

Correlations and field theory inside the arctic circle [or Arctic quenches]

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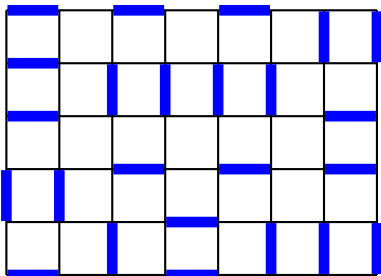
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in collaboration with N. Allegra, J. Dubail, M. Haque and J. Viti.

Outline

- 1 Arctic circle in Statistical mechanics
 - Dimers
 - Six vertex
- 2 A simple toy model
 - Model and Motivations
 - Correlations in the bulk
 - Dirac action in curved space
- 3 Consequences and generalizations
 - Dimers and vertex models
 - Back to real time

Classical Dimers in 2d



- dimers with hardcore constraint.
- Exactly solvable: free fermions.
- $Z = \det(\dots)$ [Kasteleyn, Fisher]
- Critical system

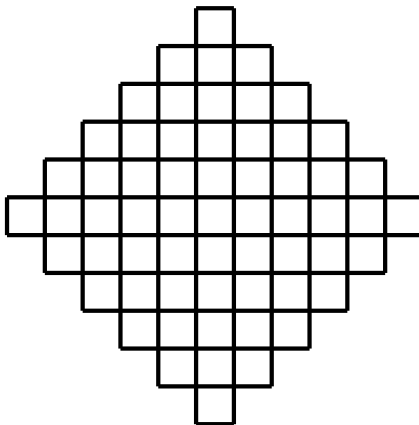
Long distance limit: Dirac field or free gaussian compact field

$$S = \frac{g}{4\pi} \int dx dy (\nabla\varphi)^2, \quad \varphi = \varphi + 2\pi$$

$$C_{dd}(\mathbf{r}, \mathbf{r}') = |\mathbf{r} - \mathbf{r}'|^{-1/g}, \quad C_{mm}(\mathbf{r}, \mathbf{r}') = |\mathbf{r} - \mathbf{r}'|^{-g}$$

Aztec diamond, and the arctic circle

[Jokusch, Propp and Shor, 1995]



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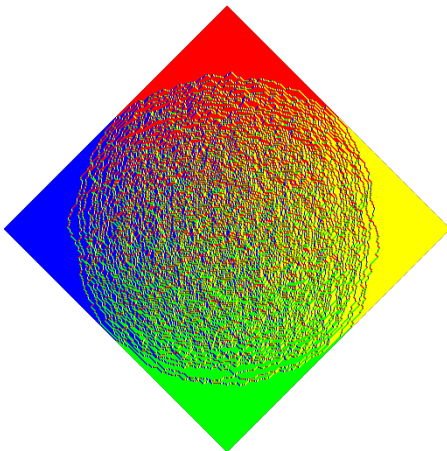
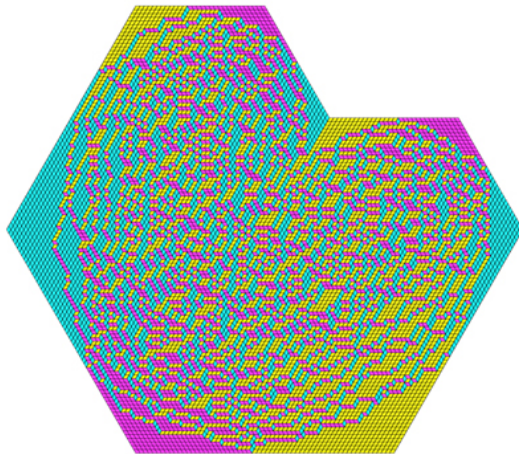


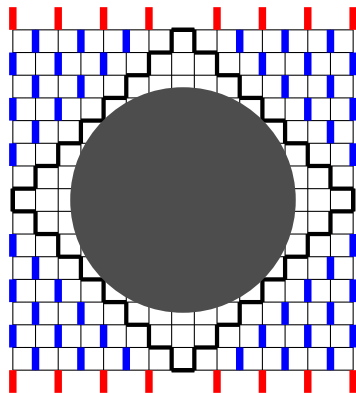
Image at <http://tuvalu.santafe.edu/~moore/gallery.html>

Lots of variations on this



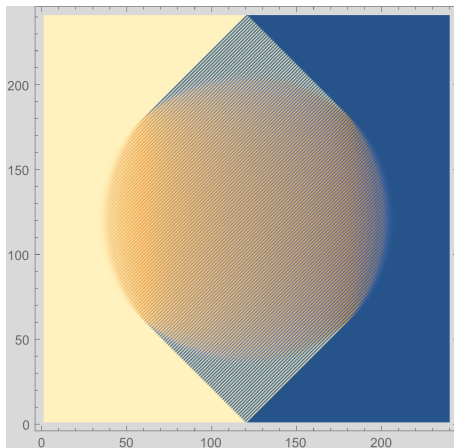
Theory for the shape [Kenyon, Okounkov, Sheffield]

Can be induced by boundary conditions



$$Z = \langle \psi_0 | T^{L_y} | \psi_0 \rangle$$

Density profile for dimers

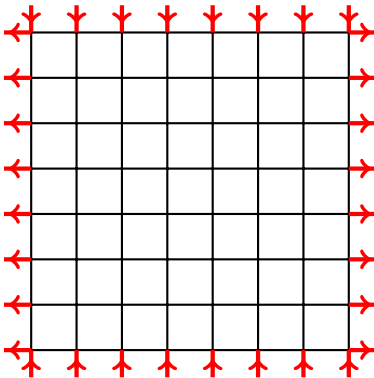


$$\rho(x, y) = \frac{1}{2} + \frac{1}{\pi} \arctan \left(\frac{2y-1}{\sqrt{1-2x^2-2y^2}} \right)$$

[Cohn, Elkies and Propp, Duke. Math. Journ 1996]

Six vertex with domain wall boundary conditions

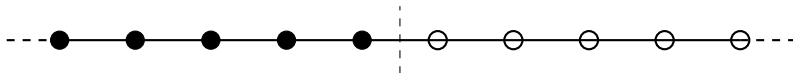
[Korepin, Izergin, Zinn-Justin, Colomo, Pronko, ...]



Conjecture for the arctic curve [Colomo & Pronko, J. Stat. Phys 2010]

Field theory inside the circle?

A simple toy model



$$|\psi_0\rangle = \prod_{x<0} c_x^\dagger |0\rangle$$

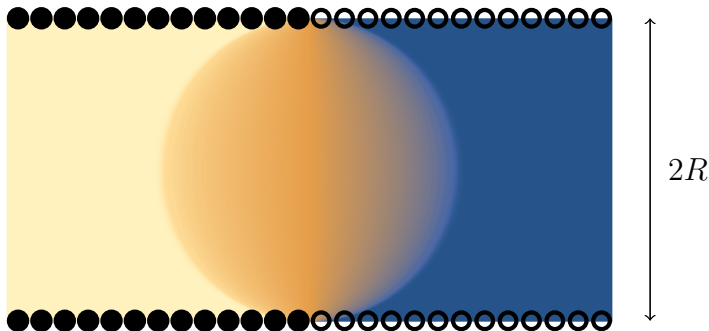
$$H = -\frac{1}{2} \sum_x \left(c_x^\dagger c_{x+1} + h.c \right)$$

Imaginary time evolution

$$Z = \langle \psi_0 | e^{-2RH} | \psi_0 \rangle$$

Motivations (1/3)

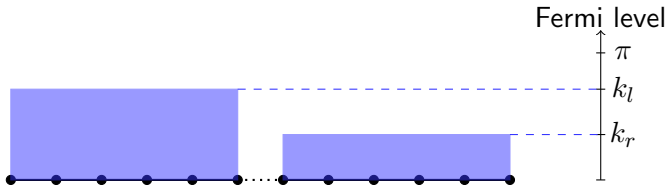
- TASEP in continuous time [Rost, Zeit. Wahrs. 1981].
- Arctic circle phenomenology.



- Quantum mechanics from a domain wall initial state.
[Antal Rácz Rákos Schütz 1999 ; Antal Krapivsky Rákos 2008 ; ...]

Motivation (2/3): Filling fraction quenches

$$H = - \sum_j \left(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j \right) = \sum_k \varepsilon_k c_k^\dagger c_k$$

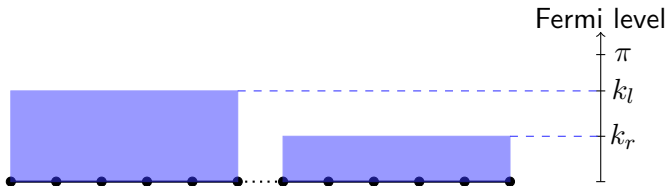


Fermions, with certain Fermi levels (k_l, k_r)

$$|\psi_0\rangle = |k_l\rangle \otimes |k_r\rangle$$

and let evolve with $H(\frac{k_l+k_r}{2})$ at time $t > 0$

Motivation (2/3): Filling fraction quenches



Limiting cases:

- $k_l = \pi$, $k_r = 0$ is the domain wall quench.
- $k_l = k_r$

[Eisler, Karevski, Platini & Peschel, 2008] [Calabrese & Cardy, 2008]

[JMS & Dubail, 2011]

Motivation (3/3) low energy local quenches

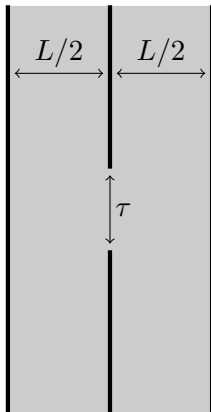
$$k_l = k_r$$

$$\mathcal{L}(\tau) = |\langle \psi_0 | e^{-\tau H_{\text{tot}}} | \psi_0 \rangle|^2$$

Keep in mind

$\tau \rightarrow it$, but only at the end.

$\mathcal{F}(\tau) = -\ln \mathcal{L}(\tau)$ is a free energy! [JMS & Dubail, 2011]

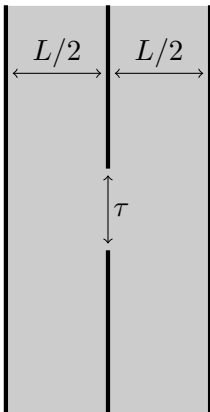


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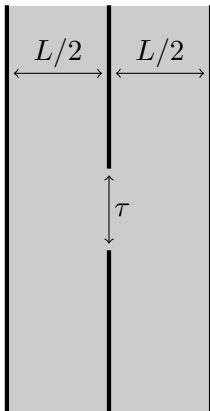
Loschmidt echo

$$\mathcal{F}(\tau) = \frac{c}{4} \ln \left| \frac{L}{\pi} \sinh \left(\frac{\pi \tau}{L} \right) \right|$$

Motivation (3/3) low energy local quenches $k_l = k_r$

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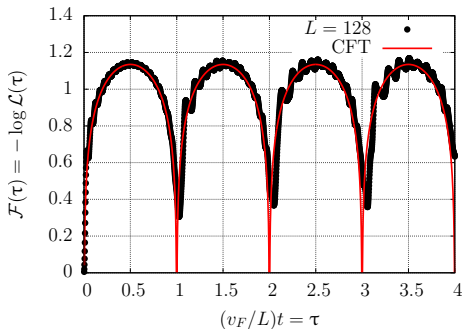
Back to real time

$$\mathcal{F}(t) = \frac{c}{4} \ln \left| \frac{L}{\pi} \sin \left(\frac{\pi t}{L} \right) \right|$$

Symmetric case $L_A = L_B = L/2$

The Loschmidt echo is periodic

$$\mathcal{F}(t) = \frac{c}{4} \ln \left| \frac{L}{\pi} \sin \left(\frac{\pi t}{L} \right) \right|$$



Non symmetric case $L_A = L/3$

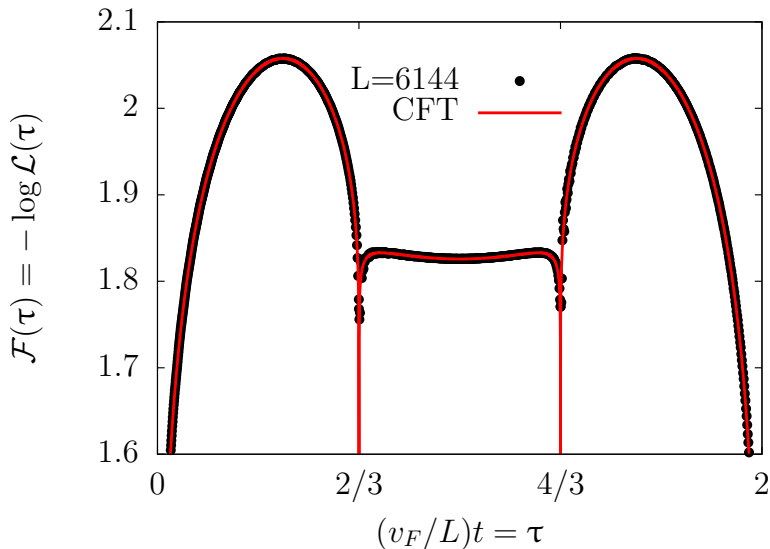
Loschmidt echo

$$\mathcal{F}(a) = \frac{c}{4} \ln L + \frac{c}{24} \ln \left| \frac{a^3(a+1)^6(a+2)(2a+1)}{(a-1)^7} \right|$$

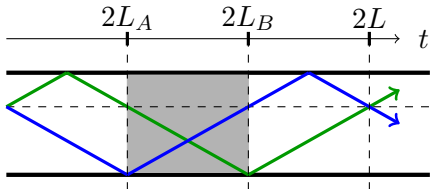
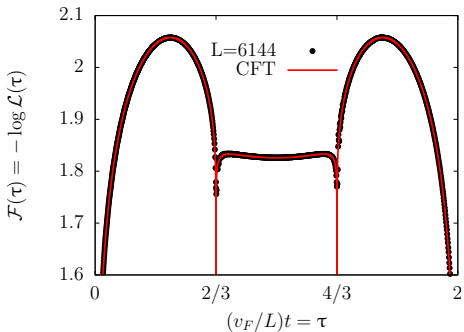
a is one of the solutions of

$$it = \frac{2L}{\pi} \left[\frac{1}{3} \ln \left(\frac{b-1}{b+1} \right) + \frac{2}{3} \ln \left(\frac{a-b}{a+b} \right) \right]$$

$$b^2 = a \frac{a+2}{2a+1}$$

Non symmetric case $L_A = L/3$ 

Non symmetric case $L_A = L/3$



Motivation (end)

Here it is not that simple, because (need both!)

- 1 Conservation of the number of particles
- 2 Inhomogeneous initial state

Naive low energy-field theory does not work.

Correlations inside the “circle”

Want to compute

$$\left\langle c_x^\dagger(y) c_{x'}(y') \right\rangle = \frac{\langle \psi_0 | e^{-(R+y)H} c_x^\dagger e^{-(y-y')H} c_{x'} e^{-(R-y')H} | \psi_0 \rangle}{\langle \psi_0 | e^{-2RH} | \psi_0 \rangle}$$

$$c^\dagger(k) = \sum_{x \in \mathbb{Z}} e^{-ikx} c_x^\dagger \quad , \quad [H, c^\dagger(k)] = \varepsilon(k) c^\dagger(k)$$

Appearance of the Hilbert transform

$$\langle c^\dagger(k, y)c(k', y') \rangle = \frac{\langle \psi_0 | e^{-(R-y)H} c^\dagger(k) e^{-(y'-y)H} c(k') e^{-(R+y)H} | \psi_0 \rangle}{\langle \psi_0 | e^{-2RH} | \psi_0 \rangle}$$

One can show that

$$\langle c^\dagger(k, y)c(k', y') \rangle = \frac{e^{-iR[\tilde{\varepsilon}(k) - \tilde{\varepsilon}(k')]} e^{y\varepsilon(k) - y'\varepsilon(k')}}{2i \sin\left(\frac{k-k'}{2} + i0^+\right)}$$

$$\tilde{\varepsilon}(k) = \text{Hilbert transform of } \varepsilon(k) = \text{pv} \int_{-\pi}^{\pi} \frac{dq}{2\pi} \varepsilon(q) \cot \frac{k-q}{2}.$$

Bosonisation trick

$$c^\dagger(k) \rightarrow : e^{i\varphi(k)} :$$

$$c(k) \rightarrow : e^{-i\varphi(k)} :$$

$$c^\dagger(k)c(k) \rightarrow \partial\varphi(k)$$

Fusion of two vertex operators:

$$: e^{\alpha\varphi(k)} :: e^{\beta\varphi(k')} := e^{\alpha\beta\langle\varphi(k)\varphi(k')\rangle} : e^{\alpha\varphi(k)+\beta\varphi(k')} :$$

Need to define

- Normal order : $c_x^\dagger c_x$:
- $\langle\varphi(k)\varphi(k')\rangle = \log \sin \frac{k-k'}{2}$

Comments

- The result is exact.
- This completely solves the problem in principle.
- Real space correlations: inverse Fourier transform+stationary phase approximation.
- Stationary points

$$x + iy \frac{d\varepsilon(k)}{k} + R \frac{d\tilde{\varepsilon}(k)}{dk} = 0$$

Density profile

$$\langle c_x^\dagger(y) c_x(y) \rangle = \frac{1}{\pi} \arccos \frac{x}{\sqrt{R^2 - y^2}}$$



$$Z = e^{-R^2/2}$$

Dirac in curved space

$$c_x^\dagger = \frac{e^{-i(\pi/4+\theta(x,y))}}{\sqrt{2\pi}} \psi^\dagger(x,y) + \frac{e^{i(\pi/4+\theta^*(x,y))}}{\sqrt{2\pi}} \bar{\psi}^\dagger(x,y)$$

where

$$\langle \psi^\dagger(x,y) \psi(x',y') \rangle = \frac{e^{-\frac{1}{2}[\sigma(x,y)+\sigma(x',y')]}{\sin\left(\frac{z(x,y)-z(x',y')}{2}\right)}$$

$$z(x,y) = \arcsin \frac{x}{\sqrt{R^2 - y^2}} + i \operatorname{arcth} \frac{y}{R}$$

$$\theta(x,y) = -z(x,y)x - \sqrt{R^2 - x^2 - y^2}$$

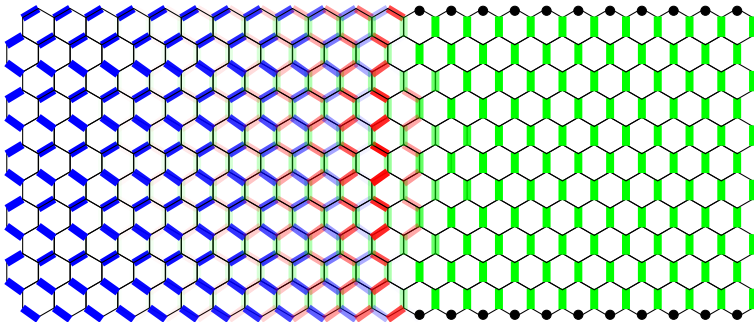
$$\sigma(x,y) = \log \sqrt{R^2 - x^2 - y^2}$$

Dirac theory with a metric $ds^2 = e^{2\sigma}(dx^2 + dy^2)$

Dimers on honeycomb

$$T^2 = \exp \left[- \int \frac{dk}{2\pi} \varepsilon(k) c(k)^\dagger c(k) \right]$$

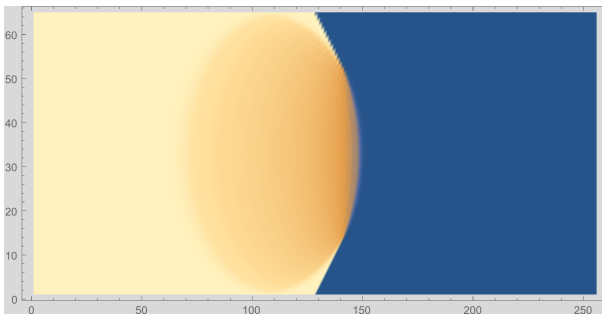
$$\varepsilon(k) = \log(1 + u^2 + 2u \cos k)$$



Dimers on honeycomb

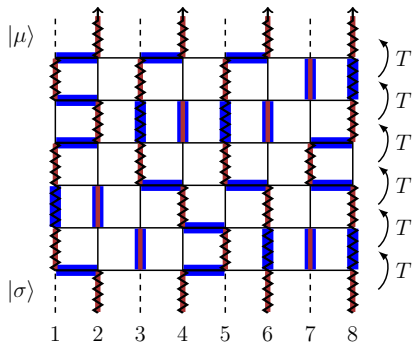
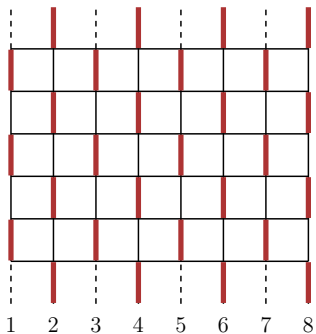
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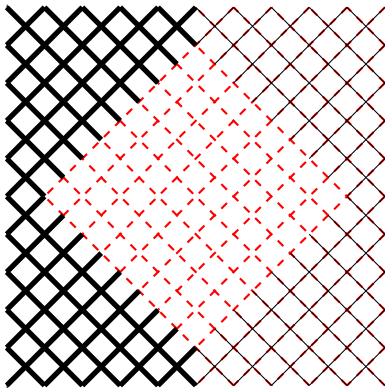


Dimers on square

Mapping: two bands



Six vertex



a



a



b



b



c



c

Back to the toy model

Is the analytic continuation $y = it$, $R \rightarrow 0$ consistent?

Analytic continuation

- Loschmidt echo

$$L(\tau) = \langle \psi_0 | e^{-\tau H} | \psi_0 \rangle = e^{\tau^2/8} \quad \longrightarrow \quad L(t) = e^{-t^2/8}$$

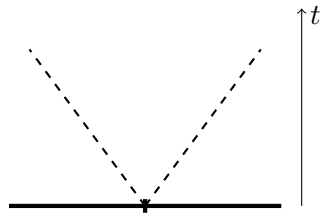
- Density

$$\rho(x, y, R) = \frac{1}{\pi} \arccos \frac{x}{\sqrt{R^2 - y^2}} \quad \longrightarrow \quad \rho(x, t) = \frac{1}{\pi} \arccos \frac{x}{t}$$

Analytic continuation of the circle



= light cone.



Analytic continuation

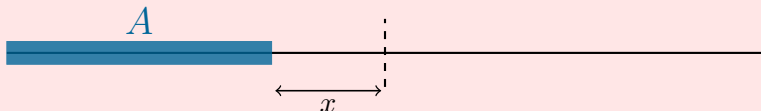
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Prediction for the entanglement entropy



$$S(x, t) = \frac{1}{6} \log \left[t (1 - x^2/t^2)^{3/2} \right]$$

Conclusion

- Arctic circle revisited: correlations.
- Inhomogeneous system, so inhomogeneous field theory.
- Other initial states.
- Interactions?
- Logarithmic terms in six-vertex DWBC?

Thank you!