The Entropy of Six-Vertex Model with Variety of Different Boundary Conditions

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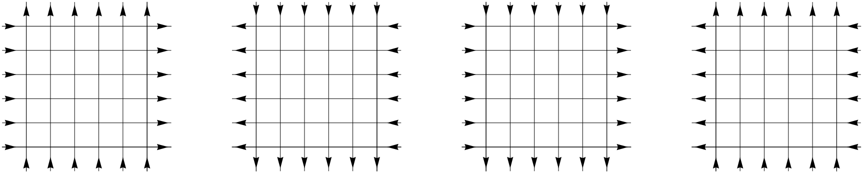
A non-trivial problem in combinatorics

- Six-vertex model was proposed as a 2D realization of the counting problem of ice residual entropy
- Solved by Lieb under periodic (toroidal) boundary condition: $S = \frac{3}{2} \ln \left(\frac{4}{3} \right)$.
- Why Periodic Boundary Conditions? Should we always expect intensive properties to be independent of boundary conditions?



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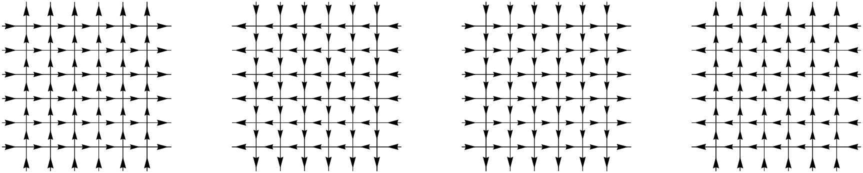
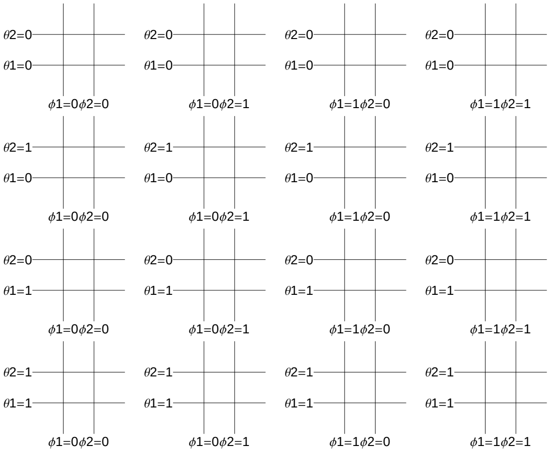


- The first Counter-examples! Are they exceptions to the rule?

- Brascamp et al prove that, for rectangular lattices with even number of sites, the free-energy of free boundary conditions and periodic boundary conditions are the same(1973).
- Batchelor et al prove that toroidal boundary conditions with antiperiodic closing on the horizontal and periodic closing on the vertical still gives the same free-energy as PBC(1995). Nevertheless, the number of lines must be even otherwise partition function is zero.
- Korepin and Zinn-Justin prove that Domain-Wall boundary conditions gives a different free-energy. The residual entropy is $S = \frac{1}{2}\ln(\frac{3^3}{2^4})$.
- What is really happening? Are those kinds of boundary really exceptions?

- Arrows can be either equal or opposite at closing!
- Horizontal: $0 \Rightarrow T^{(0)} = A + D$, $1 \Rightarrow T^{(1)} = B + C$
- Vertical: $0 \Rightarrow \mathcal{G}^{(0)} = Id$, $1 \Rightarrow \mathcal{G}^{(1)} = \sigma_x$

$$Z_{free} = \sum_{\phi_k, \theta_j = 0,1} \mathrm{Tr}_V \left[\bigotimes_{k=1}^{L} \mathcal{G}_{V_k}^{(\phi_k)} \prod_{j=1}^{N} T^{(\theta_j)}(\lambda_j) \right] = \mathrm{Tr}_V \left[\bigotimes_{k=1}^{L} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_k (A(\lambda) + D(\lambda) + B(\lambda) + C(\lambda))^N \right], \quad (1)$$



- Each component of the previous sum can be viewed as a particular toroidal boundary condition, which mix periodic and anti-periodic closings.

One may organize these contributions in a matrix $M_{N,L}$ whose elements are the partitions $Z_{j,k}$ such

$$j-1=\theta_1 2^0+\theta_2 2^1+\cdots+\theta_N 2^{N-1},$$
$$k-1=\phi_1 2^0+\phi_2 2^1+\cdots+\phi_L 2^{L-1}$$

$$M_{N,L}=\left(\begin{array}{cccc} Z_{1,1} & Z_{1,2} & \cdots & Z_{1,2^L} \\ Z_{2,1} & Z_{2,2} & \cdots & Z_{2,2^L} \\ \vdots & \vdots & \ddots & \vdots \\ Z_{2^N,1} & Z_{2^N,2} & \cdots & Z_{2^N,2^L} \end{array}\right), \qquad Z_{free}=\sum_{j=1}^{2^N}\sum_{k=1}^{2^L} Z_{j,k}. \qquad (2)$$

- $a = b = c = 1$

$$M_{2,2} = \left(\begin{array}{cc|cc} 18 & 0 & 0 & 8 \\ 0 & 10 & 10 & 0 \\ \hline 0 & 10 & 10 & 0 \\ 8 & 0 & 0 & 8 \end{array}\right) \quad M_{2,3} = \left(\begin{array}{cccc|cccc} 44 & 0 & 0 & 20 & 0 & 20 & 20 & 0 \\ 0 & 26 & 24 & 0 & 26 & 0 & 0 & 16 \\ \hline 0 & 26 & 24 & 0 & 26 & 0 & 0 & 16 \\ 26 & 0 & 0 & 20 & 0 & 20 & 20 & 0 \end{array}\right)$$

$$M_{3,2} = \left(\begin{array}{cc|cc} 44 & 0 & 0 & 26 \\ 0 & 26 & 26 & 0 \\ 0 & 24 & 24 & 0 \\ 20 & 0 & 0 & 20 \\ \hline 0 & 26 & 26 & 0 \\ 20 & 0 & 0 & 20 \\ 20 & 0 & 0 & 20 \\ 0 & 16 & 16 & 0 \end{array}\right) \quad M_{3,3} = \left(\begin{array}{cccc|cccc} 148 & 0 & 0 & 84 & 0 & 84 & 84 & 0 \\ 0 & 94 & 84 & 0 & 94 & 0 & 0 & 72 \\ 0 & 84 & 80 & 0 & 84 & 0 & 0 & 72 \\ 84 & 0 & 0 & 74 & 0 & 72 & 74 & 0 \\ \hline 0 & 94 & 84 & 0 & 94 & 0 & 0 & 72 \\ 84 & 0 & 0 & 72 & 0 & 76 & 72 & 0 \\ 84 & 0 & 0 & 74 & 0 & 72 & 74 & 0 \\ 0 & 72 & 72 & 0 & 72 & 0 & 0 & 68 \end{array}\right)$$

- selection rule $\mathrm{Mod}\,[\Phi - \Theta, 2] = 0$
- $Z_{j,k}^{N\times L} = Z_{k,j}^{L\times N}$
- $Z_{1,1} = \Omega_{P,P}$ is the largest element for $\Delta = \frac{1}{2}$

$$\Omega_{PP} \leq \Omega_{free} \leq 2^{L+N-1}\Omega_{PP} \;\Rightarrow\; S_{PP} = S_{free}. \tag{3}$$

- $\Delta \neq \frac{1}{2}$

We have more generally that the largest element is Z_{PP} for $\Delta \geq -1$ and

	Largest contribution for $\Delta < -1$
L even, N even	Z_{PP}
L even, N odd	Z_{PA}
L odd, N even	Z_{AP}
L odd, N odd	Z_{AA}

- This scenario was verified for L, N up to six.

$$F_{free} = F_{\max} \tag{4}$$

The homogenous toroidal boundaries are those where there is no change from periodic to anti-periodic along the horizontal or the vertical direction. They are:

$$Z_{11} = Z_{PP} = \mathrm{Tr}_V \left[\left(T^{(0)} \right)^N \right] \tag{5}$$

$$Z_{2N_1} = Z_{AP} = \mathrm{Tr}_V \left[\left(T^{(1)} \right)^N \right] \tag{6}$$

$$Z_{12^L} = Z_{PA} = \mathrm{Tr}_V \left[\Pi^x \left(T^{(0)} \right)^N \right] \tag{7}$$

$$Z_{2N_2L} = Z_{AA} = \mathrm{Tr}_V \left[\Pi^x \left(T^{(1)} \right)^N \right] \tag{8}$$

Both $T^{(0)}$ and $T^{(1)}$ can be diagonalized, and due to the discrete symmetries:

$$\left[T^{(0)}(\lambda), \Pi^x \right] = \left[T^{(0)}(\lambda), \Pi^z \right] = 0, \tag{9}$$

$$\left[T^{(1)}(\lambda), \Pi^x \right] = \left[T^{(1)}(\lambda), \Pi^z \right]_+ = 0, \tag{10}$$

$$\Pi^x \Pi^z = (-1)^L \Pi^z \Pi^x, \tag{11}$$

where $\Pi^x = \bigotimes_{m=1}^{L} \sigma^x$ is the reflection operator and $\Pi^z = \bigotimes_{m=1}^{L} \sigma^z$ is the parity operator, we can see that all four free-energies above can be obtained.

The homogenous toroidal boundaries are those where there is no change from periodic to anti-periodic along the horizontal or the vertical direction. They are:

$$Z_{11} = Z_{PP} = \mathrm{Tr}_V \left[\left(T^{(0)} \right)^N \right] = \sum_{j=1}^{2^L} \left(\Lambda_j^{(0)} \right)^N \tag{5}$$

$$Z_{2N_1} = Z_{AP} = \mathrm{Tr}_V \left[\left(T^{(1)} \right)^N \right] = (1 + (-1)^N) \sum_{j=1}^{2^{L-1}} \left(\Lambda_j^{(1)} \right)^N \tag{6}$$

$$Z_{12^L} = Z_{PA} = \mathrm{Tr}_V \left[\Pi^x \left(T^{(0)} \right)^N \right] = \sum_{j=1}^{2^L} (-1)^{p_j^x} \left(\Lambda_j^0 \right)^N \tag{7}$$

$$Z_{2N_{2^L}} = Z_{AA} = \mathrm{Tr}_V \left[\Pi^x \left(T^{(1)} \right)^N \right] = (1 + (-1)^{N+L}) \sum_{j=1}^{2^{L-1}} (-1)^{p_j^x} \left(\Lambda_j^{(1)} \right)^N \tag{8}$$

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Batchelor et. al. $\Rightarrow F_{AP} = F_{PP}$ for $\Delta < -1$, therefore

$$\lim_{L\to\infty} \frac{1}{L} \ln \Lambda^{(1)}_{\max} = \lim_{L\to\infty} \frac{1}{L} \ln \Lambda^{(0)}_{\max} \qquad \Delta < -1, \tag{12}$$

hence we have

$$F_{PP} = F_{AP} = F_{PA} = F_{AA}, \tag{13}$$

whenever they are allowed by selection rule. Therefore

$$F_{free} = F_{PP} \qquad \forall\Delta \tag{14}$$

- Note that there is no restriction over the parity of lattice size

- The First Row of $M_{N,L}$ is given by:

$$Z_{1,j} = \mathrm{Tr}_V \left[\bigotimes_{m=1}^{L} \mathcal{G}_{V_m}^{(\phi_m)} (T^{(0)})^N \right] = \sum_{g=1}^{2^L} \left(\Lambda_g^{(0)} \right)^N f_{L,g}^{\{\phi_m\}}, \tag{15}$$

$$f_{L,g}^{\{\phi_m\}} = \left\langle g^{(0)} \right| \prod_{m=1}^{L} \mathcal{G}_{V_m}^{(\phi_m)} \left| g^{(0)} \right\rangle \tag{16}$$

- Since $T^{(0)}$ commutes with S^z, we can choose eigenvectors to live in a definite sector of S^z. Therefore we have to have Φ even, otherwise $f_{L,g}^{\{\phi_m\}}$ will be zero.
- Perron-Frobenius theorem $\Rightarrow$ $f_{L,g}^{\{\phi_m\}}$ is non-negative for maximal eigenvectors of each sector.
- How could $f_{L,g}^{\{\phi_m\}}$ change the free-energy? It should decay as fast as $\mathrm{e}^{-\delta LN}$. But this impossible since it only depends on L!

- $\Delta \leq 1 \quad \Rightarrow \quad n = \lfloor \frac{L}{2} \rfloor \quad \Rightarrow \quad F_{1,j} = -\lim_{L\to\infty} \frac{1}{L} \ln \Lambda_{\max}^{(0)} = F_{PP}$
- $\Delta > 1 \quad \Rightarrow \quad n = 0 \quad \Rightarrow \quad F_{1,j} = -\lim_{L\to\infty} \frac{1}{L} \ln \Lambda_{\max,\frac{\Phi}{2}}^{(0)} = F_{PP}?$

Because for $\Delta > 1$ we have $\Lambda_{\max,n=0} > \Lambda_{\max,n=1} \ldots > \Lambda_{\max,n=\lfloor \frac{L}{2} \rfloor}$, but $f_{\max,n<\frac{\Phi}{2}} = 0$.

- Bethe ansatz solution reveals that $\lim_{L\to\infty} \frac{1}{L} \ln \Lambda_{\max,n}^{(0)} = \lim_{L\to\infty} \frac{1}{L} \ln \Lambda_{\max,0}^{(0)}$

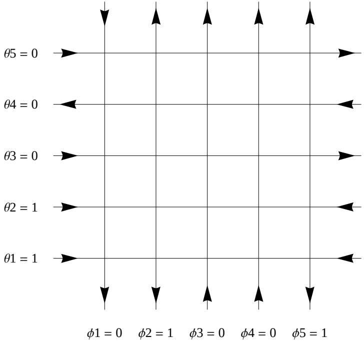
$$F_{1,j} = F_{PP} \qquad \forall \Delta \tag{17}$$

Conjecture:

$$F_{i,j} = F_{PP} \qquad \forall \Delta \tag{18}$$

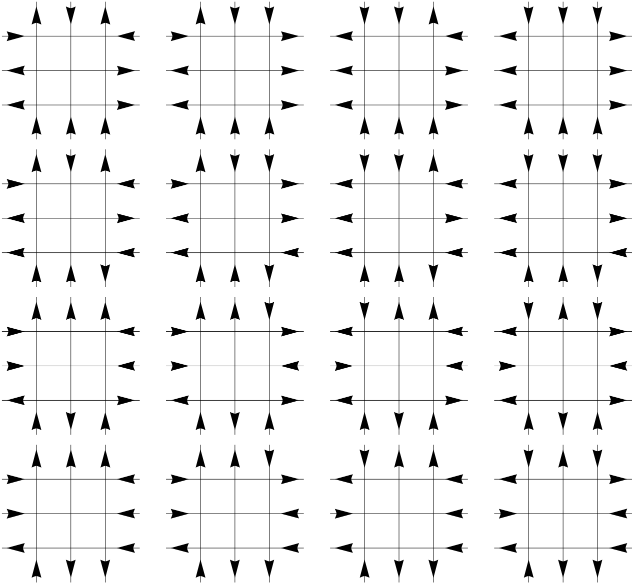
Second Part: Fixed Boundaries

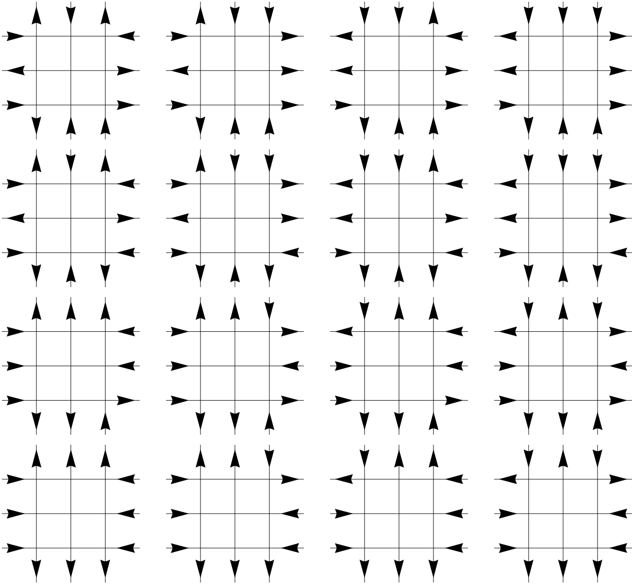
- Although is very probable that $F_{i,j} = F_{PBC}$ for all mixtures of local periodicity and anti-periodicity, we already know some fixed boundary conditions with intensive properties differing from PBC. Therefore we should search for different types of fixed boundary conditions!

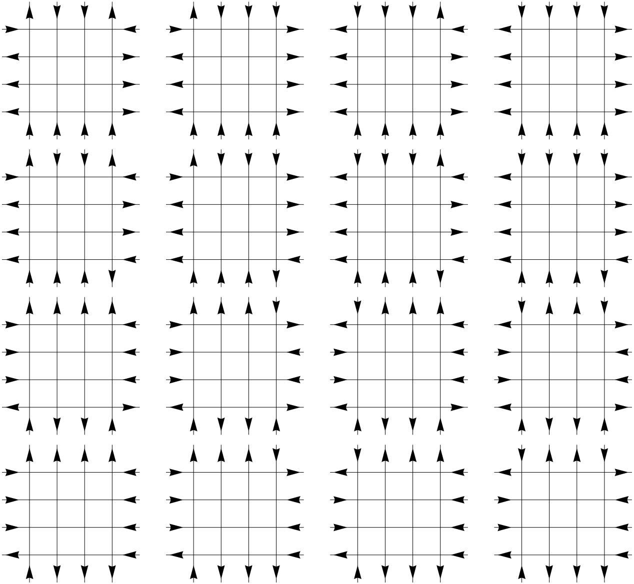


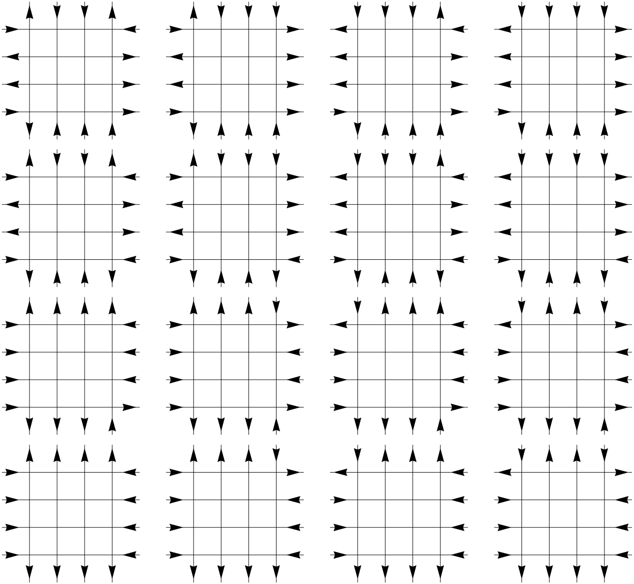
This Boundary can be viewed as one term in the summation of $Z_{10,25}$
$\{0,1,0,0,1\}_2 = 10 - 1$
$\{1,1,0,0,0\}_2 = 25 - 1$

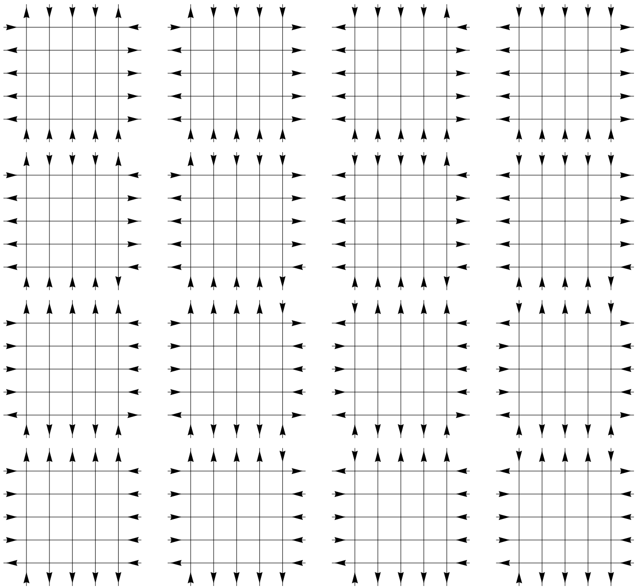
- Since there are so many fixed boundary conditions, we have chosen to search for boundaries with the same number of configurations as Domain-Wall(DW). Scanning over all fixed boundaries with $N = 3$, $N = 4$, $N = 5$ we found:

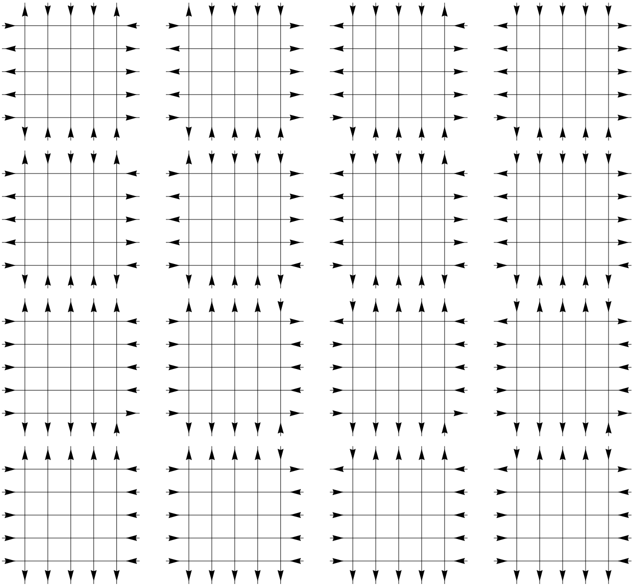




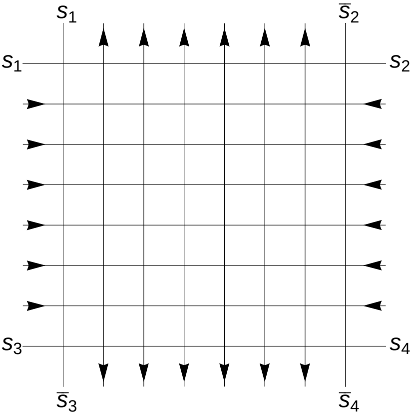








- Scanning over all fixed boundaries with $N = 3$, $N = 4$ and $N = 5$ gives 32 different boundaries with the same number of configurations of DW, $\Omega = 7$, $\Omega = 42$ and $\Omega = 429$,respectively.
- Is there a pattern for these boundaries? Can we group them in a family?



- Corner edges satisfy an isolated arrow conservation
- 2^4 corners $\times$ 2 DW $=$ 32

Can we find a determinant formula for the partition function of these boundaries?

The simplest example: $s_1 = s_2 = s_4 = -$, and $s_3 = +$. In QISM formulation

$$Z_N(\{\lambda\},\{\mu\}) = \langle \Downarrow_N | \, B(\lambda_N) \ldots B(\lambda_2) D(\lambda_1) \, | {\downarrow} \Uparrow_{N-1} \rangle \, ,$$

using two-site decomposition (Bogoliubov, Pronko, Zvonarev.2002):

$$\mathcal{T}_{\mathcal{A}}(\lambda) = \begin{pmatrix} A_{N-1}(\lambda) & B_{N-1}(\lambda) \\ C_{N-1}(\lambda) & D_{N-1}(\lambda) \end{pmatrix} \begin{pmatrix} A_1(\lambda) & B_1(\lambda) \\ C_1(\lambda) & D_1(\lambda) \end{pmatrix}, \tag{19}$$

we can derive the following relation,

$$Z_N(\{\lambda\},\{\mu\}) = \sum_{j=1}^{N} r_j \, \langle \Downarrow_{N-1} | \prod_{k=j+1}^{\overset{\curvearrowleft}{N}} B_{N-1}(\lambda_k) D_{N-1}(\lambda_j) \prod_{k=1}^{\overset{\curvearrowleft}{j-1}} B_{N-1}(\lambda_k) \, | \Uparrow_{N-1} \rangle \, , \tag{20}$$

where

$$r_1 = a(\lambda_1 - \mu_1) \prod_{m=2}^{N} b(\lambda_m - \mu_1), \qquad r_2 = \frac{c(\lambda_1 - \mu_1) c(\lambda_2 - \mu_1)}{a(\lambda_1 - \mu_1) b(\lambda_2 - \mu_1)} r_1,$$

$$r_j = \frac{a(\lambda_{j-1} - \mu_1) c(\lambda_j - \mu_1)}{c(\lambda_{j-1} - \mu_1) b(\lambda_j - \mu_1)} r_{j-1} \qquad j = 3, \ldots, N. \tag{21}$$

Using Yang-Baxter Algebra

$$D(\lambda_j)\overset{\curvearrowleft}{\prod_{m=1}^{j-1}} B(\lambda_m) = \sum_{k=1}^{j} \beta_{jk} \overset{\curvearrowleft}{\prod_{\substack{m=1\\ m\neq k}}^{j}} B(\lambda_m) D(\lambda_k), \tag{22}$$

where

$$\beta_{jk} = \begin{cases} -\dfrac{c(\lambda_j-\lambda_k)}{b(\lambda_j-\lambda_k)} \displaystyle\prod_{\substack{i=1\\ i\neq k}}^{j} \dfrac{a(\lambda_k-\lambda_i)}{b(\lambda_k-\lambda_i)}, & k \neq j, \\ \displaystyle\prod_{i=1}^{j-1} \dfrac{a(\lambda_j-\lambda_i)}{b(\lambda_j-\lambda_i)}, & k = j. \end{cases} \tag{23}$$

The recursion relation:

$$Z_N(\{\lambda\},\{\mu\}) = \sum_{k=1}^{N} Z_{N-1}^{DWBC}(\{\lambda\}\setminus\lambda_k, \{\mu\}\setminus\mu_1)\left[(b(\lambda_k))^{N-1}\sum_{j=k}^{N} r_j \beta_{jk}\right]. \tag{24}$$

Using DW determinant formula(Izergin, Coker, Korepin.1992)

$$Z_N^{DWBC}(\{\lambda\},\{\mu\}) = f_N(\{\lambda\},\{\mu\}) \det\left[\rho(\lambda_i,\mu_j)\right]_{i=1,\dots,N}^{j=1,\dots,N}, \tag{25}$$

in the relation (24)

We finally obtain that

$$Z_N(\{\lambda\},\{\mu\}) = \begin{vmatrix} \delta_1 & \rho(\lambda_1,\mu_2) & \rho(\lambda_1,\mu_3) & \dots & \rho(\lambda_1,\mu_N) \\ \delta_2 & \rho(\lambda_2,\mu_2) & \rho(\lambda_2,\mu_3) & \dots & \rho(\lambda_2,\mu_N) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_N & \rho(\lambda_N,\mu_2) & \rho(\lambda_N,\mu_3) & \dots & \rho(\lambda_N,\mu_N) \end{vmatrix}, \tag{26}$$

where δ_k, $\rho(\lambda,\mu)$ and $f_N(\{\lambda\},\{\mu\})$ are given by

$$\delta_k = (-1)^{1+k} f_{N-1}(\{\lambda\}\setminus\lambda_k,\{\mu\}\setminus\mu_1) b^{N-1}(\lambda_k) \sum_{j=k}^{N} r_j \beta_{jk}, \tag{27}$$

$$\rho(\lambda,\mu) = \frac{c(\lambda-\mu)}{a(\lambda-\mu)b(\lambda-\mu)}, \tag{28}$$

$$f_N(\{\lambda\},\{\mu\}) = \frac{\displaystyle\prod_{\substack{i,j=1\\ i<j}}^{N} \frac{(c_{ij}c_{ji}b_{ii}b_{jj}+c_{ii}c_{jj}a_{ij}a_{ji})(c_{ii}c_{jj}b_{ij}b_{ji}+c_{ij}c_{ji}a_{ii}a_{jj})}{\rho_{ii}\rho_{jj}(c_{ij}c_{ji}b_{ii}b_{jj}+c_{ii}c_{jj}a_{ij}a_{ji})-\rho_{ij}\rho_{ji}(c_{ij}c_{ji}a_{ii}a_{jj}+c_{ii}c_{jj}b_{ij}b_{ji})}}{\prod_{i=1}^{N}(a_{ii}b_{ii})^{N-2}}, \tag{29}$$

and we have denoted $a_{ij} = a(\lambda_i - \mu_j)$ and so on.

Further comments on dDWBC

- Isolated arrow conservation rule extension

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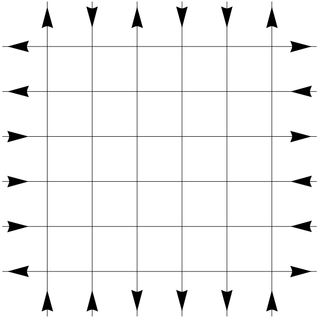
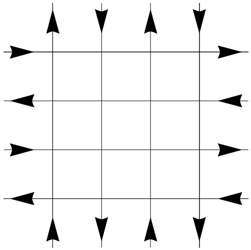
- N larger than 5



- $N = 6 \;\Rightarrow\; \Omega = 7436$, the DW number!
- There are a total of 160 boundaries sharing the same Ω!

- A natural question: Which fixed boundary condition has the largest Ω?

Looking at $N = 3$ could lead us to wrong conclusions. In this case the dDWBC are the family with the largest number of configurations. At $N = 4$ this is not the case anymore. While dDWBC has 42 configurations, the largest number is 64. The related boundary is



- This new boundary satisfy the isolated arrow conservation rule!
- 32 boundaries with the same Ω.

- Here the number 32 of boundaries with largest number of configurations remains the same for $N = 6$!

N	3	4	5	6
Ω	7	64	1322	64934

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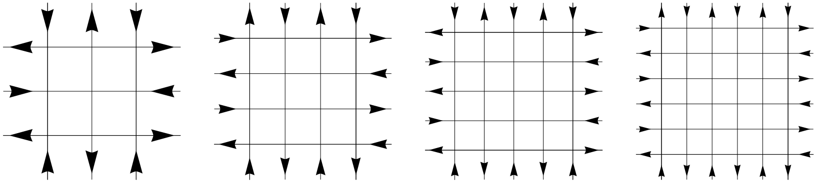
- Separation of families!



Is it possible to obtain the number of configurations as a function of n? Can we find a determinant formula?
QISM Representation:

$$Z_N^{NE}(\{\lambda\},\{\mu\}) = \langle \uparrow\downarrow \dots \uparrow\downarrow | D(\lambda_N)A(\lambda_{N-1}) \cdots D(\lambda_2)A(\lambda_1) | \uparrow\downarrow \dots \uparrow\downarrow \rangle \, , \tag{30}$$

$$[A(\lambda), D(\mu)] \neq 0 \tag{31}$$

$$[D(\lambda)A(\lambda), D(\mu)A(\mu)] \neq 0 \tag{32}$$

$$\left[D(\lambda)A(\lambda), (D(\lambda)A(\lambda))^t\right] \neq 0 \tag{33}$$

- No Universal eigenstates!
- No normality, except at infinite temperature point!

$$[A(\lambda), S^z] = [D(\lambda), S^z] = [D(\lambda)A(\mu), S^z] = 0. \tag{34}$$

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- Nevertheless we still can get some information without actually calculating the exact number of configurations.

We have the following inequality

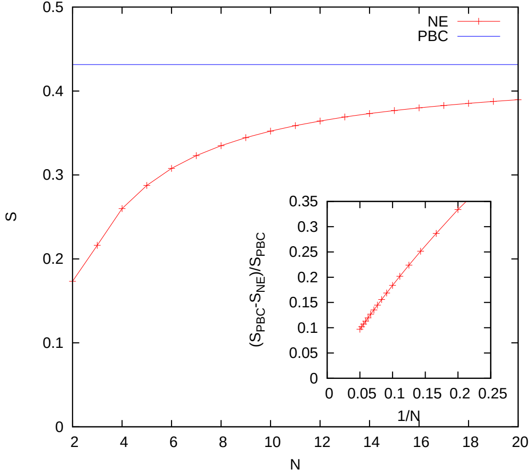
$$\Omega_{max,fix} \leq \Omega_{free} \leq 2^{2N+2L}\Omega_{max,fix} \quad \Rightarrow \quad S_{free} = S_{PBC} = S_{max,fix} \qquad (35)$$

- Our conjecture is: $max, fix \;\equiv\; NE$.

N	number of states
1	1
2	2
3	7
4	64
5	1322
6	64914
7	7474305
8	2033739170
9	1305583070738
10	1981880443295788
11	7111657020627320662
12	60382974032926242142168
13	1213039653244899907872180826
14	57687270950680153355854587442676
15	6494209210696211480439308528411663853
16	1731204438495421321106461120147832169010790
17	1092829001103470428650265862752651675963745966742
18	1633892840599915791908254127642749411000513938128114064
19	5785898354977820698935460290451680551971080689572072829375890
20	48534629904275880189653389798729712740901732087151544103619504415896

Table: Number of configurations for Néel boundary condition.

We have tried some sequence solvers, but they could'nt obtain the general term nor predict the next number.



$$S_{NE} = S_{PBC}(1 - \frac{\gamma}{N}) \qquad \gamma \sim 2 \tag{36}$$

Still not convinced?

$$S_N = \frac{\log \Omega}{N^2} = \frac{\log(c_N \omega)}{N^2} = \frac{\log c_N}{N^2} + \frac{\log \omega}{N^2} \approx \frac{\log \omega}{N^2}, \tag{37}$$

where c_N is of order of the unity.

- Therefore, we should count digits or some other exponential growth!

N	number of digits	difference
3	1	
4	2	1
5	4	2
6	5	1
7	7	2
8	10	3
9	13	3
10	16	3
11	19	3
12	23	4
13	28	5
14	32	4
15	37	5
16	43	6
17	49	6
18	55	6
19	61	6
20	68	7

$$\begin{aligned} S_{4k+3} &\approx \frac{1 + \left(\sum_{j=1}^{k} 6j\right)}{(4k+3)^2} \log(10) \\ &= \frac{2 + (1+6k)k}{2(4k+3)^2} \log(10), \end{aligned} \tag{38}$$

taking thermodynamic limit $k \to \infty$, we find

$$S = \frac{3}{16} \log(10) = 0.431735\ldots \tag{39}$$

compare with

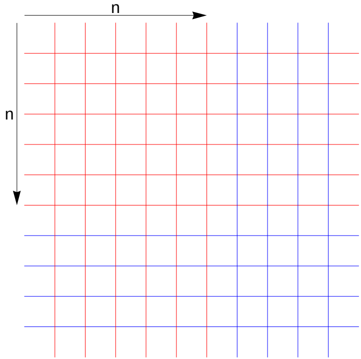
$$S_{PBC} = \frac{3}{2} \log\left(\frac{4}{3}\right) = 0.431523\ldots \tag{40}$$

- By now we know that there are at least three possible outcomes for the entropy of 6 V:

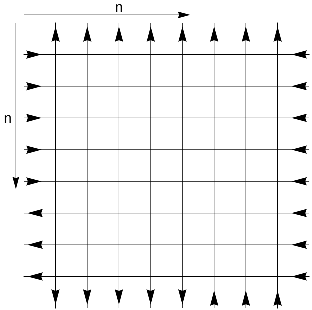
$$S_{FE} = 0 \qquad S_{dDWBC} = \frac{1}{2}\ln(27/16) \qquad S_{PBC} = \frac{3}{2}\ln(4/3), \tag{41}$$

Are all the values between S_{FE} and S_{PBC} accessible?

- To answer this question we introduced what we call Merge-type boundaries



DW-FE Fusion



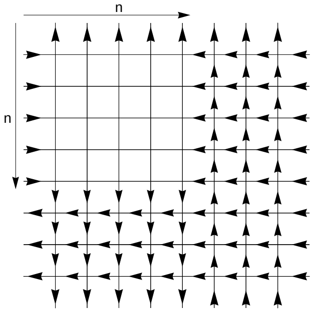
$$\frac{Z_N^{DWFE}}{\left[\prod_{i=1}^{N}\prod_{j=n+1}^{N} b(\lambda_i-\mu_j)\right]\left[\prod_{i=n+1}^{N}\prod_{j=1}^{N} a(\lambda_i-\mu_j)\right]} = Z_n^{DW} \quad (42)$$

$$S^{DWFE} = \lim_{N\to\infty}\left(\frac{n}{N}\right)^2 S^{DW} \quad (43)$$

- Choosing a suitable sequence $n(N)$, one can obtain any value of entropy S such that

$$S^{FE} \leq S \leq S^{DW} \quad (44)$$

DW-FE Fusion



$$\frac{Z_N^{DWFE}}{\left[\prod_{i=1}^{N}\prod_{j=n+1}^{N} b(\lambda_i-\mu_j)\right]\left[\prod_{i=n+1}^{N}\prod_{j=1}^{N} a(\lambda_i-\mu_j)\right]} = Z_n^{DW} \tag{42}$$

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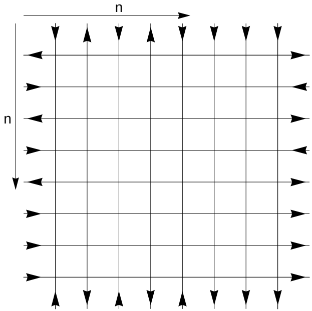
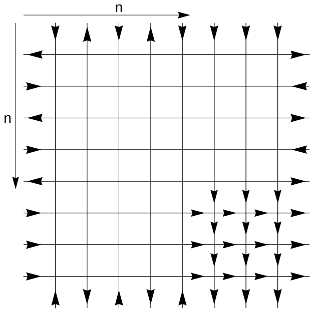
NE-FE Fusion



NE-FE Fusion

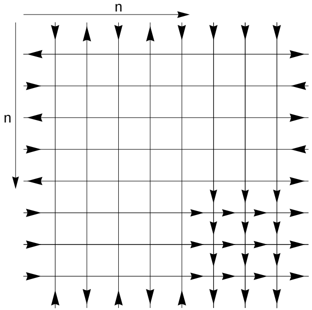
- Conjecture: $0 \leq S^{NEFE} \leq S^{PBC}$ for a suitable chosen sequence $n(N)$

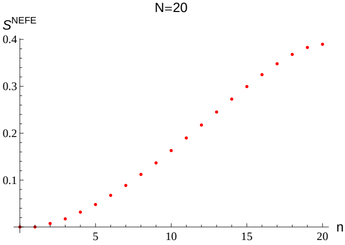


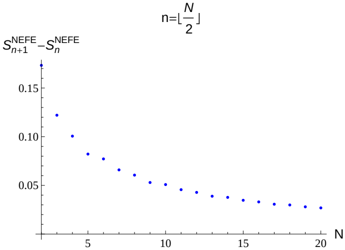
- L-shaped partition function. Colomo and Pronko(2015)

NE-FE Fusion

- Conjecture: $0 \leq S^{NEFE} \leq S^{PBC}$ for a suitable chosen sequence $n(N)$



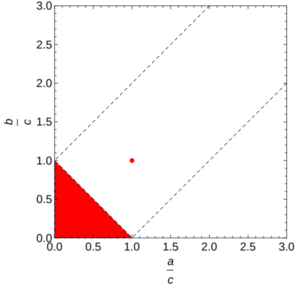




- L-shaped partition function. Colomo and Pronko(2015)

$$S^{NEFE}_{n+1} - S^{NEFE}_{n} \leq O(\frac{1}{N}) \tag{45}$$

- We argued that the free-energy for boundaries mixing periodic and anti-periodic closings should be the same as PBC and free-boundary.
- We found a family of 32 fixed boundary conditions with same number of configurations as DW. There are other fixed boundary conditions whose number of configurations coincide with DW, but we were not able to classify them in the same family.
- We introduced the Néel boundary condition, whose number of configurations we believe to be the largest one among all fixed boundaries. From that we conclude $S^{NE} = S^{PBC}$, but there is no rigorous proof that Néel is indeed maximal.



- Assuming the last result to be true, is $F^{NE} = F^{PBC}$ for all a, b, c?
- Is it possible to find the exact number of configurations for all N? Does integrability play any role for this boundary?

- We introduced the Merge type boundaries and proved that the entropy may take any value between S^{FE} and S^{DW}.
- If one can prove the "continuity" of entropy for merge-type boundaries and that NE is maximal, then we can extend S^{DW} to S^{PBC} in the above assertion.

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