Bruhat and Tamari Orders in Integrable Systems

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Contents of this talk

Part I: KdV and KP soliton interactions in a "tropical limit"

- KdV solitons and (weak) Bruhat orders
- Higher Bruhat orders $B(N,n)$
  (Manin&Schechtman 1986; Ziegler; Kapranov&Voevodsky ...)
- (From higher Bruhat to) Tamari orders $T(N,n)$
  (expected to be equivalent to *higher Stasheff-Tamari orders*:
  Kapranov&Voevodsky; Edelman&Reiner ...)
- Physical realization by KP solitons

Part II: Simplex equations and "Polygon equations"

- $B(3,1)$ and the Yang-Baxter equation
- Simplex equations
  $B(N+1, N-1) \Longrightarrow$ N-*simplex equation*
- Polygon equations: $T(N, N-2) \Longrightarrow N$-gon equation
  Sequence of equations analogous to simplex equations,
  generalizing the well-known pentagon equation ($N = 5$)

Part I

KdV and KP soliton interactions in a "tropical limit"

Our original contact with Bruhat and Tamari orders

Dimakis & M-H

- J. Phys. A: Math. Theor. **44** (2011) 025203
- Chapter in Tamari Festschrift "Associahedra, Tamari Lattices and Related Structures", Progress in Mathematics **299** (2012) 391-423
- J. Phys.: Conf. Ser. **482** (2014) 012010

KdV Solitons and Bruhat Orders

KdV equation: $4\,u_t - u_{xxx} - 6\,u\,u_x = 0$ $\qquad u = 2\,(\log\tau)_{xx}$

M-soliton solution:

$$\boxed{\tau = \sum_{A\in\{-1,1\}^M} e^{\Theta_A}}$$

$$\Theta_A = \sum_{j=1}^{M} \alpha_j\,\theta_j + \log\Delta_A$$

$$\theta_j = p_j\,x + p_j^3\,t + c_j \quad = \sum_{k=1}^{M} p_j^{2k-1}\,t^{(k)}$$

$$0 < p_1 < p_2 < \cdots < p_M$$

$$A = (\alpha_1,\ldots,\alpha_M) \qquad \alpha_j \in \{\pm 1\}$$

$$\Delta_A = |\Delta(\alpha_1 p_1,\ldots,\alpha_M p_M)|$$

$\nwarrow$ Vandermonde determinant

Tropical limit

$$\log\tau = \Theta_B + \log \sum_{A\in\{-1,1\}^M} e^{-(\Theta_B-\Theta_A)} \cong \max\{\Theta_A \,|\, A \in \{-1,1\}^M\}$$

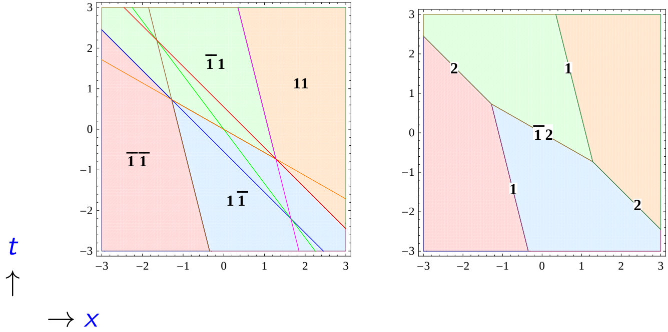
Θ_B is linear in x $\curvearrowright$ $u = 2\,(\log\tau)_{xx}$ is localized along the boundaries of non-empty "dominating phase regions"

$$\mathcal{U}_B = \left\{ \boldsymbol{t} \in \mathbb{R}^M \,\middle|\, \max\{\Theta_A(\boldsymbol{t}) \,|\, A \in \{-1,1\}^M\} = \Theta_B(\boldsymbol{t}) \right\}$$

- Determine the boundaries $\{\Theta_A = \Theta_B\}$ for pairs of phases. We have to solve linear algebraic equations.
- Determine their *visible* parts: compare phases.
- Determine coincidence events of more than two phases and their visibility.

A KdV soliton solution corresponds to a piecewise linear graph in 2d space-time.

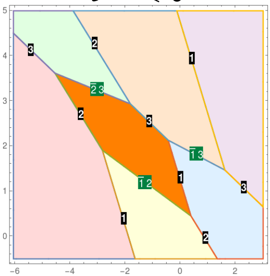
Interaction of two KdV solitons



4 phases and $\binom{4}{2} = 6$ boundary lines, displayed in the left plot.
The right plot only shows the *visible* parts of these lines.
Interaction by exchange of a "virtual" soliton.
$(1,2) \rightarrow (2,1)$ (weak) **Bruhat order** $B(2,1)$.

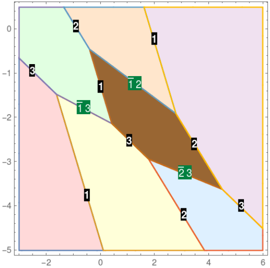
Interaction of three KdV solitons

$t^{(3)} < 0$



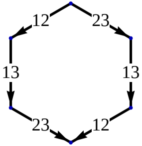
$$(3,2,1) \uparrow (2,3,1) \uparrow (2,1,3) \uparrow (1,2,3)$$

$t^{(3)} > 0$

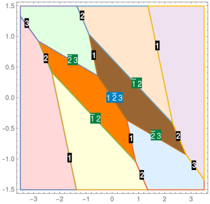


$$(3,2,1) \uparrow (3,1,2) \uparrow (1,3,2) \uparrow (1,2,3)$$

only 2-particle exchanges, (weak) **Bruhat order** $B(3,1)$



For $t^{(3)} = 0$ also
3-particle exchange.
More complicated
processes ...
Still to be explored ...



Higher Bruhat Orders

$[N] = \{1, 2, \ldots, N\}$

$\binom{[N]}{n}$ set of n-element subsets of $[N]$

A **linear order** (permutation) of $\binom{[N]}{n}$ is called **admissible** if, for any $K \in \binom{[N]}{n+1}$, the packet $P(K) := \{n\text{-element subsets of } K\}$ is contained in it in **lexicographical** or in **reverse lexicographical** order

$A(N, n)$ set of admissible linear orders of $\binom{[N]}{n}$

Equivalence relation on $A(N, n)$: $\rho \sim \rho'$ if they only differ by exchange of two neighboring elements, not both contained in some packet.

Higher Bruhat order: $B(N, n) := A(N, n)/\sim$

Partial order via inversions of lexicographically ordered packets:

$$\overrightarrow{P}(K) \longmapsto \overleftarrow{P}(K)$$

$B(4,2)$

$B(4,2)$ consists of the two maximal chains

$$
\begin{array}{ccccccccccccc}
12 & & 12 & & 23 & & 23 & & 23 & & 23 & & 34 \\
13 & & 13 & & 13 & & 13 & & 24 & & 24 & & 24 \\
14 & \overset{\sim}{\to} & 23 & \xrightarrow{123=\hat{4}} & 12 & \xrightarrow{124=\hat{3}} & 24 & \overset{\sim}{\to} & 13 & \xrightarrow{134=\hat{2}} & 34 & \xrightarrow{234=\hat{1}} & 23 \\
23 & & 14 & & 14 & & 14 & & 14 & & 14 & & 14 \\
24 & & 24 & & 24 & & 12 & & 34 & & 13 & & 13 \\
34 & & 34 & & 34 & & 34 & & 12 & & 12 & & 12
\end{array}
$$

$$
\begin{array}{ccccccccccccc}
12 & & 12 & & 12 & & 34 & & 34 & & 34 & & 34 \\
13 & & 13 & & 34 & & 12 & & 24 & & 24 & & 24 \\
14 & \xrightarrow{234=\hat{1}} & 14 & \xrightarrow{134=\hat{2}} & 14 & \overset{\sim}{\to} & 14 & \xrightarrow{124=\hat{3}} & 14 & \xrightarrow{123=\hat{4}} & 14 & \overset{\sim}{\to} & 23 \\
23 & & 34 & & 13 & & 24 & & 12 & & 23 & & 14 \\
24 & & 24 & & 24 & & 13 & & 13 & & 13 & & 13 \\
34 & & 23 & & 23 & & 23 & & 23 & & 12 & & 12
\end{array}
$$

Here they are resolved into elements of $A(4,2)$. These are in correspondence with the maximal chains of $B(4,1)$, which forms a permutahedron.

Tamari orders

Splitting of packet:

$$P(K) = P_o(K) \cup P_e(K)$$

$P_o(K)$ ($P_e(K)$) half-packet consisting of elements with odd (even) position in the lexicographically ordered $P(K)$.
Inversion operation in case of Tamari orders:

$$\overrightarrow{P}_o(K) \longmapsto \overleftarrow{P}_e(K)$$

We have to eliminate those elements in the linear orders that are not in accordance with the splitting of packets and with this rule. (See Dimakis & M-H, SIGMA **11** (2015) 042, for the precise rules.)

The simplest physical example

$$P(123) = \{12, 13, 23\},\ P(123)_o = \{12, 23\},\ P(123)_e = \{13\}$$

$B(3,2)$:

$$\begin{matrix} 12 & & 23 \\ 13 & \xrightarrow{123} & 13 \\ 23 & & 12 \end{matrix}$$

$T(3,2)$:

$$\begin{matrix} 12 & \xrightarrow{123} & 13 \\ 23 & & \end{matrix}$$

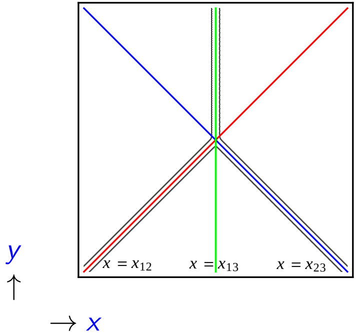


Figure shows a snapshot of a *line soliton solution* (thick lines) of the *KP equation*, in the xy-plane. Passing from bottom to top (i.e., in y-direction), thin lines (coincidences of 2 phases) realize $B(3,2)$. Only the thick parts are "visible" in the soliton solution. They realize the Tamari order $T(3,2)$.

From higher Bruhat to Tamari orders

via a 3-color decomposition. For $B(4,2)$:

$$\begin{matrix} 12 \\ 13 \\ 14 \\ 23 \\ 24 \\ 34 \end{matrix} \overset{\sim}{\to} \begin{matrix} 12 \\ 13 \\ 23 \\ 14 \\ 24 \\ 34 \end{matrix} \overset{123}{\to} \begin{matrix} 23 \\ 13 \\ 12 \\ 14 \\ 24 \\ 34 \end{matrix} \overset{124}{\to} \begin{matrix} 23 \\ 13 \\ 24 \\ 14 \\ 12 \\ 34 \end{matrix} \overset{\sim}{\to} \begin{matrix} 23 \\ 24 \\ 13 \\ 14 \\ 34 \\ 12 \end{matrix} \overset{134}{\to} \begin{matrix} 23 \\ 24 \\ 34 \\ 14 \\ 13 \\ 12 \end{matrix} \overset{234}{\to} \begin{matrix} 34 \\ 24 \\ 23 \\ 14 \\ 13 \\ 12 \end{matrix}$$

and correspondingly for the second chain. This contains the Tamari order $T(4,2)$:

$$\begin{matrix} 12 \\ 23 \\ 34 \end{matrix} \overset{123}{\to} \begin{matrix} 13 \\ 34 \end{matrix} \overset{134}{\to} 14 \qquad \begin{matrix} 12 \\ 23 \\ 34 \end{matrix} \overset{234}{\to} \begin{matrix} 12 \\ 24 \end{matrix} \overset{124}{\to} 14$$

KP solitons and Tamari lattices

KP: $(-4u_t + u_{xxx} + 6uu_x)_x + 3u_{yy} = 0$ $\qquad u = 2\,(\log\tau)_{xx}$

Subclass of line soliton solutions:

$$\tau = \sum_{j=1}^{M+1} e^{\theta_j} \qquad \theta_j = p_j\, x + p_j^2\, y + p_j^3\, t + c_j = \sum_{r=1}^{M} p_j^r\, t^{(r)}$$

$t^{(1)} = x,\ t^{(2)} = y,\ t^{(3)} = t$ $\qquad t^{(r)},\ r > 3$ KP *hierarchy* “times”

In the *tropical limit*:

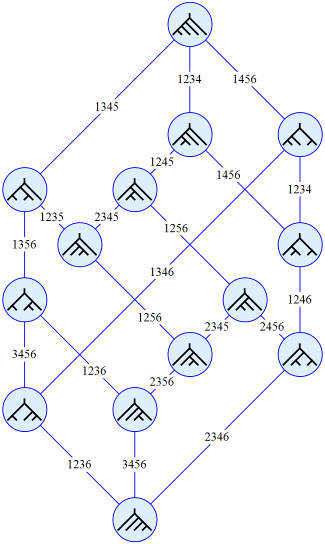
- At fixed time, density distribution in the xy-plane is a *rooted (generically binary) tree*.
- Any evolution (with fixed M) starts with the same tree and ends with the same tree.
- Time evolution corresponds to *right rotation in tree*.

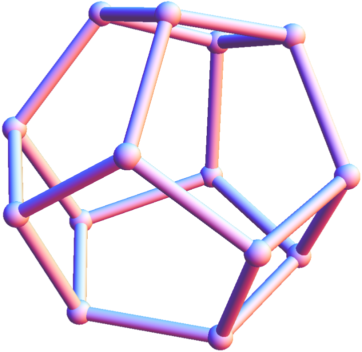
$\Longrightarrow$ maximal chain of Tamari lattice $T(M+1, 3)$

$T(5,3)$ forms a pentagon.

All Tamari orders are realized via the above class of KP solitons !

Tamari lattice $T(6,3)$ (associahedron) realized by KP





An open problem

KP solitons
tropically

↙ ↘

parallel line solitons (KdV) tree-shaped solitons

↓ ↓

(Higher?) Bruhat orders ? **Tamari orders**

The combinatorics underlying the **full** set of KP solitons (forming networks at a fixed time) is more involved and not ruled by Bruhat and Tamari orders.

Part II

From Bruhat and Tamari orders to Simplex and Polygon equations

Dimakis & M-H, SIGMA **11** (2015) 042

$B(3,1)$ and the Yang-Baxter Equation

$B(3,1)$ consists of the two maximal chains

$$\begin{matrix}1\\2\\3\end{matrix} \overset{12}{\to} \begin{matrix}2\\1\\3\end{matrix} \overset{13}{\to} \begin{matrix}2\\3\\1\end{matrix} \overset{23}{\to} \begin{matrix}3\\2\\1\end{matrix} \qquad\qquad \begin{matrix}1\\2\\3\end{matrix} \overset{23}{\to} \begin{matrix}1\\3\\2\end{matrix} \overset{13}{\to} \begin{matrix}3\\1\\2\end{matrix} \overset{12}{\to} \begin{matrix}3\\2\\1\end{matrix}$$

A set-theoretical realization

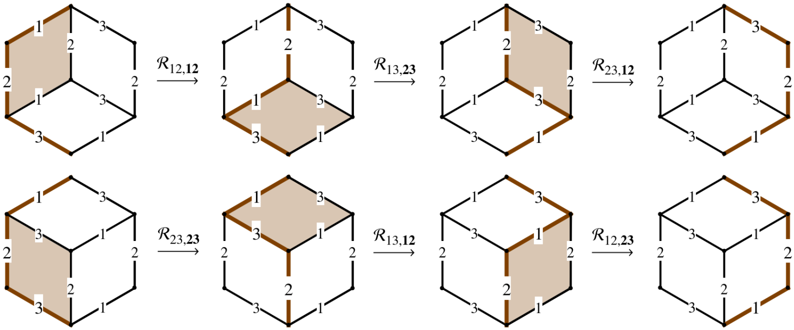
$$i \mapsto \mathcal{U}_i\,, \qquad (i,j,k) \mapsto \mathcal{U}_i \times \mathcal{U}_j \times \mathcal{U}_k$$
$$ij \mapsto \mathcal{R}_{ij} : \mathcal{U}_i \times \mathcal{U}_j \to \mathcal{U}_j \times \mathcal{U}_i$$

(or a realization using vector spaces and tensor products) leads to the **Yang-Baxter equation**

$$\mathcal{R}_{23,\mathbf{12}}\,\mathcal{R}_{13,\mathbf{23}}\,\mathcal{R}_{12,\mathbf{12}} = \mathcal{R}_{12,\mathbf{23}}\,\mathcal{R}_{13,\mathbf{12}}\,\mathcal{R}_{23,\mathbf{23}}$$

The (boldface) **position indices** are read off from the diagrams.

Visualization of the YB equation on the cube $B(3,0)$



"Deformation" of maximal chains of $B(3,0)$.
The elements of $A(3,1)$ (here equal to $B(3,1)$) are in bijection with the maximal chains of $B(3,0)$, the Boolean lattice of $\{1,2,3\}$.

Simplex Equations

Zamolodchikov (1980), Bazhanov & Stroganov (1982), Maillet & Nijhoff (1989), ...
Manin & Schechtman (1985)

N-simplex equation = realization of $B(N+1, N-1)$

We consider again a set-theoretical framework.
With $K \in \binom{[N+1]}{N}$, we associate a map $\mathcal{R}_K : \mathcal{U}_{\overrightarrow{P}(K)} \to \mathcal{U}_{\overleftarrow{P}(K)}$ where $\overrightarrow{P}(K)$ ($\overleftarrow{P}(K)$) is the (reverse) lexicographically ordered packet. $\mathcal{R}_K$ realizes an inversion.
With an exchange we associate the respective transposition map $\mathcal{P}$.

$N = 2 \curvearrowright B(3,1) \curvearrowright$ **Yang-Baxter equation**

N=3

$\curvearrowright$ $B(4,2)$. Here is one of its two chains:

$$\begin{array}{ccccccccccccc}
12 & & 12 & & 23 & & 23 & & 23 & & 23 & & 34 \\
13 & & 13 & & 13 & & 13 & & 24 & & 24 & & 24 \\
14 & \overset{\sim}{\to} & 23 & \overset{123}{\to} & 12 & \overset{124}{\to} & 24 & \overset{\sim}{\to} & 13 & \overset{134}{\to} & 34 & \overset{234}{\to} & 23 \\
23 & & 14 & & 14 & & 14 & & 14 & & 14 & & 14 \\
24 & & 24 & & 24 & & 12 & & 34 & & 13 & & 13 \\
34 & & 34 & & 34 & & 34 & & 12 & & 12 & & 12
\end{array}$$

$$\mathcal{R}_{ijk} : \mathcal{U}_{ij} \times \mathcal{U}_{ik} \times \mathcal{U}_{jk} \to \mathcal{U}_{jk} \times \mathcal{U}_{ik} \times \mathcal{U}_{ij}, \qquad i < j < k$$

$$\mathcal{R}_{234,\mathbf{1}}\, \mathcal{R}_{134,\mathbf{3}}\, \mathcal{P}_{\mathbf{5}}\, \mathcal{P}_{\mathbf{2}}\, \mathcal{R}_{124,\mathbf{3}}\, \mathcal{R}_{123,\mathbf{1}}\, \mathcal{P}_{\mathbf{3}} = \mathcal{P}_{\mathbf{3}}\, \mathcal{R}_{123,\mathbf{4}}\, \mathcal{R}_{124,\mathbf{2}}\, \mathcal{P}_{\mathbf{4}}\, \mathcal{P}_{\mathbf{1}}\, \mathcal{R}_{134,\mathbf{2}}\, \mathcal{R}_{234,\mathbf{4}}$$

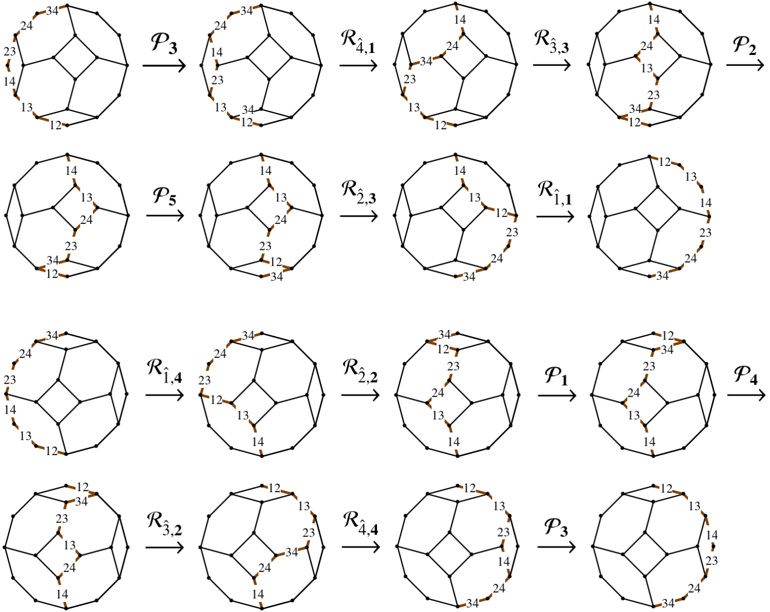
with transposition map $\mathcal{P}_{\boldsymbol{a}} := \mathcal{P}_{\boldsymbol{a},\boldsymbol{a}+1}$. In terms of $\hat{\mathcal{R}} = \mathcal{R}\,\mathcal{P}_{\mathbf{13}}$:

Zamolodchikov (tetrahedron) equation

$$\hat{\mathcal{R}}_{\hat{1},\mathbf{123}}\, \hat{\mathcal{R}}_{\hat{2},\mathbf{145}}\, \hat{\mathcal{R}}_{\hat{3},\mathbf{246}}\, \hat{\mathcal{R}}_{\hat{4},\mathbf{356}} = \hat{\mathcal{R}}_{\hat{4},\mathbf{356}}\, \hat{\mathcal{R}}_{\hat{3},\mathbf{246}}\, \hat{\mathcal{R}}_{\hat{2},\mathbf{145}}\, \hat{\mathcal{R}}_{\hat{1},\mathbf{123}}$$

using complementary notation (also in following figures).

Tetrahedron equation on $B(4,1)$ (permutahedron)



"Integrability" of simplex equations

$B(N+1, N-1)$	$\longleftrightarrow$	N-simplex equation
$\downarrow$		$\downarrow$
$B(N+2, N-1)$		contains system of N-simplex equations "localize" it: $\mathcal{L}(u_{N+1}) \cdots \mathcal{L}(u_1) = \mathcal{L}(v_1) \cdots \mathcal{L}(v_{N+1})$
$\downarrow$		$\Downarrow$
$B(N+2, N)$	$\longleftrightarrow$	consistency condition: $(N+1)$-simplex equation for the map $\mathcal{R} : (u_1, \ldots, u_{N+1}) \mapsto (v_{N+1}, \ldots, v_1)$

Polygon Equations

In analogy with the case of simplex equations we set

$$\boxed{N\textbf{-gon equation} = \text{realization of } T(N, N-2)}$$

The two maximal chains of $T(5,3)$ can be resolved to

$$\begin{matrix}123\\134\\145\end{matrix} \xrightarrow{1234} \begin{matrix}234\\124\\145\end{matrix} \xrightarrow{1245} \begin{matrix}234\\245\\125\end{matrix} \xrightarrow{2345} \begin{matrix}345\\235\\125\end{matrix} \qquad \begin{matrix}123\\134\\145\end{matrix} \xrightarrow{1345} \begin{matrix}123\\345\\135\end{matrix} \xrightarrow{\sim} \begin{matrix}345\\123\\135\end{matrix} \xrightarrow{1235} \begin{matrix}345\\235\\125\end{matrix}$$

Here we are dealing with maps

$$\mathcal{T}_{ijkl} : \mathcal{U}_{ijk} \times \mathcal{U}_{ikl} \to \mathcal{U}_{jkl} \times \mathcal{U}_{ijl} \qquad i < j < k < l$$

Using complementary notation, the **pentagon equation** is thus

$$\mathcal{T}_{\hat{1},\mathbf{12}}\, \mathcal{T}_{\hat{3},\mathbf{23}}\, \mathcal{T}_{\hat{5},\mathbf{12}} = \mathcal{T}_{\hat{4},\mathbf{23}}\, \mathcal{P}_{\mathbf{12}}\, \mathcal{T}_{\hat{2},\mathbf{23}}$$

In terms of $\hat{\mathcal{T}} := \mathcal{T}\,\mathcal{P}$, it takes the form

$$\hat{\mathcal{T}}_{\hat{1},\mathbf{12}}\, \hat{\mathcal{T}}_{\hat{3},\mathbf{13}}\, \hat{\mathcal{T}}_{\hat{5},\mathbf{23}} = \hat{\mathcal{T}}_{\hat{4},\mathbf{23}}\, \hat{\mathcal{T}}_{\hat{2},\mathbf{12}}$$

Beyond the pentagon equation

Hexagon equation $\mathcal{T} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathcal{U} \times \mathcal{U}$

$$\mathcal{T}_{\hat{2},\mathbf{1}} \, \mathcal{P}_{\mathbf{3}} \, \mathcal{T}_{\hat{4},\mathbf{2}} \, \mathcal{T}_{\hat{6},\mathbf{1}} \, \mathcal{P}_{\mathbf{3}} = \mathcal{T}_{\hat{5},\mathbf{2}} \, \mathcal{P}_{\mathbf{1}} \, \mathcal{T}_{\hat{3},\mathbf{2}} \, \mathcal{T}_{\hat{1},\mathbf{4}}$$

Heptagon equation $\mathcal{T} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \to \mathcal{U} \times \mathcal{U} \times \mathcal{U}$

$$\mathcal{T}_{\hat{1},\mathbf{1}} \, \mathcal{T}_{\hat{3},\mathbf{3}} \, \mathcal{P}_{\mathbf{5}} \, \mathcal{P}_{\mathbf{2}} \, \mathcal{T}_{\hat{5},\mathbf{3}} \, \mathcal{T}_{\hat{7},\mathbf{1}} \, \mathcal{P}_{\mathbf{3}} = \mathcal{P}_{\mathbf{3}} \, \mathcal{T}_{\hat{6},\mathbf{4}} \, \mathcal{P}_{\mathbf{3}} \, \mathcal{P}_{\mathbf{2}} \, \mathcal{P}_{\mathbf{1}} \, \mathcal{T}_{\hat{4},\mathbf{3}} \, \mathcal{P}_{\mathbf{2}} \, \mathcal{P}_{\mathbf{3}} \, \mathcal{T}_{\hat{2},\mathbf{4}}$$

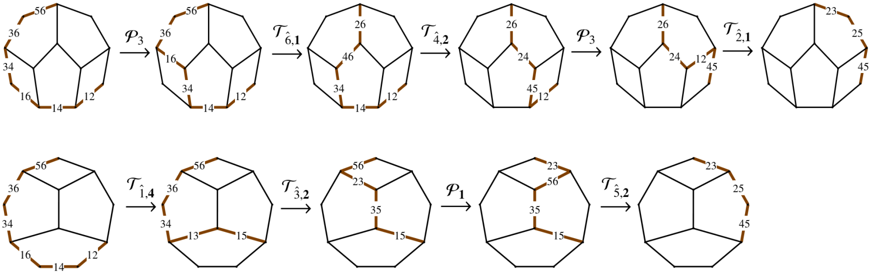
or in terms of $\hat{\mathcal{T}} := \mathcal{T} \, \mathcal{P}_{\mathbf{13}}$,

$$\hat{\mathcal{T}}_{\hat{1},\mathbf{123}} \, \hat{\mathcal{T}}_{\hat{3},\mathbf{145}} \, \hat{\mathcal{T}}_{\hat{5},\mathbf{246}} \, \hat{\mathcal{T}}_{\hat{7},\mathbf{356}} = \hat{\mathcal{T}}_{\hat{6},\mathbf{356}} \, \hat{\mathcal{T}}_{\hat{4},\mathbf{245}} \, \hat{\mathcal{T}}_{\hat{2},\mathbf{123}}$$

etc

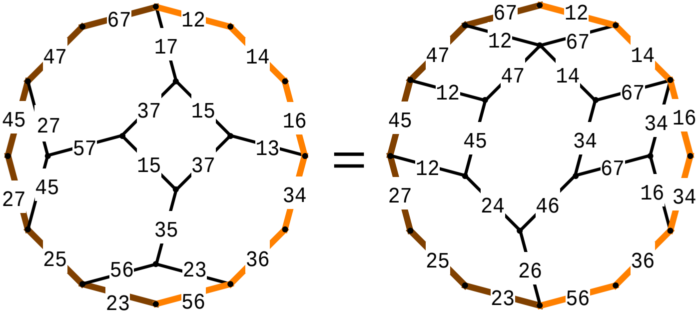
Polygon equations are related by the same kind of **"integrability"** that connects simplex equations !

Hexagon equation



The left (right) hand side of the hexagon equation corresponds to a sequence of maximal chains on the front (back) side of the **associahedron** (Stasheff polytope) in three dimensions, formed by the **Tamari lattice** $T(6,3)$.

Heptagon equation



The equality represents the heptagon equation on complementary sides of the Edelman-Reiner polytope formed by $T(7,4)$.

Summary and further remarks

- Simplex equations are an attempt towards higher- (than two-) dimensional (quantum) integrable systems or models of statistical mechanics. (Zamolodchikov 1980; Bazhanov & Stroganov; Maillet & Nijhoff; ...) Underlying combinatorics: *higher Bruhat orders* (Manin & Schechtman 1986).
- In the same way as simplex equations generalize the *Yang-Baxter equation* $\hat{\mathcal{R}}_{12}\,\hat{\mathcal{R}}_{13}\,\hat{\mathcal{R}}_{23} = \hat{\mathcal{R}}_{23}\,\hat{\mathcal{R}}_{13}\,\hat{\mathcal{R}}_{12}$, the "polygon equations" generalize the *pentagon equation* $\hat{\mathcal{T}}_{12}\,\hat{\mathcal{T}}_{13}\,\hat{\mathcal{T}}_{23} = \hat{\mathcal{T}}_{23}\,\hat{\mathcal{T}}_{12}$. Underlying: *Tamari orders*.
- Very little is known so far about the polygon equations beyond the pentagon equation. This is essentially new terrain. Distinguishing property: "integrability" in the sense that the $(N+1)$-gon equation arises as consistency condition of a system of localized N-gon equations.
- Solutions via tropical KP solitons ?

Relevance of pentagon equation:

- 3-cocycle condition in Lie group cohomology.
- Identity for fusion matrices in CFT.
- Appearance in a category-theoretical framework (Street 1998).
- Any finite-dimensional Hopf algebra is characterized by an invertible solution of the pentagon equation (Militaru 2004).
- Multiplicative unitary (Baaj & Skandalis 1993).
- Consistency condition for *associator* in quasi-Hopf algebras.
- Quantum dilogarithm (Faddeev, Kashaev 1993).
- Invariants of 3-manifolds via triangulations, realization of Pachner moves (Korepanov 2000).

Relevance of higher polygon equations ?

A version of the *hexagon equation* appeared in work of Korepanov (2011), Kashaev (2014): realizations of Pachner moves of triangulations of a 4-manifold. A step to higher dimensions ...

Grazie per l'attenzione !