Spontaneous Breaking of U(N) symmetry in invariant Matrix Models & Ergodicity Breaking

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Support by:

MARIE CURIE ACTIONS

SEVENTH FRAMEWORK PROGRAMME

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Outlook

- Consider a Quantum System
- Localization/extendedness of wavefunctions is a basis-dependent property
- However, eigen-energy statistics (Poisson/Wigner Dyson) characterizes insulating/conducting systems
- Seeking for a basis independent, general structure of (Anderson) insulators

Results

- The U(N) symmetry matrix models are endowed with can be spontaneously broken
- Thermodynamic limit also takes symmetry's rank to infinity
- Eigenvectors encode non-trivial information!
- Certain models break U(N) in a critical way: similarity with Metal/Insulator Transition
- These models are in the family of CS/ABJM

Outline

1. Intro 1: Disorder & Localization
2. Intro 2: Matrix Models
3. Spontaneous Symmetry Breaking:
   - Geometrical argument
   - Symmetry Breaking term
   - Numerical finite size detection
4. Weakly Confined Matrix Models
   - Spectral Statistics (known)
   - Energy landscape (new)
5. Conclusions & Outlook

Part 1

Introduction
on
Localization
due to Disorder

Disorder & Localization

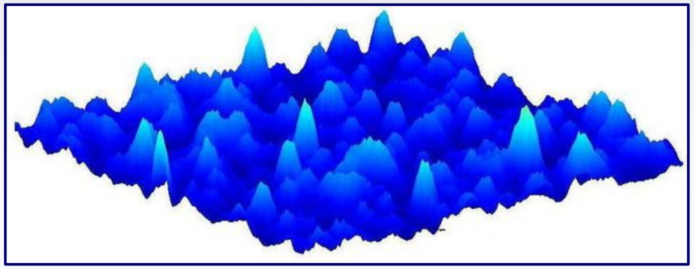
- Anderson Model: $\mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[c_l^\dagger c_j + c_j^\dagger c_l \right]$ (Anderson. '58)
- Tight-binding model (nearest neighbor hopping)
- Random on-site energies: $\epsilon_j \in [-W, +W]$
- 1 (& 2) Dimensions: localized for any $W \neq 0$
- Higher D:
  - Small W: conducting (weak localization, Random Matrices)
  - $W > W_c$: insulating (localized at low energies)
- Hard problem (uncontrolled perturbation expansion)

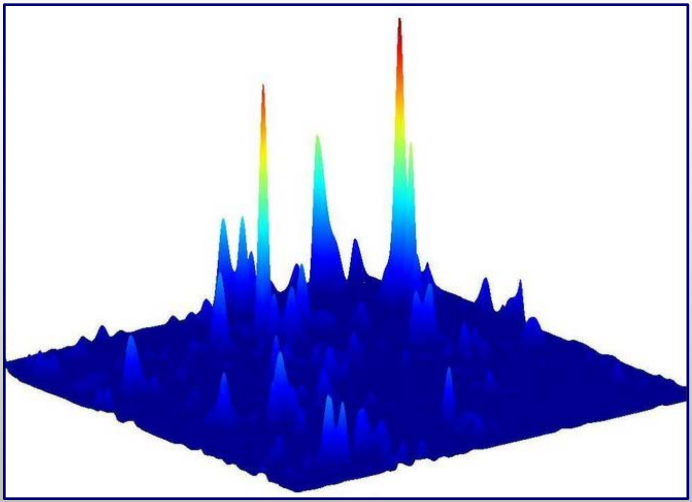
Metal/Insulator Transition

$$\mathcal{H} = \sum_j \epsilon_j c_j^\dagger c_j + \sum_{\langle j,l \rangle} \left[c_l^\dagger c_j + c_j^\dagger c_l \right]$$

$$\epsilon_j \in [-W, +W] \qquad W < W_c \qquad D \geq 3$$

- At $E = E_m$:Mobility Edge separating extended from localized states
- Transition as Intermediate state (multifractal)

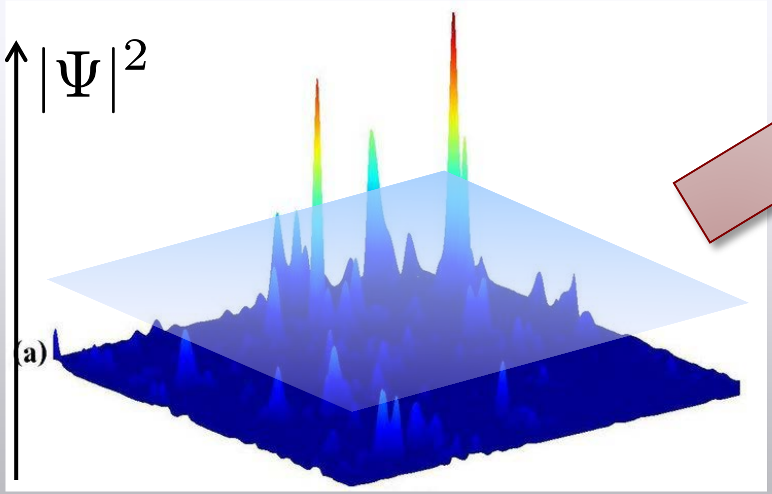


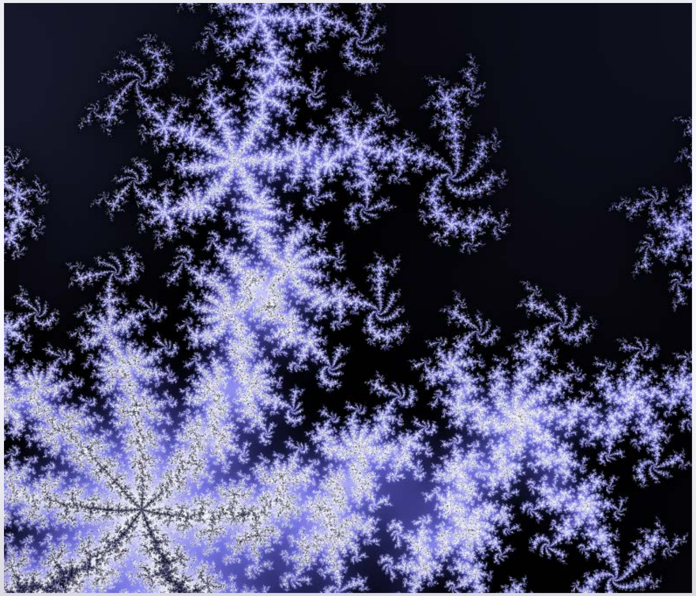


Van Tiggelen group (PRL 2009)

Multifractality

- At each height $|\Psi|^2 = \alpha$, the wavefunction's amplitude draws a "curve" with a different fractal dimension $f(\alpha)$





Multi-fractal Spectrum

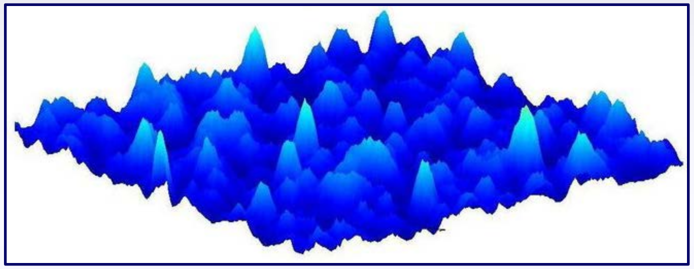
- To characterize localization: $\mathrm{IPR}_q = \sum_j^N |\Psi_j|^{2q}$, $N \propto L^d$
  - Extended: $\mathrm{IPR}_q \simeq N^{1-q} = L^{-d(q-1)}$
  - Localized: $\mathrm{IPR}_q \simeq \mathrm{const}$
  - Critical state:

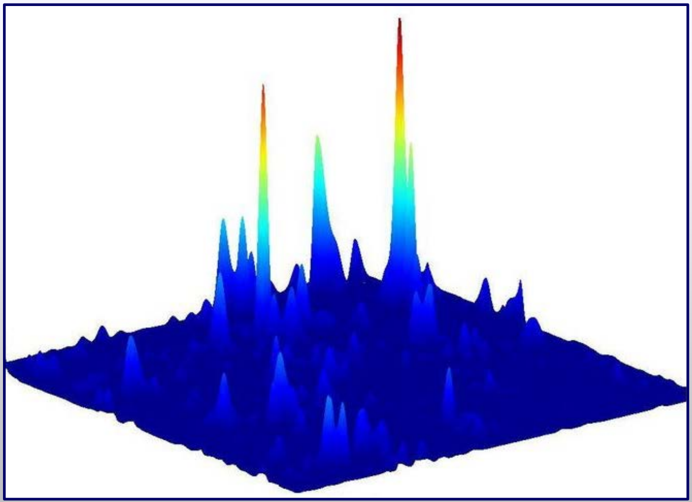
$$\mathrm{IPR}_q \simeq L^{-d_q(q-1)}$$

$$= \int N^{-q\alpha+f(\alpha)} d\alpha$$

$0 < d_q < d$: fractal dimensions

$f(\alpha)$: multi-fractal spectrum



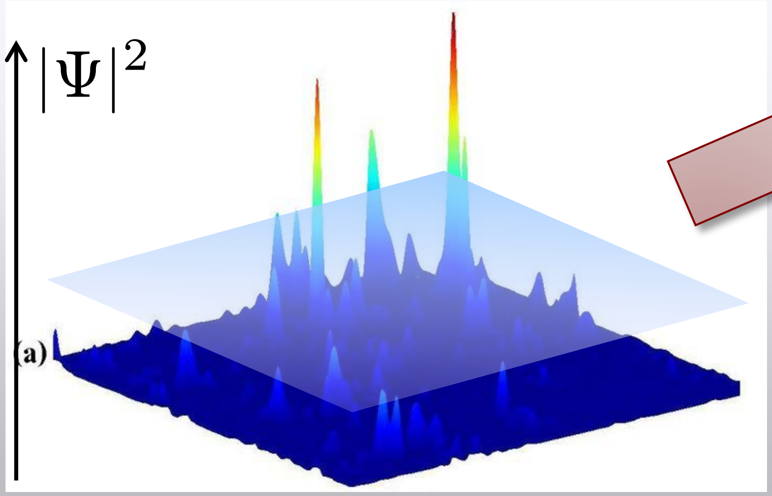


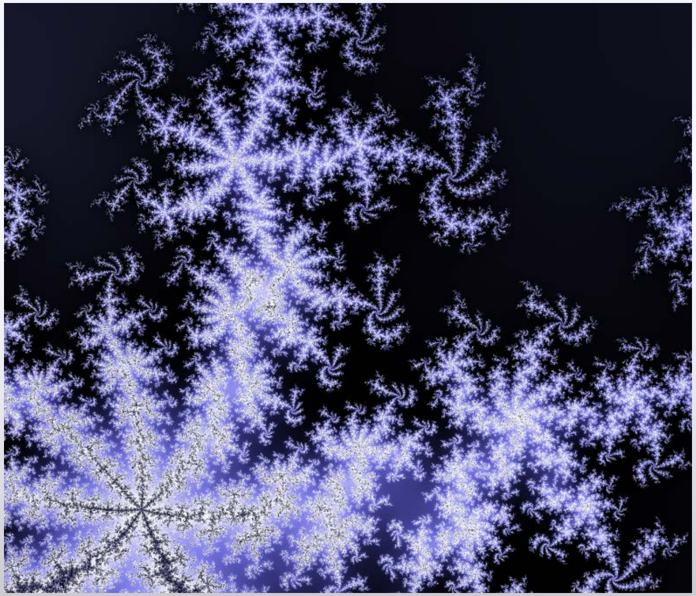
Van Tiggelen group (PRL 2009)

Multifractality

$$L^{-d_q(q-1)} = \int N^{-q\alpha+f(\alpha)} d\alpha$$

- Fractal dimensions d_q /fractal spectrum $f(\alpha)$ known analytically only in perturbative regimes ($d_q \simeq 0, d$)





Landau Zener Picture

- Qualitative picture on eigenvalue/eigenvector connection
- 2-level system: $\begin{pmatrix} \epsilon_1 & V \\ V^* & \epsilon_2 \end{pmatrix} \longrightarrow \begin{pmatrix} E_1 & 0 \\ 0 & E_2 \end{pmatrix}$

$$\delta = E_1 - E_2 = \sqrt{(\epsilon_1 - \epsilon_2)^2 + |V|^2}$$

"Localized"

$$\boxed{V \ll \epsilon_1 - \epsilon_2}$$

$$\delta \simeq \epsilon_1 - \epsilon_2$$

$$\Psi_{1,2} \simeq \psi_{1,2} + \mathcal{O}\left(\frac{1}{\epsilon_1 - \epsilon_2}\right) \psi_{2,1}$$

"Extended"

$$\boxed{V \gg \epsilon_1 - \epsilon_2}$$

$$\delta \simeq |V|$$

$$\Psi_{1,2} \simeq \psi_{1,2} \pm \psi_{2,1}$$

Part 2

Introduction
on
Matrix Models

Matrix Models

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-W(\mathbf{M})}$$

matrix-valued action

- Several applications: nuclear theory, mesoscopic conduction, 2-D quantum gravity, string theory, statistical physics, econophysics, neuroscience, chaos theory, number theory, integrability…
- Reflects a large universality
- Matrices can be link between points; fields in adjoint or bi-fundamental, representation of operators in many-body theory (Hamiltonians, Scattering…)…

Random Matrices

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-W(\mathbf{M})} \qquad \mathcal{F} = \ln \mathcal{Z}$$

- If W(M) real: statistical model
- Consider M as a Hamiltonian:
  - Interaction between every degree of freedom
  - Matrix entries randomly from a distribution
- Describe quantum "chaotic" systems
- Universality determined by symmetry:

  Orthogonal, Unitary, Symplectic,… ensembles

Invariant Ensembles

- Action invariant under rotations: $W(\mathbf{M}) = \mathrm{Tr} V(\mathbf{M})$

- Switch to eigenvalues/eigenvectors: $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, \Delta^\beta(\{\lambda\}) \, e^{-\sum_j V(\lambda_j)}$$

Eigenvectors uniformly distributed over the N-dimensional sphere (Hilbert space): independent from V(λ)

Van der Monde Determinant:

$$\Delta(\{\lambda\}) = \prod_{j>l}^{N} (\lambda_j - \lambda_l)$$

(from Jacobian)

Coulomb Gas Picture

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, \Delta^\beta(\{\lambda\}) \, e^{-\sum_j V(\lambda_j)}$$

- Jacobian introduces interaction between eigenvalues
- Effective Coulomb gas:

$$\mathcal{L} = -\beta \sum_{j>l} \ln|\lambda_j - \lambda_l| + \sum_j V(\lambda_j)$$

- Eigenvalues as 1-D particles with
  - logarithmic interaction
  - external confining potential $V(\lambda)$
- Eigenvalue distribution from equilibrium configuration

$\beta = 1, 2, 4$

for

Orthogonal

Unitary

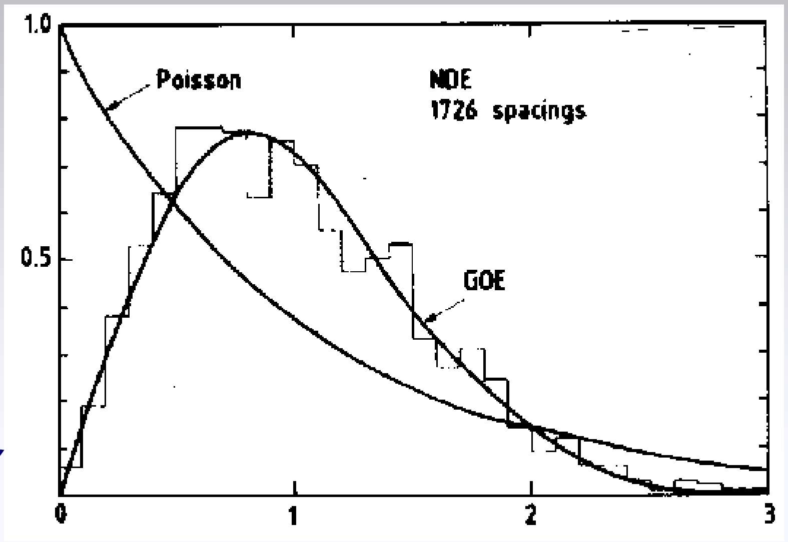
Symplectic

Wigner-Dyson Universality

$$\mathcal{L} = -\beta \sum_{j>l} \ln |\lambda_j - \lambda_l| + \sum_j V(\lambda_j)$$

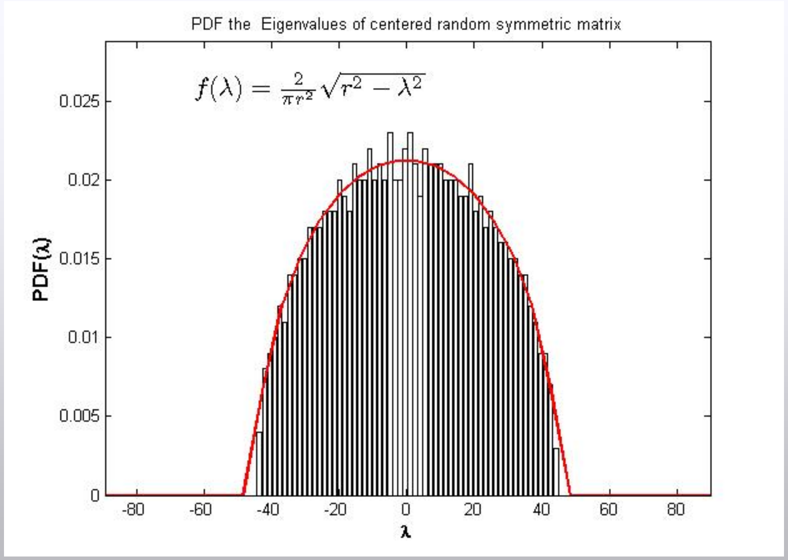
- Distribution of the distance between n.n. eigenvalues (level spacing) universal:

$$P(s) \propto s^{\beta} e^{-A(\beta)s^2}$$

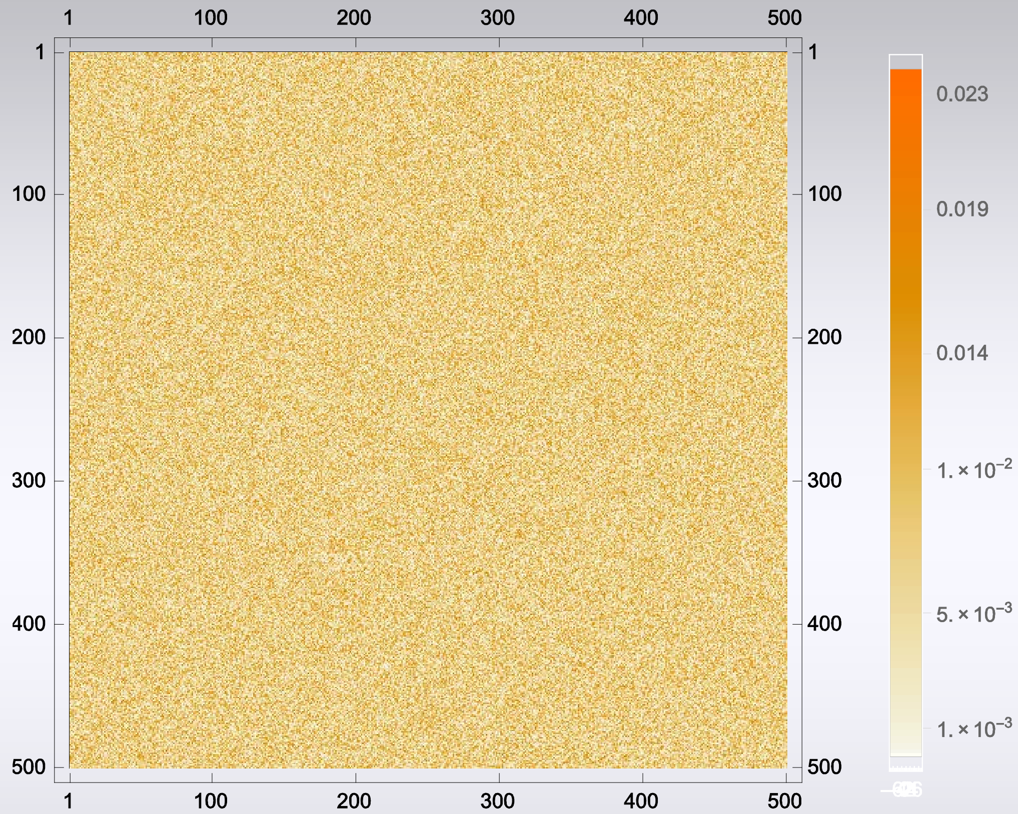


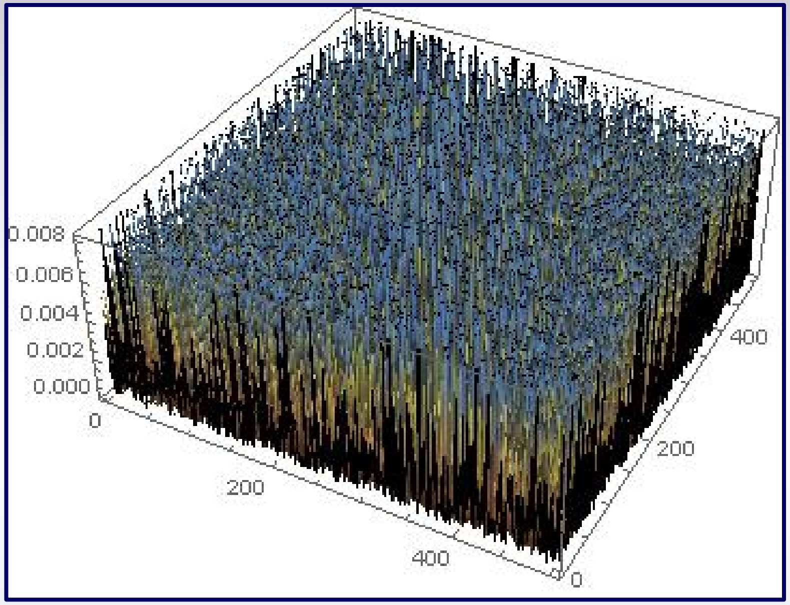
- Universality captured by Gaussian ensemble: $V(\lambda) = \frac{\lambda^2}{2}$

- Valid for any polynomial V(λ)



The Haar Measure





- Entries of Unitary matrix follow the Porter-Thomas Distribution: $$\mathcal{P}\left(\left|\tilde{U}_{ij}\right|^2\right) = N \exp\left[-N\left|\tilde{U}_{ij}\right|^2\right]$$

Invariant Ensembles

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda \, \Delta^\beta(\{\lambda\}) \, e^{-\sum_j V(\lambda_j)}$$

- Wigner Dyson distribution & level repulsion:

  Jacobian introduces interaction between eigenvalues

- Extended states/conducting phases:

  uniform distribution means eigenvectors typically

  have all non-vanishing entries

- Eigenvalues interact through their eigenvectors:

$$\boxed{\textbf{WD} \Leftrightarrow \textbf{extended states}}$$

Non-Invariant Ensembles

- To study localization problems, introduce non-invariant random matrix ensembles (Random Banded Matrices)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\sum_{j,l} A_{jl} \, |M_{jl}|^2} \Rightarrow \langle M_{nm}^2 \rangle = A_{nn}^{-1}$$

$$A_{nm} = \mathrm{e}^{|n-m|/B}$$

→ Localized states (Poisson statistics)

(Mirlin et al. '96)

$$A_{nm} = 1 + \frac{(n-m)^2}{B^2}$$

→ Multi-Fractal states (Critical Statistics)

(Evers & Mirlin, '00)

Invariant vs. non-Invariant Ensembles

- Invariant: basis <u>independent</u>

  → Wigner-Dyson eigenvalue statistics
  de-Haar measure for eigenvector

  ⇒ delocalized systems
  <u>analytical</u> techniques

- Non-Invariant: basis <u>dependent</u>

  → Poisson/critical eigenvalue statistics
  eigenvector connected with eigenvalue

  ⇒ localized/critical systems
  mostly <u>numerical</u> approaches

Loophole: Spontaneous Breaking of Rotational Invariance

- Invariant models are endowed with superior (non-perturbative) analytical techniques
- Spontaneous breaking of rotational invariance:
  - $\Rightarrow$ Eigenvectors contain non-trivial information
  - $\Rightarrow$ Invariant machinery for localization problems!
- Recall a ferromagnet:
  - From partition function, rotational invariance $\rightarrow$ no spontaneous magnetization
  - Need symmetry breaking term

PART 3

Spontaneous Breaking of Rotational Symmetry in Invariant Multi-Cuts Matrix Model

Multi-Cut Solutions

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int d^N\lambda\, e^{-\sum_j V(\lambda_j) + 2\sum_{j>l} \ln|\lambda_j - \lambda_l|}$$

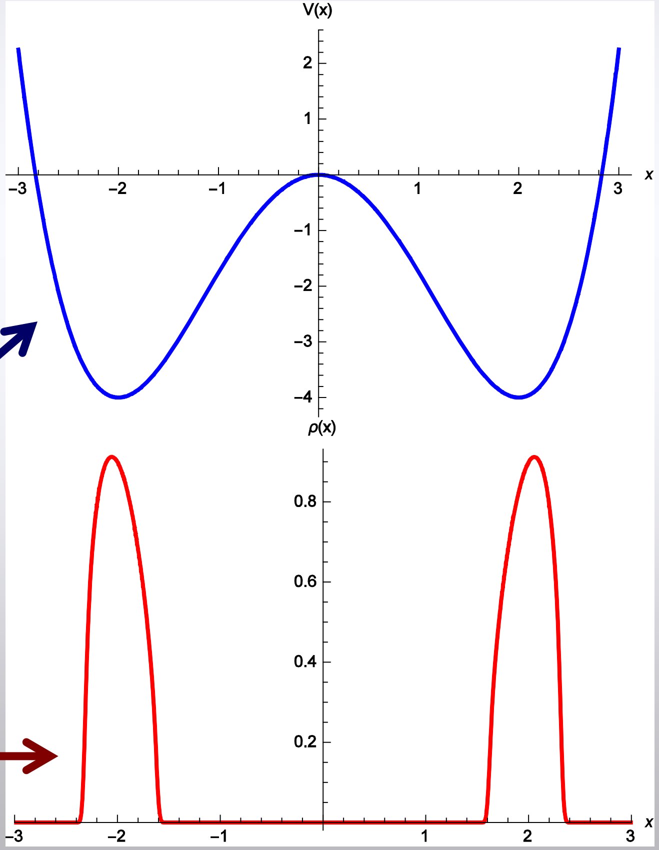
- V(x) with several, well separated, minima

  $\Rightarrow$ disconnected support for eigenvalues (multi-cuts)

- For example: double well potential

$$V_{2\mathrm{W}}(x) = \frac{1}{4}x^4 - \frac{t}{2}x^2$$

(2-cuts for $t > 2$)



Level Density: $\rho(x) = \sum_{j}^{N} \delta(x - \lambda_j)$

Understanding the matrix SSB

- Geometrical argument: line element

$$\boxed{\beta = 2\,,\mathrm{Unitary}}$$

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2\, |dA_{jl}|^2$$

$$d\mathbf{A} \equiv \mathbf{U}^\dagger d\mathbf{U}$$

$$\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$

- Angular degrees of freedom live on spheres of radii $r_{jl} = |\lambda_j - \lambda_l|$

➢ Two lengths scales:
- Eigenvalues spacing: $\mathcal{O}\left(\frac{1}{N}\right)$
- Support of distribution: $\mathcal{O}\,(1)$

➢ Small arc lengths:
- $r_{jl} \sim \mathcal{O}\left(\frac{1}{N}\right) \ \rightarrow dA_{jl} \sim \mathcal{O}\,(1)$
- $r_{jl} \sim \mathcal{O}\,(1) \ \rightarrow dA_{jl} \sim \mathcal{O}\left(\frac{1}{N}\right)$

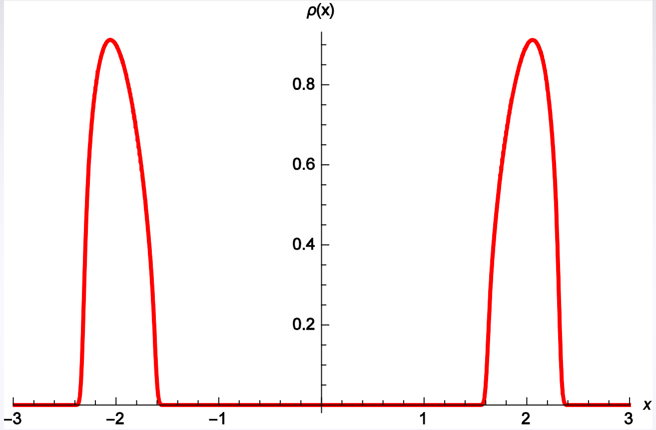
Multi-Cuts SSB

- Level repulsion resolves degeneracy:

  $\Rightarrow$ each of the n cuts contains m_j eigenvalues

- Gap between cuts breaks rotational invariance: $\boxed{U(N) \stackrel{N\to\infty}{\to} \prod_{j=1}^{n} U\left(m_j\right)}$

- Three Arguments:
  - ★ Brownian motion;
  - ★ Numerical finite size analysis;
  - ★ Symmetry Breaking Term



Double well

$$U(N) \stackrel{N\to\infty}{\to} U(N/2) \times U(N/2)$$

(assume N even)

F.F. arXiv:1412.6523

Symmetry Breaking Term

$$\boxed{W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\, \mathbf{T} - \mathbf{M}\, \mathbf{S})|}}$$

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

- Calculate (dis-)order parameter:
  - $\frac{d}{dJ} \lim_{N \to \infty} W(J) \Big|_{J=0} = 0 \longrightarrow$ Symmetry is Broken!
  - Finite N: $\frac{dW(J)}{dJ} \Big|_{J=0} = \langle |\mathrm{Tr}\,(\mathbf{\Lambda}\, \mathbf{T} - \mathbf{M}\, \mathbf{S})| \rangle \neq 0$ Eigenvectors misaligned

Symmetry Breaking Term

$$W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\ \mathbf{T} - \mathbf{M}\ \mathbf{S})|}$$

$$\mathrm{M} = \mathbf{U}^{\dagger} \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^{\dagger} \mathbf{T} \mathbf{V}$$

- Calculate (dis-)order parameter:
  - $\frac{d}{dJ} \lim_{N \to \infty} W(J) \Big|_{J=0} = 0$
  - Finite N: $\frac{dW(J)}{dJ} \Big|_{J=0} \neq 0$

Instantons:
- Pairs of eigenvalues tunneling between cuts
- Restore broken symmetries

$$\int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\ \mathbf{T} - \mathbf{M}\ \mathbf{S})|} \propto \mathcal{Z}_0 + \mathcal{Z}_1 \left(e^{-2JN|\lambda_j - \lambda'_l|} \right) + \ldots$$

Multi-Cuts SSB: Conclusions

- Gap in the eigenvalue distribution
  → Deviation from WD universality
  → Spontaneous breaking of rotational symmetry
  → Eigenvectors localized in patch of Hilbert space spanned by the other eigenvectors in the same cut

- Broken symmetries restored by instantons

- Abstract characterization of localization without reference to basis: IPR not sufficient, need response under perturbation (application to MBL?)

Part 4

Weakly Confined Matrix Models & The Metal/Insulator Transition

Weakly Confined Invariant Models

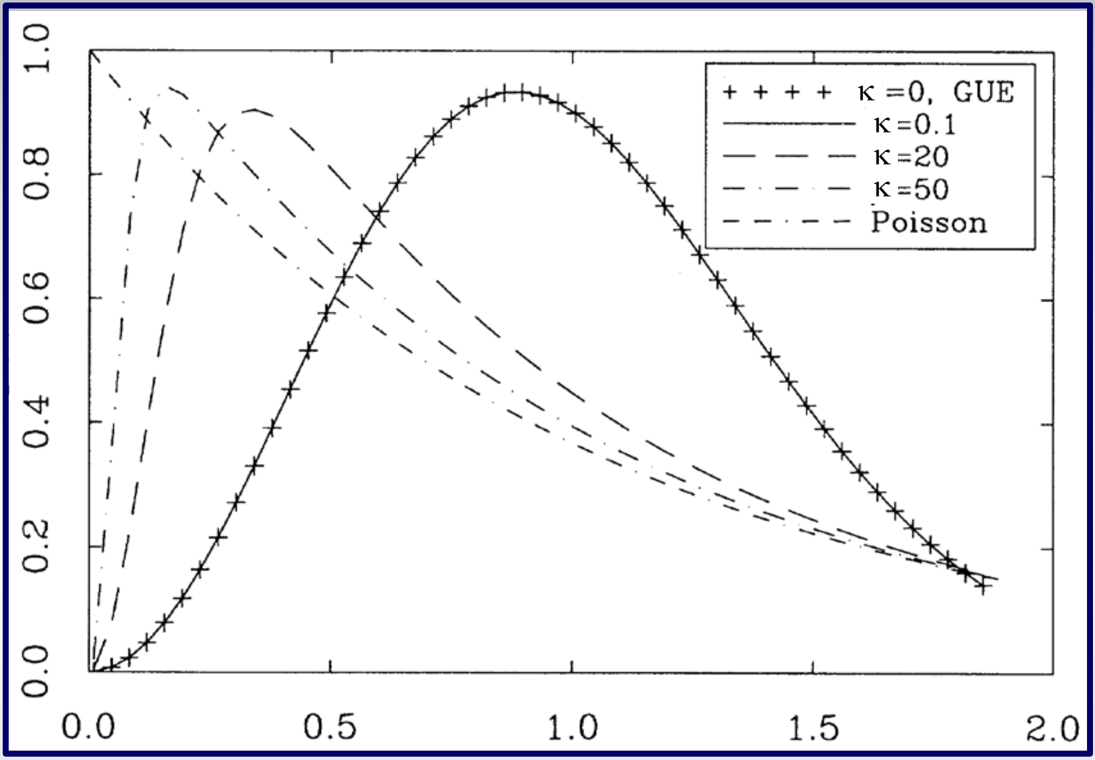
$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} \, , \; V(\lambda) \stackrel{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa} \ln^2 |\lambda|$$

- Soft confinement sets them apart from usual polynomial potentials
  $\rightarrow$ WD universality does not apply
  $\rightarrow$ Indeterminate moment problem
- Arise in localization limit of Chern-Simons/ABJM: $\kappa \propto \dfrac{i}{g_s}$
  (Marino '02; Kapustin et a. '10; …)
- Solvable through orthogonal polynomials:
  q-deformed Hermite/Laguerre Polynomials
  (Muttalib et al. '93; Tierz'04)

Weakly Confined Matrix Models

$$V(\lambda) \overset{|\lambda|\to\infty}{\simeq} \frac{1}{2\kappa} \ln^2|\lambda|$$

- Intermediat level spacing statistics
- Same eigenvalue correlations as power law (critical) Random Banded Matrices

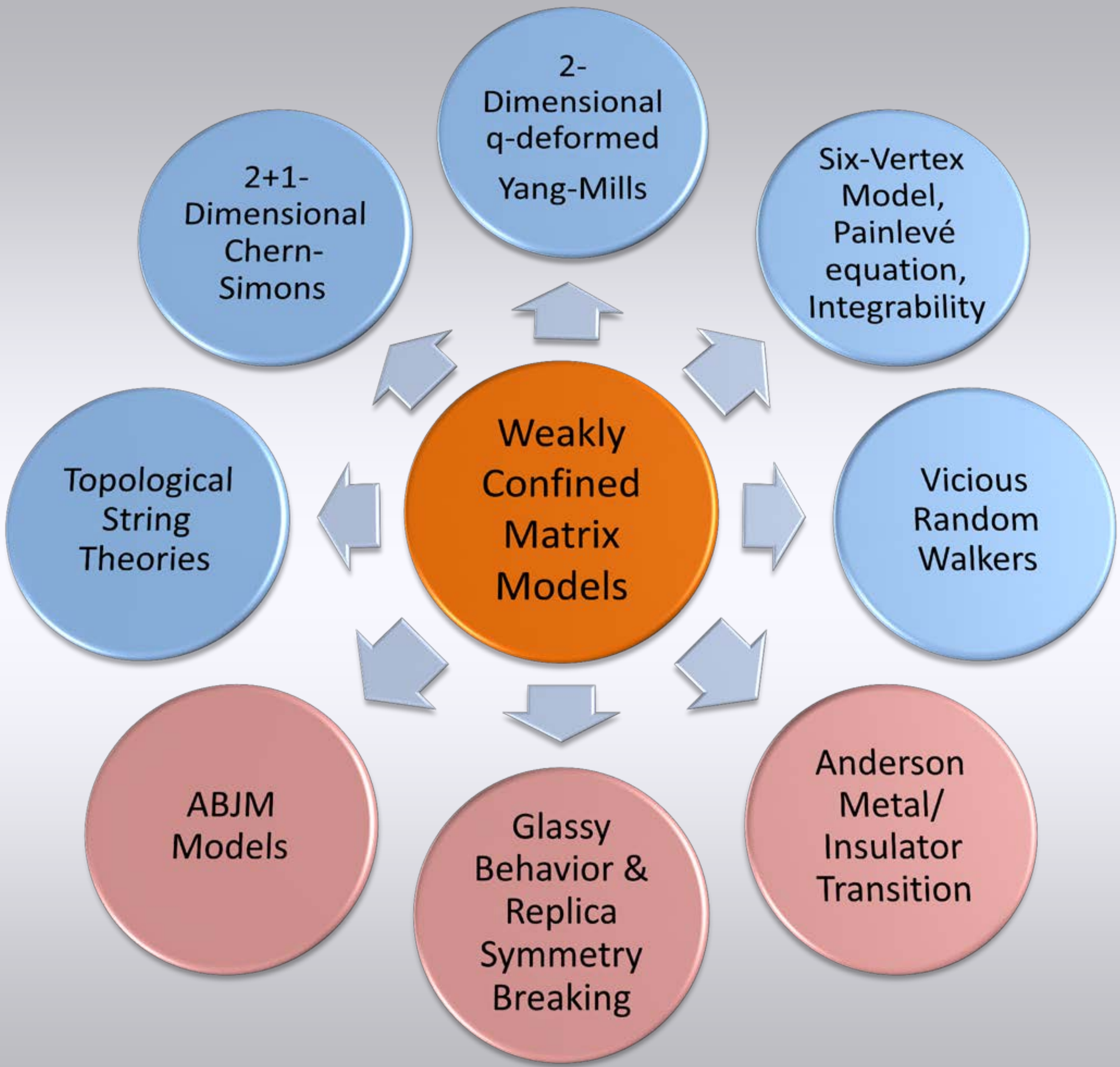


(Muttalib et al. '93)

- Critical level statistics signals fractal eigenstates?
- Critical Spontaneous Breaking of U(N) Invariance?

(Canali, Kravtsov, '95)

Weakly Confined Matrix Models & their applica-tions



WCMM Energy Landscape

- Take exactly log-normal ensemble (positive eigenvalues)

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int_{\lambda>0} d^N\lambda \, \Delta(\{\lambda\}) \, e^{-\frac{1}{2\kappa}\sum_j \ln^2 \lambda_j}$$

- Exponential mapping: $\boxed{\lambda_j = e^{\kappa x_j}}$

$$\mathcal{Z} \propto \int d^N x_j \prod_{n<m} \left(e^{\kappa x_n} - e^{\kappa x_m}\right)^2 \; e^{-\frac{\kappa}{2}\sum_{l=1}^N \left[x_l^2 - 2x_l\right]}$$

- Each term of the Van der Monde shifts the equilibrium of the parabolic potential: different effective potential felt by each eigenvalue for each term for the VdM

F.F. arXiv:1503.03341

WCMM Energy Landscape

- Full partition function known (orthogonal polynomials)

$$\mathcal{Z} \propto e^{\frac{\kappa}{6}N(4N^2-1)} (2\pi\kappa)^{N/2} N! \prod_{n=1}^{N-1} (1-q^n)^{N-n}$$

- Each term of the expansion of the product
  - Corresponds to a different saddle (equilibrium conf.)
  - Has the same leading energy (differ for the powers of q)
  - q^j fugacity of the instantons
- Instantons restore broken symmetries: from the $U(1)^N$ configuration, to the full $U(N)$ when all instantons act

F.F. arXiv:1503.03341

WCMM Outlook

$$\mathcal{Z} = \int \mathcal{D}\mathbf{U} \int_{\lambda>0} d^N\lambda\, \Delta\left(\{\lambda\}\right) e^{-\frac{1}{2\kappa}\sum_j \ln^2 \lambda_j}$$

- Critical eigenvalue statistics from complex landscape
- Each saddle can be interpreted as endowed with a reduced symmetry with respect to $U(N)$
- To do: Employ replica approach for WCMM:

  Anderson transition as a full-RSB
- Conjecture: calculate IPR and multi-fractal spectrum from contributions of the different saddles

SSB Structure

- Each saddle point corresponds to a different SSB

- Unitary matrix from Hermitian matrix: $\mathbf{U} = e^{i\mathbf{A}}$

$$ds^2 = \mathrm{Tr}\,(dM)^2 = \sum_{j=1}^{N} (d\lambda_j)^2 + 2\sum_{j>l}^{N} (\lambda_j - \lambda_l)^2\,|dA_{jl}|^2$$

$$d\mathbf{A} \equiv \mathbf{U}^\dagger d\mathbf{U}$$

- $U(1)^N$ saddle has all $dA_{ij} = 0$

- Conjecture:

Each instanton $-q^n$ "turns on" one element: $dA_{i,i+n} \neq 0$

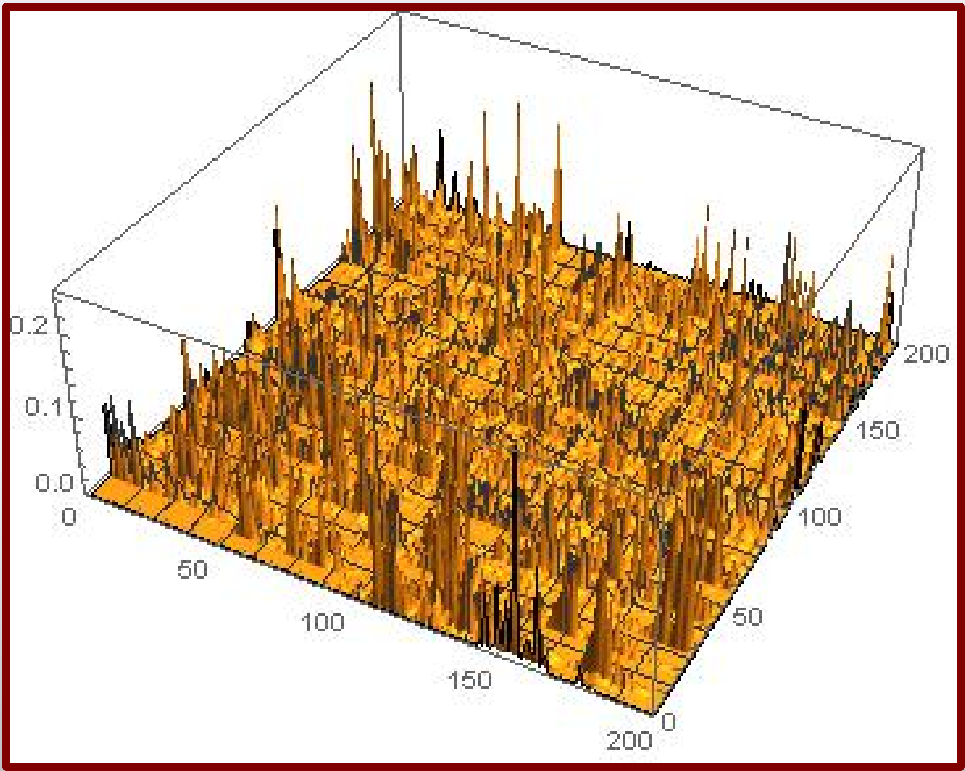
Multi-fractal Spectrum

- Numerical check of conjecture
- Unitary matrix from Hermitian matrix: $\mathbf{U} = e^{i\mathbf{A}}$
- Generate each element A_{jl}

with probability

q^{j-l} $\rightarrow$ sample A_{jl} uniformly

$1 - q^{j-l}$ $\rightarrow$ take $A_{jl} = 0$

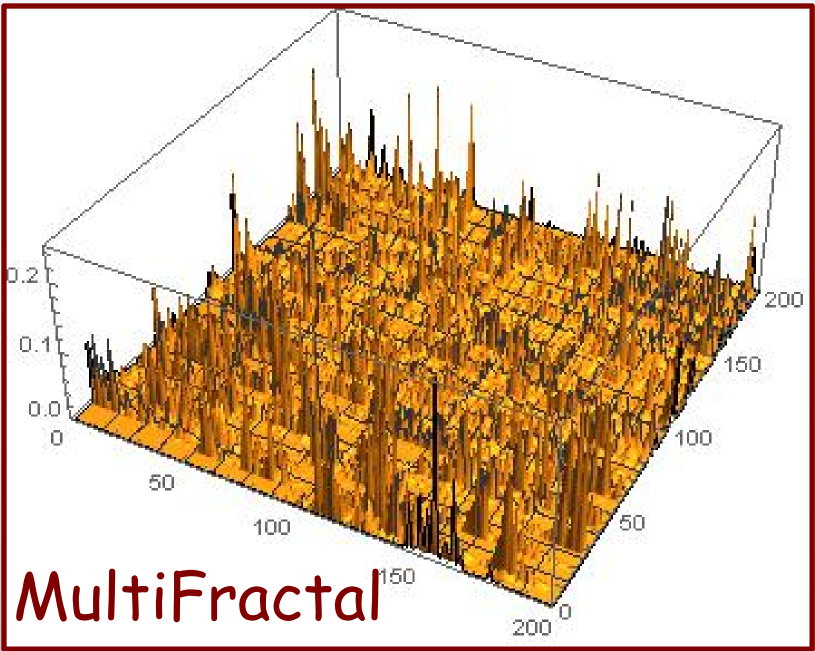


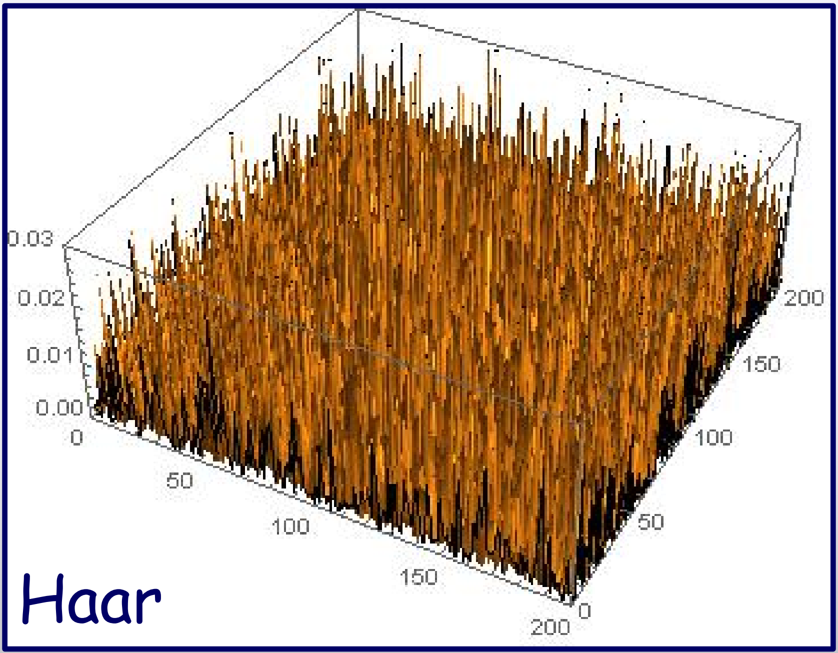
$\Longrightarrow$ MULTIFRACTALITY!

Multi-fractal Spectrum

- Inverse Participation Ratios of $\mathbf{U} = e^{i\mathbf{A}}$ scale with fractional power of N

→ $\boxed{\text{Multi-fractal spectrum from invariant matrix model!}}$





- Matrix $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$ has power-law Gaussian elements!

Conclusions

- Invariant Matrix Models usually applied only to extended/conducting states: eigenvectors discarded
- Deviation of eigenvalue statistics from Wigner-Dyson signals loos of ergodicity: gap between eigenvalues mutually localize their eigenvectors: U(N) broken
- Invariant Models techniques for localization problems!
- WCMM has complex energy landscape $\rightarrow$ critical SSB

Outlook

- Matching WCMM critical SSB with Metal/Insulator Transition multi-fractal spectrum?
- Critical exponents of SSB
- Direct characterization of eigenvector behavior
- Connection between SSB & Replica Symmetry Breaking
- WCMM & Matrix models arise in string theory: meaning of the $N \to \infty$ $U(N)$ symmetry breaking?
- …

Thank you!

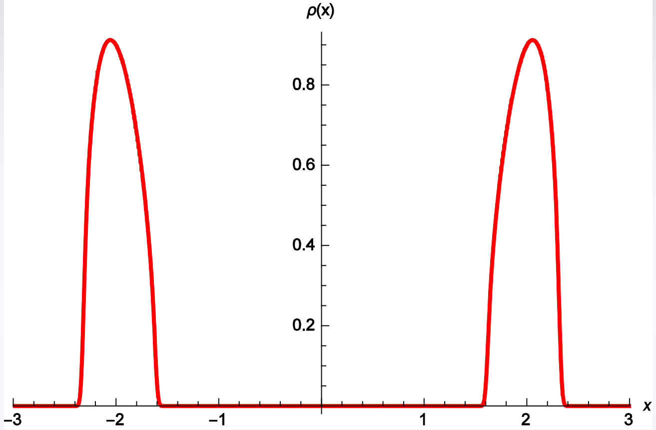
Brownian Motion Picture

- Level repulsion resolves degeneracy:

  $\Rightarrow$ each of the n cuts contains m_j eigenvalues

- Gap between cuts breaks rotational

  invariance: $U(N) \overset{N\to\infty}{\to} \prod_{j=1}^{n} U(m_j)$



- Dyson Brownian Motion for equilibrium distribution shows scale separation:

$$d\lambda_j = -\frac{dV(\lambda_j)}{d\lambda_j}\,dt + \frac{1}{N}\sum_{l\neq j}\frac{dt}{\lambda_j-\lambda_l} + \frac{1}{\sqrt{N}}\,dB_j(t)$$

$$d\vec{U}_j(t) = -\frac{1}{2N}\sum_{l\neq j}\frac{dt}{(\lambda_j-\lambda_l)^2}\,\vec{U}_j + \frac{1}{\sqrt{N}}\sum_{l\neq j}\frac{dW_{jl}(t)}{\lambda_j-\lambda_l}\,\vec{U}_l$$

$\boxed{dB_j, dW_{jl} \text{ delta-corr. stochastic sources}}$

Generating a Random Matrix

- Gaussian Models: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr}\mathbf{M}^2} = \int \prod dM_{jl} e^{-\sum_{jl} M_{jl}^2}$

  → each matrix entries sampled independently

- One-Cut Models: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} V(\mathbf{M})} = \int \mathcal{D}\mathbf{M} e^{-\mathrm{Tr} \sum_k g_k \mathbf{M}^k}$

  → entries correlated: generated as perturbation of Gaussian case in a Metropolis scheme

- Multi-Cut Solutions: Gaussian case unstable

  → start from initial seed and evolve it to equilibrium

  → SSB: final configuration has memory of eigenvectors of initial seed

Symmetry Breaking Term

- To detect SSB introduce symmetry breaking term
- Most natural one is $\mathrm{Tr}\left([\mathbf{M},\mathbf{S}]\right)^2$, but too hard to handle

$\mathbf{S}$: given Hermitian Matrix
Favors alignment of eigenvectors

- We introduce: $$W(J) = \ln \int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}$$

J : source strength

$$\mathrm{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^\dagger \mathbf{T} \mathbf{V}$$

Absolute value can be removed by sorting eigenvalues in increasing order

Symmetry Breaking: Double Well

$$W(J) = \ln \int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN |\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|}$$

$$\mathrm{M} = \mathbf{U}^{\dagger}\mathbf{\Lambda}\mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^{\dagger}\mathbf{T}\mathbf{V}$$

- Double well: $U(N) \overset{N\to\infty}{\to} U(N/2) \times U(N/2)$

  (assume N even)

- Take $\mathbf{S}$ with 2 sets of N/2-degenerate eigenvalues: $t = \pm 1$ to induce correct symmetry breaking

- Calculate (dis-)order parameter:

$$\left.\frac{dW(J)}{dJ}\right|_{J=0} = \langle |\mathrm{Tr}\,(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})| \rangle \begin{cases} = 0 & \text{Symmetry Broken} \\ \neq 0 & \text{Eigenvectors misaligned} \end{cases}$$

Symmetry Breaking Term

$$W(J) = \ln \int d\mathbf{M} e^{-N\mathrm{Tr}V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\ \mathbf{T} - \mathbf{M}\ \mathbf{S})|}$$

$$\mathrm{M} = \mathbf{U}^{\dagger}\mathbf{\Lambda}\mathbf{U}$$
$$\mathrm{S} = \mathbf{V}^{\dagger}\mathbf{T}\mathbf{V}$$

- Use Itzykson-Zuber formula: (Itzykson & Zuber, '80)

$$\int d\mathbf{U} e^{\mathrm{Ttr}\mathbf{AUBU}^{\dagger}} \propto \frac{\det\left[e^{a_j b_l}\right]}{\Delta(\{a\})\,\Delta(\{b\})}$$

- After regularization for degenerate eigenvalues:

$$\int d\mathbf{U} e^{JN\mathrm{Tr}\mathbf{M}\ \mathbf{S}} \propto \frac{1}{\Delta(\{\lambda\})} \sideset{}{'}\sum_{\{\alpha\}\cup\{\alpha'\}=\{\lambda\}} e^{-JN\sum_j(\alpha_j - \alpha'_j)}\,\Delta(\{\alpha\})\,\Delta(\{\alpha'\})$$

Sum over ways to partition eigenvalues of M according to degeneracies of S

Symmetry Breaking Term

$$\int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|} \propto$$

$$\int_{\lambda>0,\lambda'<0} d^{\frac{N}{2}}\lambda d^{\frac{N}{2}}\lambda' e^{-N\sum_j V(\lambda_j) - N\sum_l V(\lambda'_l)} \times \Delta^2(\{\lambda\})\,\Delta^2(\{\lambda'\}) \prod_{j,l}^{N} (\lambda_j - \lambda_l) \times$$

$$\times \left[1 + \sum_{j,l=1}^{N/2} e^{-2JN(\lambda_j - \lambda'_l)} \prod_{p=1}^{N/2}\prod_{q=1}^{N/2} \frac{(\lambda_l - \lambda'_p)(\lambda_j - \lambda'_q)}{(\lambda'_j - \lambda_p)(\lambda'_l - \lambda_q)} + \dots\right]$$

- Hence: $\frac{d}{dJ} \lim_{N\to\infty} W(J)\Big|_{J=0} = 0$
- At finite N: $\langle |\mathrm{Tr}\,(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})| \rangle \neq 0$

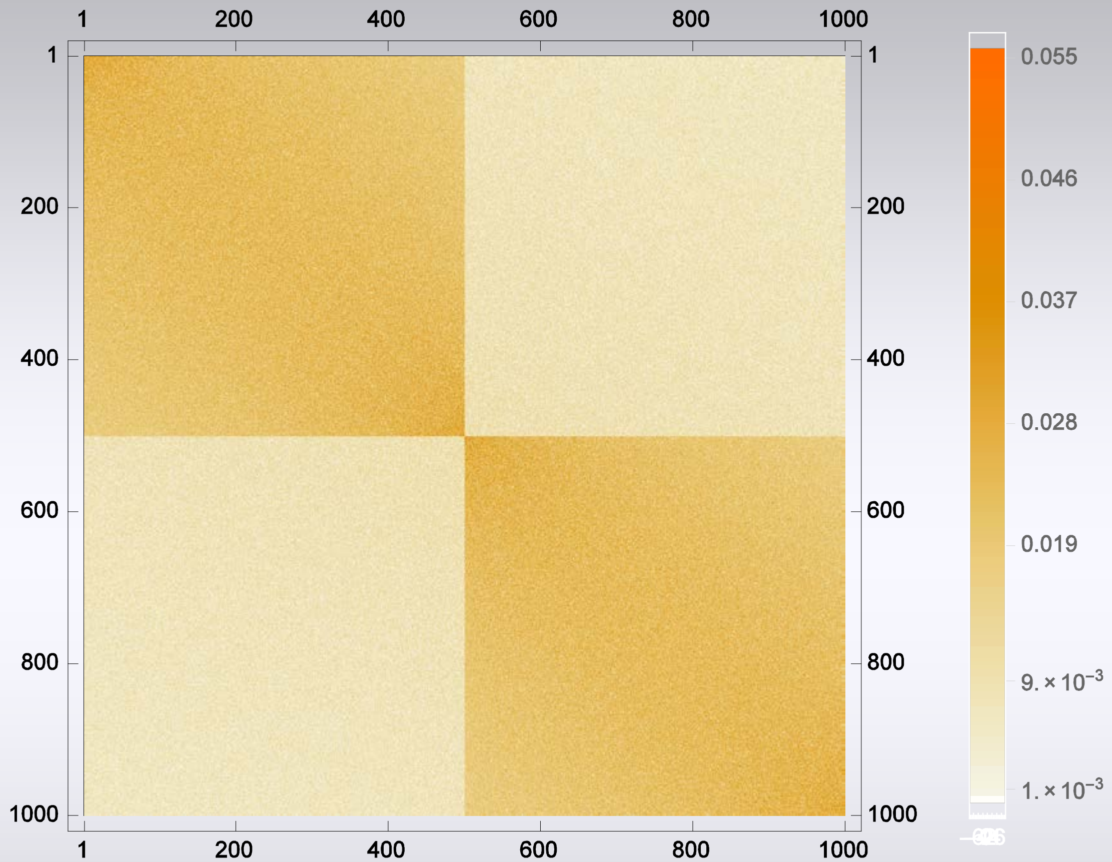
Instantons:
- Pairs of eigenvalues tunneling between wells
- Restore broken symmetries

$$\int d\mathbf{M} e^{-N \mathrm{Tr} V(\mathbf{M}) + JN|\mathrm{Tr}(\mathbf{\Lambda}\,\mathbf{T} - \mathbf{M}\,\mathbf{S})|} \propto \mathcal{Z}_0 + \mathcal{Z}_1 \left(e^{-2JN|\lambda_j - \lambda'_l|}\right) + \dots$$

Finite Size Analysis

- Without preferred, reference basis; localization means rigidity of eigenvectors under perturbations
- Take double well matrix model: $\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N\mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$
- Generate a representative matrix: $\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$
- Apply perturbation $\mathbf{\Delta M}$ (sparse Gaussian Matrix)
- Find eigenvectors of perturbed matrix: $\mathbf{M} + \mathbf{\Delta M} = \mathbf{U}'^\dagger \mathbf{\Lambda}' \mathbf{U}'$
- Consider eigenvectors of perturbed matrix in original eigenvector basis (rotation due to perturbation): $\tilde{\mathbf{U}} = \mathbf{U}'\mathbf{U}^\dagger$

Finite Size Analysis



$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N\mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$$

Equilibrium conf. from Coulomb gas

$$\mathbf{M} = \mathbf{U}^\dagger \mathbf{\Lambda} \mathbf{U}$$

Randomly generated

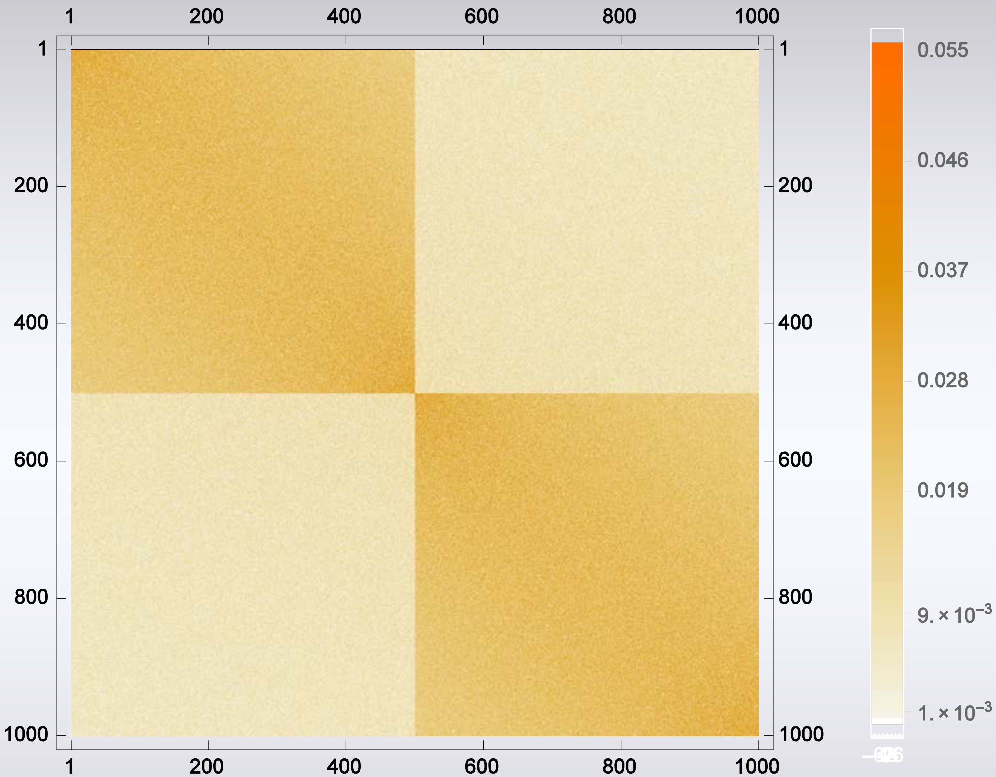
$$\mathbf{M} + \mathbf{\Delta M} = \mathbf{U}'^\dagger \mathbf{\Lambda}' \mathbf{U}'$$

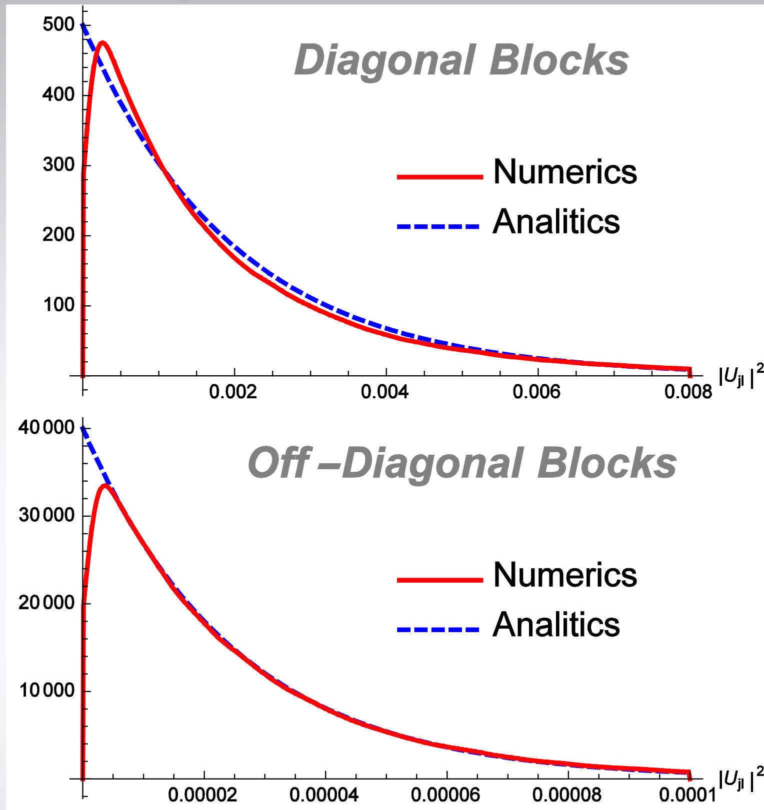
Sparse Gaussian Matrix

Typical Unitary matrix $\tilde{\mathbf{U}} = \mathbf{U}'\mathbf{U}^\dagger$ connecting the eigenvectors before and after the perturbation (t=4; N=1000; sparse matrix with n=200 non zero elements, drawn from Gaussian with zero mean and variance N)

Finite Size Analysis

$$\mathcal{Z} = \int \mathcal{D}\mathbf{M} e^{-N\mathrm{Tr}\left[\frac{1}{4}\mathbf{M}^4 - \frac{t}{2}\mathbf{M}^2\right]}$$



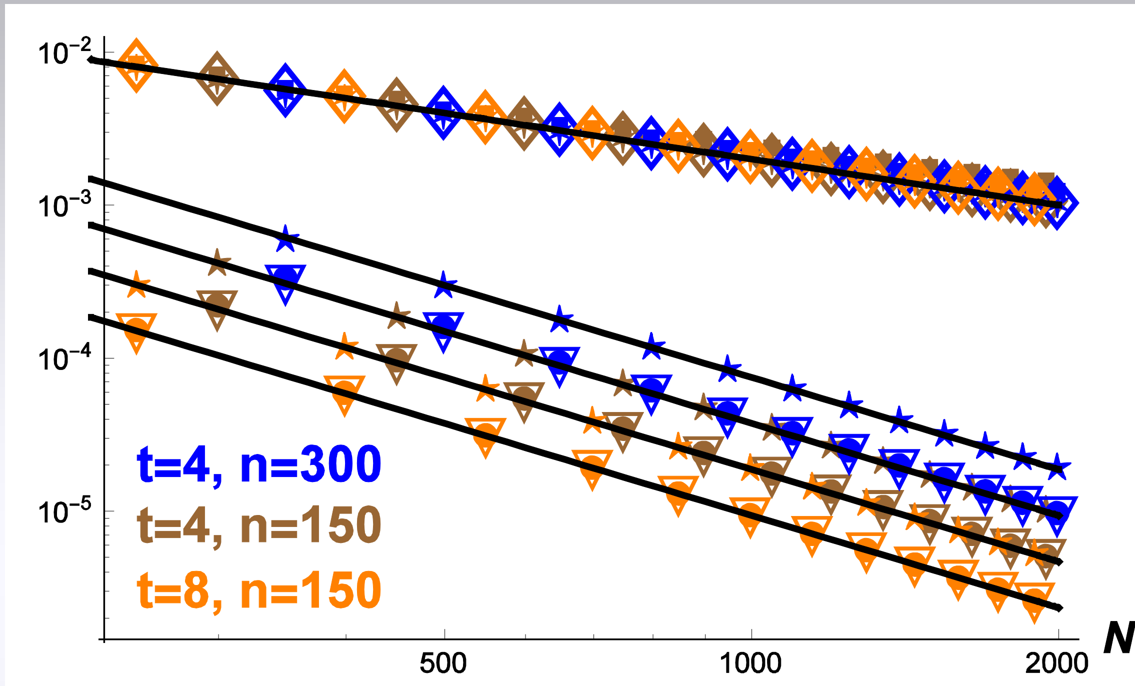


$$\mathcal{P}\left(\left|\tilde{U}_{ij}\right|^2\right) = \chi \exp\left[-\chi\left|\tilde{U}_{ij}\right|^2\right]$$

$$\chi_{\mathrm{D}} = \frac{N}{2} \qquad \chi_{\mathrm{OD}} = \frac{2tN^2}{n}$$

t=4, N=1000, sparse matrix with n=200 non zero elements, drawn from Gaussian with zero mean and variance N)

Finite Size Analysis



$$O_{jl} = \sum_m^N \left|\tilde{U}_{mj}\right|^2 \left|\tilde{U}_{ml}\right|^2$$

Overlaps between eigenstates

$$\langle |O_{jl}| \rangle_{\mathrm{D}} = \langle \left|\tilde{U}_{jl}\right| \rangle_{\mathrm{D}} = \langle \left|\Delta\tilde{U}_{jl}\right| \rangle_{\mathrm{D}} = \frac{1}{\chi_{\mathrm{D}}} = \frac{2}{N}$$

$$\langle |O_{jl}| \rangle_{\mathrm{OD}} = 2\langle \left|\tilde{U}_{jl}\right| \rangle_{\mathrm{OD}} = 2\langle \left|\Delta\tilde{U}_{jl}\right| \rangle_{\mathrm{OD}} = \frac{2}{\chi_{\mathrm{OD}}} = \frac{n}{tN^2}$$

Off-diagonal blocks suppressed as 1/N compared to diagonal ones

Onset of localizations!

WCMM Energy Landscape

$$\mathcal{Z} \propto \int d^N x \prod_{n<m} (e^{\kappa x_n} - e^{\kappa x_m})^2 \; e^{-\frac{\kappa}{2}\sum_{l=1}^N [x_l^2 - 2x_l]}$$

- Consider simplex of eigenvalues in increasing order

$$\mathcal{Z} \propto N! \int d^N x \prod_{n<m} \left[1 - e^{\kappa(x_n - x_m)}\right]^2 \; e^{-\frac{\kappa}{2}\sum_{l=1}^N [x_l^2 - 2(2l-1)x_l]}$$

- In the $\kappa \to \infty$ limit, all terms left inside the VdM vanish
- Eigenvalue crystalize on a lattice

(Bogomolny et al. '97)

$$\lim_{\kappa\to\infty} \mathcal{Z} \propto N! e^{\frac{\kappa}{6}N(4N^2-1)} \int d^N x e^{-\frac{\kappa}{2}\sum_{l=1}^N (x_l + 1 - 2l)^2}$$

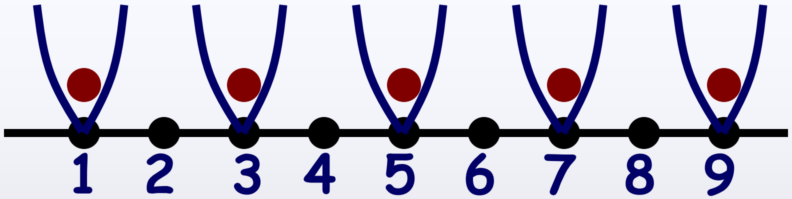
Eigenvalue Crystallization

$$\mathcal{Z} \propto N! \int d^N x \prod_{n<m} \left[1 - e^{\kappa(x_n - x_m)}\right]^2 e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2(2l-1)x_l\right]}$$

- Eigenvalue crystallization for $\kappa \to \infty$

$$\mathcal{Z} \propto \int d^N x \left[e^{-\frac{\kappa}{2}\sum_{l=1}^{N}(x_l + 1 - 2l)^2} + \ldots\right]$$

(Bogomolny et al. ‘97)



- Corresponds to SSB: $\boxed{U(N) \to U(1)^N}$

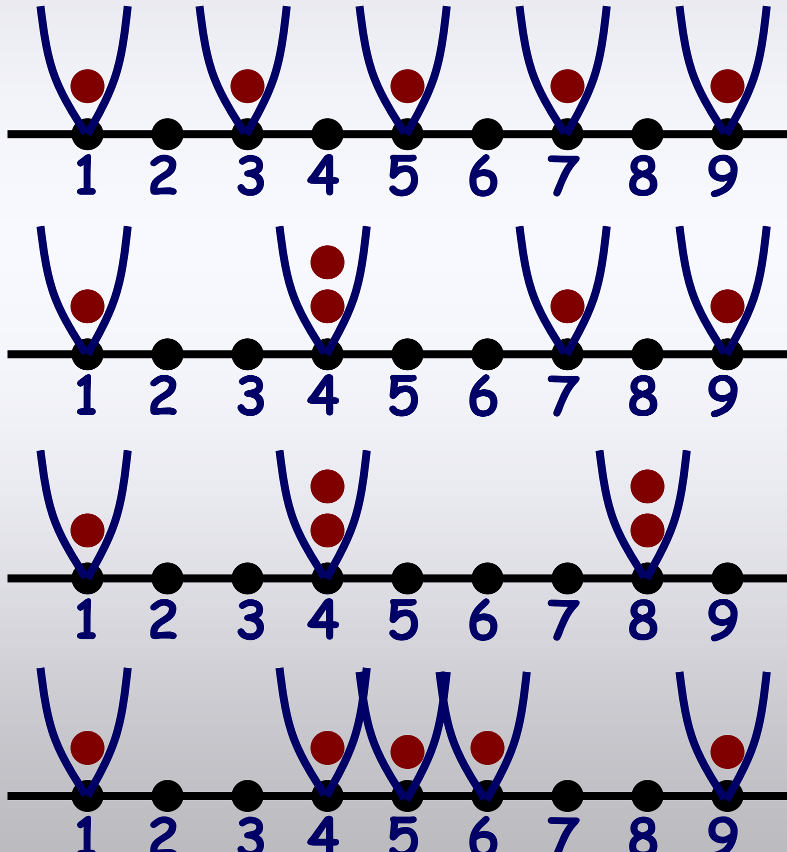
(Exponential separation between eigenvalues completely freezes eigenstates dynamics)

(Pato, ‘00)

WCMM Energy Landscape

$$\mathcal{Z} \propto N! \int d^N x \prod_{n<m} \left[1 - e^{\kappa(x_n - x_m)}\right]^2 e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2(2l-1)x_l\right]}$$

- Finite κ corrections organized in powers of $q = e^{-\kappa}$



q^0

$-q^1$ $\times(N-1)$

$+q^2$ $\times\frac{(N-2)(N-3)}{2}$

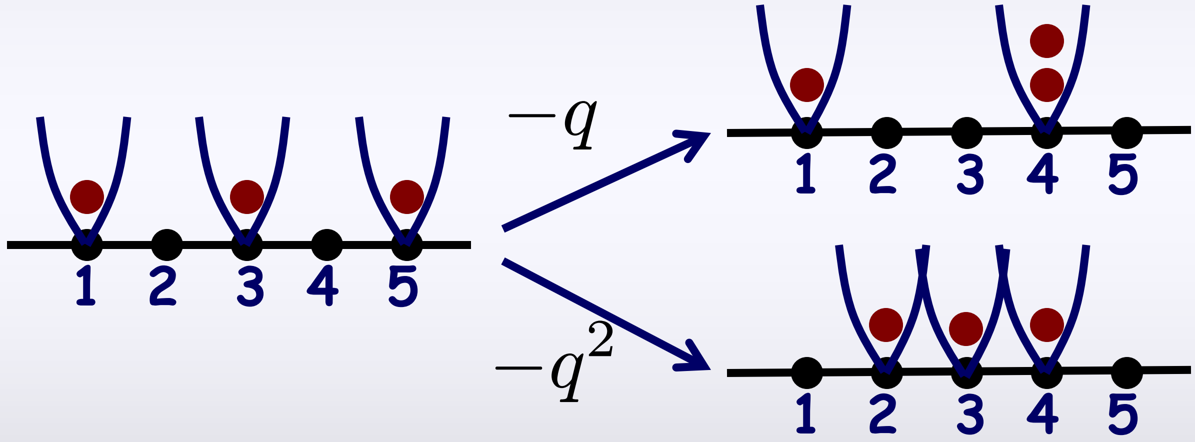
$+q^2 - q^2$ $\times(N-2)$

$\ldots$

WCMM Energy Landscape

$$\mathcal{Z} \propto N! \int d^N x \prod_{n<m} \left[1 - e^{\kappa(x_n - x_m)}\right]^2 e^{-\frac{\kappa}{2}\sum_{l=1}^{N}\left[x_l^2 - 2(2l-1)x_l\right]}$$

- Each term in VdM shifts the equilibrium point of 2 eigenvalues (one notch closer to one-another)



- Each term of the VdM generates a new equilibrium configuration (saddle point of the partition function)
- Saddles connected by instantons with weight $-q^n$

WCMM Energy Landscape

- Full partition function known (orthogonal polynomials)

$$\mathcal{Z} \propto e^{\frac{\kappa}{6}N(4N^2-1)} (2\pi\kappa)^{N/2} N! \prod_{n=1}^{N-1} (1-q^n)^{N-n}$$

- Natural interpretation in terms of instantons
  - 1 instanton connecting a pair of eigenvalues N-1 apart
  - 2 instantons connecting a pair N-2 apart
  - …
  - N-1 instantons connecting a pair of N.N. eigenvalues
- Metal/Insulator Transition as glassy phase?

WCMM Energy Landscape

$$\mathcal{Z} \propto e^{\frac{\kappa}{6}N(4N^2-1)} (2\pi\kappa)^{N/2} N! \prod_{n=1}^{N-1} (1-q^n)^{N-n}$$

- Exponential number of saddle points
- Every equilibrium configuration which preserves the center of mass is realized by the action of the instantons
- Each equilibrium configuration has the same leading energy: they differ only for the powers of q
- Instantons restore broken symmetries: from the $U(1)^N$ configuration at $\kappa \to \infty$, to the full $U(N)$ when all instantons bring each eigenvalue to the same equilibrium point