{\kappa \sum_{j=-\infty}^{\infty} \sigma_j^z + \alpha \sum_{j=-\infty}^{0} \sigma_j^z}\,\mathcal{O}\right)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(T_{\mathrm{S},\mathbf{M}}\ q^{\kappa \sum_{j=-\infty}^{\infty} \sigma_j^z + \alpha \sum_{j=-\infty}^{0} \sigma_j^z}\right)}$$

$T_{\mathrm{S},\mathbf{M}}$ is monodromy matrix with $\mathfrak{H}_{\mathrm{S}} = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2$ and $\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{j=1}}^{N} \mathbb{C}^2$

# Fermionic operators

Describe the basis of quasi-local operators via certain creation operators. Jimbo, Miwa, Smirnov, Takeyama, HB (07–09)

Creation operators $\mathbf{t}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ together with annihilation operators $\mathbf{b}, \mathbf{c}$ are constructed with help of representation theory of quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and act in space

$$\mathcal{W}^{(\alpha)} = \bigoplus_{s=-\infty}^{\infty} \mathcal{W}_{\alpha-s,s}$$

where $\mathcal{W}_{\alpha-s,s}$ is subspace of quasi-local operators of the spin $s$. They are formal power series of $\zeta^2-1$ and have the block structure

$$\begin{aligned}
\mathbf{t}^*(\zeta) \quad &: \mathcal{W}_{\alpha-s,s} \to \mathcal{W}_{\alpha-s,s}\,, \\
\mathbf{b}^*(\zeta), \mathbf{c}(\zeta) &: \mathcal{W}_{\alpha-s+1,s-1} \to \mathcal{W}_{\alpha-s,s}\,, \\
\mathbf{c}^*(\zeta), \mathbf{b}(\zeta) &: \mathcal{W}_{\alpha-s-1,s+1} \to \mathcal{W}_{\alpha-s,s}\,.
\end{aligned}$$

$\mathbf{t}^*(\zeta)$ is bosonic and generates commuting integrals of motion. It commutes with all fermionic operators $\mathbf{b}(\zeta), \mathbf{c}(\zeta)$ and $\mathbf{b}^*(\zeta), \mathbf{c}^*(\zeta)$.

# Anti-commutation relations and fermionic basis

Fermionic operators satisfy canonical anti-commutation relations

$$\left[\mathbf{c}(\zeta),\mathbf{c}^*(\zeta')\right]_+=\psi(\zeta/\zeta',\alpha),\quad \left[\mathbf{b}(\zeta),\mathbf{b}^*(\zeta')\right]_+=-\psi(\zeta'/\zeta,\alpha)$$

$$\text{with}\quad \psi(\zeta,\alpha)=\frac{1}{2}\,\zeta^{\alpha}\,\frac{\zeta^2+1}{\zeta^2-1}.$$

Annihilation operators **b** and **c** "kill" lattice "primary field" $q^{2\alpha S(0)}$

$$\mathbf{b}(\zeta)\big(q^{2\alpha S(0)}\big)=0,\quad \mathbf{c}(\zeta)\big(q^{2\alpha S(0)}\big)=0,\quad S(k)=\frac{1}{2}\sum_{j=-\infty}^{k}\sigma_j^z.$$

Space of states is generated via multiple action of $\mathbf{t}^*(\zeta),\mathbf{b}^*(\zeta),\mathbf{c}^*(\zeta)$ on "primary field" $q^{2\alpha S(0)}$. In this way we get fermionic basis.

# Mode expansions and locality

Annihilation operators are singular at $\zeta^2 \to 1$ while creation operators are regular:

$$\mathbf{b}(\zeta) = \zeta^{-\alpha-\mathbb{S}} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{b}_p, \quad \mathbf{c}(\zeta) = \zeta^{\alpha+\mathbb{S}} \sum_{p=0}^{\infty} (\zeta^2 - 1)^{-p} \mathbf{c}_p$$

$$\mathbf{b}^*(\zeta) = \zeta^{\alpha+\mathbb{S}} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{b}^*_p, \quad \mathbf{c}^*(\zeta) = \zeta^{-\alpha-\mathbb{S}} \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{c}^*_p$$

$$\mathbf{t}^*(\zeta) = \sum_{p=1}^{\infty} (\zeta^2 - 1)^{p-1} \mathbf{t}^*_p$$

Locality:

$$\mathbf{b}_p(X) = \mathbf{c}_p(X) = 0 \quad \text{for} \quad p > \text{length}(X)$$

$$\text{length}(\mathbf{b}^*_p(X)) \le \text{length}(X) + p, \quad \text{length}(\mathbf{c}^*_p(X)) \le \text{length}(X) + p$$

$$\text{length}(\mathbf{t}^*_p(X)) \le \text{length}(X) + p$$

# Jimbo-Miwa-Smirnov theorem

Important theorem was proved by Jimbo, Miwa and Smirnov (09)

$$Z^{\kappa}\left\{\mathbf{t}^*(\zeta)(X)\right\} = 2\rho(\zeta)Z^{\kappa}\{X\},$$

$$Z^{\kappa}\left\{\mathbf{b}^*(\zeta)(X)\right\} = \frac{1}{2\pi i}\oint_{\Gamma}\omega(\zeta,\xi)\, Z^{\kappa}\left\{\mathbf{c}(\xi)(X)\right\}\frac{d\xi^2}{\xi^2},$$

$$Z^{\kappa}\left\{\mathbf{c}^*(\zeta)(X)\right\} = -\frac{1}{2\pi i}\oint_{\Gamma}\omega(\xi,\zeta)\, Z^{\kappa}\left\{\mathbf{b}(\xi)(X)\right\}\frac{d\xi^2}{\xi^2}$$

where the contour $\Gamma$ goes around $\xi^2 = 1$ and $\omega$ is some explicit function. These formulae allow one to explicitly calculate

$$Z^{\kappa}\left\{\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_p^0)\mathbf{b}^*(\zeta_1^+)\cdots\mathbf{b}^*(\zeta_q^+)\mathbf{c}^*(\zeta_q^-)\cdots\mathbf{c}^*(\zeta_1^-)\left(q^{\alpha\sum\limits_{j=-\infty}^{0}\sigma_j^z}\right)\right\} =$$

$$= \prod_{i=1}^{p} 2\rho(\zeta_i^0)\det\left|\omega(\zeta_i^+,\zeta_j^-)\right|_{i,j=1,\cdots q}$$ generating function for series in $\zeta^2 - 1$

# The meaning of the JMS-theorem

The Jimbo, Miwa, Smirnov theorem states that

- any correlation function corresponding to any quasi-local operator $\mathcal{O}$ is generated by two transcendental functions $\rho$ and $\omega$. $\rho$ is related to one-point function, $\omega$ is related to nearest neighbor correlators

$$\omega(\zeta, \zeta') = Z^{\kappa}(\mathbf{b}^*(\zeta)\mathbf{c}^*(\zeta')q^{2\alpha S(0)})$$

  Both functions depend on temperature, disorder parameter and magnetic field, we call them physical part.

- In contrast to this, the basis is pure algebraic. It is built using representation theory of quantum group. We call it algebraic part.

- The basis is independent of inhomogeneities in the Matsubara direction.

# The function ω via function Φ

In more general case $\omega(\zeta,\zeta') = \omega(\zeta,\zeta'|\kappa,\kappa';\alpha)$

There are several equivalent definitions

- via deformed Abelian integrals Jimbo, Miwa, Smirnov (09)
- via solution of linear and non-linear integral equations that come from thermodynamical description of the six-vertex model Göhmann, HB (09-12)

Introduce function Φ: $\Phi(\zeta,\zeta'|\kappa,\kappa';\alpha) = \tilde{\Phi}(\zeta,\zeta'|\kappa,\kappa';\alpha) + \Delta_\zeta^{-1}\psi(\zeta/\zeta',\alpha)$

$$\tilde{\Phi}(\zeta,\zeta'|\kappa,\kappa';\alpha) = \left(\frac{\zeta}{\zeta'}\right)^{\alpha} \frac{P(\zeta,\zeta'|\kappa,\kappa';\alpha)}{A(\zeta,\kappa)A(\zeta',\kappa)}, \quad \Phi(\zeta',\zeta|\kappa,\kappa';-\alpha) = \Phi(\zeta,\zeta'|\kappa,\kappa';\alpha)$$

One must solve kind of ‘Riemann-Hilbert’ problem

$$\frac{\Delta_\zeta \Phi(\zeta,\zeta'|\kappa,\kappa';\alpha)}{\rho(\zeta|\kappa,\kappa')(1+\mathfrak{a}(\zeta,\kappa))} + \tilde{\Phi}(\zeta,\zeta'|\kappa,\kappa';\alpha) = r(\zeta,\zeta'|\kappa,\kappa';\alpha)$$

$$(\Delta_\zeta f)(\zeta) := f(q\zeta) - f(q^{-1}\zeta)$$

where the remainder $r$ is a ‘regular’ function of $\zeta$.

- The function ρ : $\rho(\zeta|\kappa,\kappa') = \frac{T(\zeta,\kappa')}{T(\zeta,\kappa)}$
- Baxter's *TQ*-relation:

$$T(\zeta,\kappa)Q(\zeta,\kappa) = d(\zeta)Q(q\zeta,\kappa) + a(\zeta)Q(q^{-1}\zeta,\kappa), \quad q = e^{i\pi\nu}$$

$$d(\zeta) = \prod_{i=1}^{N}(\zeta/\tau_j - \tau_j/\zeta),\ a(\zeta) = d(q\zeta)$$

$$Q(\zeta,\kappa) = \zeta^{-\kappa}A(\zeta,\kappa), \quad A(\zeta,\kappa) = \prod_{j=1}^{N/2}(\zeta/\zeta_j - \zeta_j/\zeta)$$

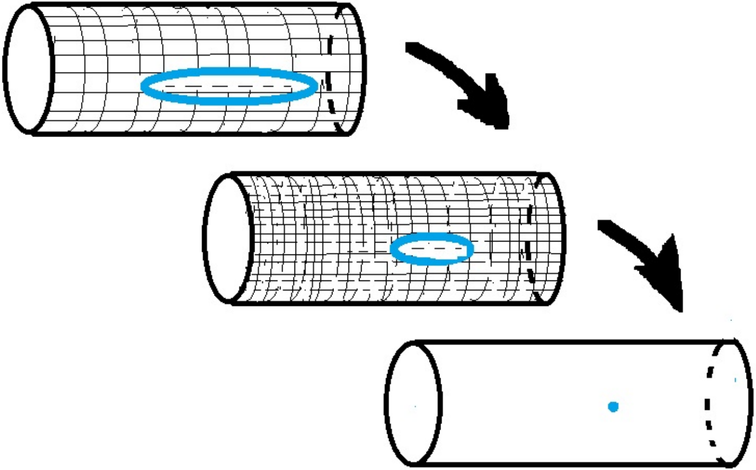
$$\mathfrak{a}(\zeta,\kappa) := \frac{d(\zeta)Q(q\,\zeta,\kappa)}{a(\zeta)Q(q^{-1}\zeta,\kappa)}, \quad \text{BAE:} \quad \mathfrak{a}(\zeta_j,\kappa) = -1, j = 1,\cdots N/2$$

- The function ω:

$$\frac{1}{4}\omega(\zeta,\zeta'|\kappa,\kappa',\alpha) = H_\zeta\ H_{\zeta'}\ \ \Phi(\zeta,\zeta'|\kappa,\kappa',\alpha)$$

$$(H_\zeta f)(\zeta) := \frac{\mathfrak{a}(\zeta,\kappa)}{1+\mathfrak{a}(\zeta,\kappa)}f(q\zeta) + \frac{1}{1+\mathfrak{a}(\zeta,\kappa)}f(q^{-1}\zeta) - \rho(\zeta|\kappa,\kappa')f(\zeta)$$

# Application to CFT: continuum limit



# Scaling limit

Jimbo, Miwa, Smirnov, HB (10), HB (11)

- Scaling or conformal limit  Introduce lattice spacing $a$ and take

  $\tau_j = q^{1/2}, \quad N \to \infty, \quad a \to 0, \quad Na = 2\pi R$ with fixed radius of cylinder $R$

  Lieb distribution gives: $\zeta_j \simeq (\pi j/N)^{\nu}$

  Spectral parameter must be re-scaled: $\zeta = \lambda \bar{a}^{\nu}, \quad \bar{a} = Ca$

Some technical point: We generalize definition of the functional $Z^{\kappa}$ by introducing "lattice screening operator" $Y_{\mathbf{M}}^{(-s)}$:

$$Z^{\kappa,s}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(Y_{\mathbf{M}}^{(-s)}\, T_{\mathrm{S},\mathbf{M}}\; q^{2\kappa S}\; \mathbf{b}^*_{\infty,s-1}\cdots \mathbf{b}^*_{\infty,0}\left(q^{2\alpha S(0)}\mathcal{O}\right)\right)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\left(Y_{\mathbf{M}}^{(-s)}\, T_{\mathrm{S},\mathbf{M}}\; q^{2\kappa S}\; \mathbf{b}^*_{\infty,s-1}\cdots \mathbf{b}^*_{\infty,0}\left(q^{2\alpha S(0)}\right)\right)}$$

$$\zeta^{-\alpha}\mathbf{b}^*(\zeta)(X) = \sum_{j=0}^{s-1}\zeta^{-2j}\mathbf{b}^*_{\infty,j}(X) + \zeta^{-\alpha}\mathbf{b}^*_{\mathrm{reg}}(\zeta)(X), \qquad X \in \mathcal{W}_{\alpha-s+1,s-1}$$

Claim: The JMS-Theorem works for new functional $Z^{\kappa,s}$ with the above functions $\rho(\zeta|\kappa,\kappa'),\omega(\zeta,\zeta'|\kappa,\kappa';\alpha)$ and kind of Dotsenko-Fateev condition

$$\kappa' = \kappa+\alpha+2s\frac{1-\nu}{\nu}$$

More precisely, the above functions $\rho$ and $\omega$ are analytical continuations of $\rho(\zeta|\kappa+\alpha-s,s)$ and $\omega(\zeta,\zeta'|\kappa,\kappa+\alpha-s,s;\alpha)$ where, for example,

$$\rho(\zeta|\kappa+\alpha-s,s)=\frac{T(\zeta,\kappa+\alpha-s,s)}{T(\zeta,\kappa)}$$

Aim is two-fold :

- to obtain the CFT with non-trivial $c=1-6\nu^2/(1-\nu)$
- to consider asymptotic series for $\kappa\to\infty$

Conjecture: $\rho^{\mathrm{sc}}(\lambda|\kappa,\kappa')=\lim\limits_{\mathrm{scaling}}\rho(\lambda\bar{a}^{\nu}|\kappa+\alpha-s,s),$

$$\omega^{\mathrm{sc}}(\lambda,\mu|\kappa,\kappa',\alpha)=\frac{1}{4}\lim_{\mathrm{scaling}}\omega(\lambda\bar{a}^{\nu},\mu\bar{a}^{\nu}|\kappa,\kappa+\alpha-s,s;\alpha)$$

# Conjectures on operators in scaling:

- The creation operators are well-defined in the scaling limit for space direction when $ja = x$ is finite

$$\tau^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{t}^*(\lambda\bar{a}^\nu), \quad \beta^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{b}^*(\lambda\bar{a}^\nu), \quad \gamma^*(\lambda) = \lim_{a\to 0}\frac{1}{2}\mathbf{c}^*(\lambda\bar{a}^\nu)$$

Asymptotic expansions at $\lambda\to\infty$ look

$$\log\bigl(\tau^*(\lambda)\bigr) \simeq \sum_{j=1}^{\infty}\tau^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}$$

$$\frac{1}{\sqrt{\tau^*(\lambda)}}\beta^*(\lambda) \simeq \sum_{j=1}^{\infty}\beta^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}, \quad \frac{1}{\sqrt{\tau^*(\lambda)}}\gamma^*(\lambda) \simeq \sum_{j=1}^{\infty}\gamma^*_{2j-1}\lambda^{-\frac{2j-1}{\nu}}.$$

# Freedom in definition of operators

- "Gauge transform":

$$\mathbf{b}^* \to e^{-\tilde{\Omega}}\, \mathbf{b}^*\, e^{\tilde{\Omega}}, \quad \mathbf{c}^* \to e^{-\tilde{\Omega}}\, \mathbf{c}^*\, e^{\tilde{\Omega}}$$

$$\tilde{\Omega} = \frac{1}{(2\pi i)^2} \oint\limits_{\Gamma} \frac{d\zeta^2}{\zeta^2} \oint\limits_{\Gamma} \frac{d\xi^2}{\xi^2} \tilde{\omega}(\zeta,\xi)\, \mathbf{c}(\xi)\mathbf{b}(\zeta), \quad \omega \to \omega + \tilde{\omega}$$

- We choose $\tilde{\omega}$ in such a way that:

$$Z_\infty(\mathcal{O} q^{\alpha S(0)}) = \frac{\langle \mathrm{vac}|\mathcal{O} q^{\alpha S(0)}|\mathrm{vac}\rangle}{\langle \mathrm{vac}|q^{\alpha S(0)}|\mathrm{vac}\rangle} = \begin{cases} 1, & \text{if } \mathcal{O} = \tau^m,\ m \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}$$

$$\zeta^{-\alpha}\mathbf{b}^*(\zeta) \to 0, \quad \zeta^{\alpha}\mathbf{c}^*(\zeta) \to 0, \quad \zeta \to 0$$

# Screening operators

- The other "gauge" choice:
  $\mathbf{b}^* \to \mathbf{b}_0^* = O(\zeta^{\alpha}), \quad \mathbf{c}^* \to \mathbf{c}_0^* = O(\zeta^{2-\alpha}), \quad \zeta \to 0$
- Acting in the subspace $\mathcal{W}_{\alpha,0}$

$$\mathbf{b}_0^*(\zeta) = \sum_{j=1}^{\infty} \zeta^{\alpha-2+2j} \mathbf{b}^*_{\mathrm{screen},j}, \quad \mathbf{c}_0^*(\zeta) = \sum_{j=1}^{\infty} \zeta^{-\alpha+2j} \mathbf{c}^*_{\mathrm{screen},j}$$

  $\mathbf{b}^*_{\mathrm{screen},j}, \mathbf{c}^*_{\mathrm{screen},j}$ are non-local.
- Scaling

$$\beta^*_{\mathrm{screen}}(\lambda) = \lim_{a\to 0} \frac{1}{2} \mathbf{b}_0^*(\lambda \bar{a}^{\nu}), \quad \gamma^*_{\mathrm{screen}}(\lambda) = \lim_{a\to 0} \frac{1}{2} \mathbf{c}^*(\lambda \bar{a}^{\nu}) \quad \text{for} \quad \lambda \to 0$$

  and for $\rho = 1$

$$\beta^*_{\mathrm{screen}}(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{\alpha+2j-2} \beta^*_{\mathrm{screen},j}, \quad \gamma^*_{\mathrm{screen}}(\lambda) \simeq \sum_{j=1}^{\infty} \lambda^{-\alpha+2j} \gamma^*_{\mathrm{screen},j}$$

# Integrals of motion

- In 1987 Alexander Zamolodchikov introduced local integrals of motion which act on local operators as

$$(\mathbf{i}_{2n-1}O)(w) = \int_{C_w} \frac{dz}{2\pi i} h_{2n}(z)O(w) \qquad (n \geq 1)$$

where the densities $h_{2n}(z)$ are certain descendants of the identity operator $I$. An important property is that

$$\langle \Delta_- | \mathbf{i}_{2n-1}\big(O(z)\big)|\Delta_+\rangle = (I^+_{2n-1} - I^-_{2n-1})\langle \Delta_-|O(z)|\Delta_+\rangle$$
$$L_n|\Delta_+\rangle = \delta_{n,0}\Delta_+|\Delta_+\rangle \quad n \geq 0, \quad \langle \Delta_-|L_n = \delta_{n,0}\Delta_-\langle \Delta_-| \quad n \leq 0$$

where $I^{\pm}_{2n-1}$ denote the vacuum eigenvalues of the local integrals of motion on the Verma module with conformal dimension $\Delta_\pm$. The Verma module is spanned by the elements

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\mathbf{l}_{-2l_1}\cdots\mathbf{l}_{-2l_q}(\phi_\alpha(0)), \quad \Delta_\alpha = \frac{\nu^2\alpha(\alpha-2)}{4(1-\nu)}$$

In case when $\Delta_+ = \Delta_-$ the space is spanned by the even Virasoro generators $\{\mathbf{l}_{-2n}\}_{n\geq 1}$.

# Asymptotic expansions

- Using the result by Bazhanov, Lukyanov, Zamolodchikov (96-99), we get asymptotic expansion

$$\log\rho^{\mathrm{sc}}(\lambda|\kappa,\kappa')\simeq\sum_{j=1}^{\infty}\lambda^{-\frac{2j-1}{\nu}}C_j\left(I^{+}_{2j-1}-I^{-}_{2j-1}\right)\rightarrow\tau^{*}_{2j-1}=C_j\mathbf{i}_{2j-1}$$

$$\omega^{\mathrm{sc}}(\lambda,\mu|\kappa,\kappa',\alpha)\simeq \qquad \text{when } \lambda^2,\mu^2\rightarrow+\infty$$

$$\sqrt{\rho^{\mathrm{sc}}(\lambda|\kappa,\kappa')}\sqrt{\rho^{\mathrm{sc}}(\mu|\kappa,\kappa')}\sum_{i,j=1}^{\infty}\lambda^{-\frac{2i-1}{\nu}}\mu^{-\frac{2j-1}{\nu}}\omega_{i,j}(\kappa,\kappa';\alpha)$$

Scaling limit of the determinant formula

$$Z_R^{\kappa,\kappa'}\left\{\tau^*(\lambda_1^0)\cdots\tau^*(\lambda_p^0)\beta^*(\lambda_1^+)\cdots\beta^*(\lambda_r^+)\gamma^*(\lambda_r^-)\cdots\gamma^*(\lambda_1^-)\left(\Phi_\alpha(0)\right)\right\}$$
$$=\prod_{i=1}^{p}\rho^{\mathrm{sc}}(\lambda_i^0|\kappa,\kappa')\times\det\left(\omega^{\mathrm{sc}}(\lambda_i^+,\lambda_j^-|\kappa,\kappa',\alpha)\right)_{i,j=1,\dots,r}.$$

Technical problem: We get coefficients $\omega_{i,j}$ by the Wiener-Hopf technique only for $\kappa=\kappa'$ when $\Delta_+=\Delta_-$ and $\rho^{\mathrm{sc}}(\zeta|\kappa,\kappa)=1$ i.e. modulo the integrals of motion.

# Correspondence to CFT 3-point correlator

- Important conjecture: it is possible to state the correspondence

$$\frac{\langle\Delta_-|P_\alpha(\{\mathbf{I}_{-k}\})\phi_\alpha(0)|\Delta_+\rangle}{\langle\Delta_-|\phi_\alpha(0)|\Delta_+\rangle}=\lim_{\mathbf{n}\to\infty,a\to 0,\mathbf{n}a=2\pi R} Z^{\kappa,s}\left\{q^{2\alpha S(0)}\mathcal{O}\right\}.$$

between a polynomial $P_\alpha(\{\mathbf{I}_{-k}\})$ and some combinations of $\beta^*_{2j-1},\gamma^*_{2j-1}$.

Introduce $\quad \beta^*_{2m-1}=D_{2m-1}(\alpha)\beta^{\mathrm{CFT}*}_{2m-1},\qquad \gamma^*_{2m-1}=D_{2m-1}(2-\alpha)\gamma^{\mathrm{CFT}*}_{2m-1}$

$$D_{2m-1}(\alpha)=\frac{1}{\sqrt{i\nu}}\Gamma(\nu)^{-\frac{2m-1}{\nu}}(1-\nu)^{\frac{2m-1}{2}}\frac{1}{(m-1)!}\frac{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2\nu}(2m-1)\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{(1-\nu)}{2\nu}(2m-1)\right)}$$

together with even and odd bilinear combinations

$$\phi^{\mathrm{even}}_{2m-1,2n-1}=(m+n-1)\frac{1}{2}\left(\beta^{\mathrm{CFT}*}_{2m-1}\gamma^{\mathrm{CFT}*}_{2n-1}+\beta^{\mathrm{CFT}*}_{2n-1}\gamma^{\mathrm{CFT}*}_{2m-1}\right),$$

$$\phi^{\mathrm{odd}}_{2m-1,2n-1}=d_\alpha^{-1}(m+n-1)\frac{1}{2}\left(\beta^{\mathrm{CFT}*}_{2n-1}\gamma^{\mathrm{CFT}*}_{2m-1}-\beta^{\mathrm{CFT}*}_{2m-1}\gamma^{\mathrm{CFT}*}_{2n-1}\right),$$

$$d_\alpha=\frac{\nu(\nu-2)}{\nu-1}(\alpha-1)$$

# Identification with Virasoro Verma-module

- If we accept an equivalence of the spaces spanned by

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\mathbf{l}_{-2l_1}\cdots\mathbf{l}_{-2l_q}\big(\phi_\alpha(0)\big) \qquad \text{and}$$

$$\mathbf{i}_{2k_1-1}\cdots\mathbf{i}_{2k_p-1}\phi^{\text{even}}_{2m_1-1,2n_1-1}\cdots\phi^{\text{even}}_{2m_r-1,2n_r-1}\phi^{\text{odd}}_{2\bar{m}_1-1,2\bar{n}_1-1}\phi^{\text{odd}}_{2\bar{m}_{\bar{r}}-1,2\bar{n}_{\bar{r}}-1}\big(\Phi_\alpha(0)\big)$$

we can identify modulo integrals of motion ($\Delta\equiv\Delta_\alpha$)

$$\phi^{\text{even}}_{1,1}\cong\mathbf{l}_{-2},\qquad \phi^{\text{even}}_{1,3}\cong\mathbf{l}^2_{-2}+\frac{2c-32}{9}\mathbf{l}_{-4},\qquad \phi^{\text{odd}}_{1,3}\cong\frac{2}{3}\,\mathbf{l}_{-4}$$

$$\phi^{\text{even}}_{1,5}\cong\mathbf{l}^3_{-2}+\frac{c+2-20\Delta+2c\Delta}{3(\Delta+2)}\mathbf{l}_{-4}\mathbf{l}_{-2}\ +\ \cdots\ \mathbf{l}_{-6}$$

$$\phi^{\text{odd}}_{1,5}\cong\frac{2\Delta}{\Delta+2}\mathbf{l}_{-4}\mathbf{l}_{-2}+\frac{56-52\Delta-2c+4c\Delta}{5(\Delta+2)}\,\mathbf{l}_{-6}$$

$$\phi^{\text{even}}_{3,3}\cong\mathbf{l}^3_{-2}+\frac{6+3c-76\Delta+4c\Delta}{6(\Delta+2)}\mathbf{l}_{-4}\mathbf{l}_{-2}\ +\ \cdots\ \mathbf{l}_{-6}$$

# Fermionic construction of primary field

Let $\mathcal{V}_\alpha$ be the subspace obtained by acting $\beta^*_{2j-1}, \gamma^*_{2j-1}$ and integrals of motion $\mathbf{i}_{2k-1}$. In case $\kappa = \kappa'$ we factor out the integrals of motion

$$\mathcal{V}^{\mathrm{quo}}_\alpha = \mathcal{V}_\alpha / \sum \mathbf{i}_{2k-1} \mathcal{V}_\alpha$$

The basis of $\mathcal{V}^{\mathrm{quo}}_\alpha$ : $\qquad \beta^*_{I^+}, \gamma^*_{I^-} \Phi_\alpha(0)$

$$\beta^*_{I^+} = \beta^*_{2k_1-1} \cdots \beta^*_{2k_n-1}, \quad \gamma^*_{I^-} = \gamma^*_{2j_n-1} \cdots \gamma^*_{2j_1-1}.$$

Acting on $\Phi_\alpha(0)$ by $\beta^*_{2j-1}, \gamma^*_{2j-1}, \beta^*_{\mathrm{screen},j}, \gamma^*_{\mathrm{screen},j}$, one gets a space $\mathcal{H}_\alpha \supset \mathcal{V}^{\mathrm{quo}}_\alpha$

The claim is: $\quad \mathcal{V}^{\mathrm{quo}}_{\alpha+2m\frac{(1-\nu)}{\nu}} \subset \mathcal{H}_\alpha, \quad m \in \mathbb{Z}_{\geq 0} \quad$ Jimbo, Miwa, Smirnov (11)

In particular: $\Phi_{\alpha+2m\frac{(1-\nu)}{\nu}}(0) \cong \beta^*_{I_{\mathrm{odd(m)}}} \gamma^*_{\mathrm{screen},I(m)} \Phi_\alpha(0)$

$$\gamma^*_{\mathrm{screen},I(m)} = \gamma^*_{\mathrm{screen},m} \cdots \gamma^*_{\mathrm{screen},1}, \quad I(m) = (1,2,\cdots,m), I_{\mathrm{odd(m)}} = (1,3,\cdots,2m-1)$$

Conformal dimensions of operators:

$$\beta^*_{2j-1}, \gamma^*_{2j-1} : \quad 2j-1, \quad \beta^*_{\mathrm{screen},j} : \quad \nu(2-\alpha-2j), \quad \gamma^*_{\mathrm{screen},j} : \quad \nu(\alpha-2j)$$

# OPE in fermionic basis

Here we take parameterization used in Liouville CFT

$$\nu = 1 + b^2, \quad c = 1 + 6Q^2, \quad Q = b + b^{-1}$$

$$\alpha = \frac{2a}{Q}, \quad \kappa = \frac{2a_1}{Q} - 1, \quad \kappa' = \frac{2a_2}{Q} - 1, \quad \Delta \to \Delta_a = a(Q-a)$$

- The OPE in the fermionic basis looks Smirnov,HB 2015

$$\Phi_{a_1}(z)\Phi_{a_2}(0) \cong$$

$$\sum_a C^a_{a_1,a_2}\ z^{\Delta_a - \Delta_{a_1} - \Delta_{a_2}} \sum_{\#(I^+)=\#(I^-)} z^{|I^+|+|I^-|}\Omega_{I^+,I^-}(a_1,a_2,a)\beta^*_{I^+}\gamma^*_{I^-}\Phi_a(0)$$

where $\cong$ means modulo integrals of motion,

$$\Omega_{I^+,I^-}(a_1,a_2,a) \equiv \Omega_{I^+,I^-}(a) = \frac{R_{I^+,I^-}(a_1,a_2,a)}{\prod_{I\in I^+} D_I(a)\prod_{I\in I^-} D_I(Q-a)}$$

$R_{I^+,I^-}(a_1,a_2,a)$ is a rational function.

# Pole structure

The poles of $R_{l^+,l^-}(a_1,a_2,a)$ are at the points

$$a = a_{m,n}, \quad a_{m,n} = -\frac{n-1}{2}b - \frac{m-1}{2}b^{-1}$$

where $mn \equiv 0 \pmod 2$. We consider only the case $n \equiv 0 \pmod 2$, the case $m \equiv 0 \pmod 2$ is obtained by duality $b \to 1/b$.

- "Resonance" poles: $a = a_{m,n}$ and $a = a_{-m,-n} = Q - a_{m,n}$ with $m,n \geq 1$
- "Unwanted" poles: $a = a_{m,n}$ with $m = 0$ or $n = 0$ or $m > 0, n < 0$ or $m < 0, n > 0$

# Null-Vectors

Introduce formally annihilation operators $\beta_j = t_j(a)^{-1}\gamma^*_{-j},\ \gamma_j = t_j(Q-a)^{-1}\beta^*_{-j}$ so that

$$[\beta_j,\beta^*_k]_+ = \delta_{j,k}, \quad [\gamma_j,\gamma^*_k]_+ = \delta_{j,k}$$

For $n \equiv 0(\mathrm{mod}2)$ the null-vectors are constructed as

$$\left(C_{m,n}\right)^{\frac{n}{2}}\beta^*_{I^+}\gamma^*_{I^-}, \quad \#(I^-) = \#(I^+) + n$$

Jimbo, Miwa, Smirnov (11)

$$C_{m,n} = \sum_{j=1}^{\infty}\beta^*_{2j-1}\gamma_{2j-1+2(n-m)} + \sum_{j=1}^{\left[\frac{n-m}{2}\right]} t_{2j-1}(a_{m,n})\gamma_{2j-1}\gamma_{2(n-m)-2j+1}, \quad n \geq m$$

$$C_{m,n} = \sum_{j=1}^{\infty}\beta^*_{2j+2(m-n)-1}\gamma_{2j-1} - \sum_{j=1}^{\left[\frac{m-n}{2}\right]} \frac{1}{t_{-2j+1}(a_{m,n})}\beta^*_{2j-1}\beta^*_{2(m-n)-2j+1}, \quad n < m$$

$$t_p(a) = \frac{1}{2\sin\frac{\pi}{Q}(p\,b^{-1} - 2a)}$$

# Recursion relations

In 1984 Alexei Zamolodchikov found recursion relations for the residues of conformal blocks. Here we have a similar relation.

- If $n-m$ is not odd positive (otherwise a bit more complicated), we have

$$\mathrm{res}_{a=a_{m,n}} \sum_{\#(I^+)=\#(I^-)} z^{|I^+|+|I^-|}\,\Omega_{I^+,I^-}(a)\;\beta^*_{I^+}\gamma^*_{I^-}\Phi_a$$

$$= W_{m,n} \sum_{\#(I^-)=\#(I^+)} z^{nm+|I^+|+|I^-|}\Omega_{I^+,I^-}(a_{m,n}+nb)\big(C_{m,n}\big)^{\frac{n}{2}}\beta^*_{I^+-2n}\gamma^*_{I^-+2n}\gamma^*_{I_{\mathrm{odd}(n)}}\Phi_a$$

where $\gamma^*_{I_{\mathrm{odd}(n)}}=\gamma^*_{2n-1}\cdots\gamma^*_1$ and $W_{m,n}$ contains

$$P_{m,n}=\prod_{j=1}^{n}\prod_{k=1}^{m}\Big(a_1+a_2-\frac{m-2k+3}{2}b^{-1}-\frac{n-2j+3}{2}b\Big)$$
$$\times\Big(a_1-a_2-\frac{m-2k+1}{2}b^{-1}-\frac{n-2j+1}{2}b\Big)$$

and a multiplier depending on $b$ only. $W_{m,n}$ is related to 3-point function of Liouville CFT.

# Conformal block

On a cylinder of radius $R = 1$ we consider the CFT with boundary conditions at $\pm\infty$ given by $\Phi_{a_2}$, $\Phi_{Q-a_2}$.
We know that the expectation values of the fermionic descendants of $\Phi_a$ on a cylinder are expressed in terms of function $\omega^{\mathrm{sc}}$ or, more precisely, in terms of coefficients in its asymptotic expansion at $\lambda,\mu \to \infty$:

$$\omega^{\mathrm{sc}}(\lambda,\mu|\kappa,\kappa',\alpha) \to \omega^{\mathrm{sc}}(\lambda,\mu|a_2,a) \simeq \sum_{i,j=1}^{\infty} \lambda^{-\frac{2i-1}{\nu}} \mu^{-\frac{2j-1}{\nu}} \omega_{2i-1,2j-1}(a_2,a)\,.$$

We know from the TBA-like approach

$$\omega_{2i-1,2j-1}(a_2,a) = D_{2i-1}(a) D_{2j-1}(Q-a) \Theta_{2i-1,2j-1}(a_2,a)\,,$$

where $\Theta_{2i-1,2j-1}(a_2,a)$ depends on $a_2$ as a polynomial in $\Delta_{a_2}$, and on $a$ as a polynomial in $\Delta_a$ with additional linear dependence on

$$d(a) = (b^{-1} - b)(Q - 2a)$$

The conformal block on the cylinder is obtained by computing the expectation values for the OPE. Introduce $\omega_{I^+,I^-}(a_2,a)$ by

$$\exp\Big\{\sum_{i,j=1}^{\infty}\omega_{2i-1,2j-1}(a_2,a)\gamma_{2i-1}\beta_{2j-1}\Big\}=\sum_{I^+,I^-}\omega_{I^+,I^-}(a_2,a)\gamma_{I^-}\beta_{I^+}$$

Then the conformal block is

$$\mathcal{F}(a_1,a_2,a,z)=z^{-2\Delta_{a_1}+\Delta_a}\sum_{I^+,I^-}z^{|I^+|+|I^-|}\Omega_{I^+,I^-}(a_1,a)\omega_{I^+,I^-}(a_2,a)$$

where $\Omega_{I^+,I^-}(a_1,a)=\Omega_{I^+,I^-}(a_1,a_1,a)$ are the above OPE-coefficients. Via the conformal mapping of the cylinder to the sphere, one can get symmetric under $a_1\leftrightarrow a_2$ combination

$$\mathcal{F}_{\mathrm{sym}}(a_1,a_2,a,z)=\Big(2\sinh\big(z/2\big)\Big)^{2\Delta_{a_1}}\mathcal{F}(a_1,a_2,a,z)$$

- Remark 1 In the conformal block $\mathcal{F}(a_1,a_2,a,z)$ or $\mathcal{F}_{\mathrm{sym}}(a_1,a_2,a,z)$ the unwanted poles cancel. So, we observe a nice duality between $\omega$ and $\Omega$.
- Remark 2 The behaviour at $a\to\infty$ is described by

$$\log\big(\mathcal{F}_{\mathrm{sym}}(a_1,a_2,a,z)\big)=$$

$$+\Delta_a\Big(\log z-\tfrac{1}{192}z^2+\tfrac{73}{1474560}z^4-\tfrac{1069}{1486356480}z^6+\tfrac{250993}{20293720473600}z^8+\cdots\Big)$$

$$+(\Delta_{a_1}+\Delta_{a_2})\Big(\tfrac{1}{16}z^2-\tfrac{5}{12288}z^4+\tfrac{17}{2949120}z^6-\tfrac{1705}{16911433728}z^8+\cdots\Big)$$

$$+(c-1)\Big(-\tfrac{1}{512}z^2+\tfrac{7}{786432}z^4-\tfrac{19}{188743680}z^6+\tfrac{16019}{10823317585920}z^8+\cdots\Big)+O(a^{-1})$$

This is in full agreement with the formula by Alexei Zamolodchikov

# Application to QFT, further perspectives

## Further applications to QFT:

- Jimbo, Miwa and Smirnov (10-13) succeeded in applying fermionic structure for computation of one-point functions of the sine-Gordon model on cylinder
- also for sinh-Gordon model at finite temperature Smirnov, Negro (13)

## Some problems are still open:

- "$\rho$-problem" of escaping the restriction $\kappa = \kappa'$ and developing new methods for general case
- Generalization to higher spin and higher rank case. Some preliminary steps were undertaken for better understanding of the functional relations Göhmann, Klümper, Nirov, Razumov, HB (10-14)

# Conclusions

- The factorization of static correlation functions of the XXZ spin chain was origially observed as a factorization of corresponding multiple integrals
- Factorized form is represented through polynomials of two transcendental functions $\rho$ and $\omega$.
- There is reach algebraic structure behind this factorization – hidden fermionic structure.
- This structure holds in conformal limit and for sine(sinh)-Gordon model.