Plane partitions with two-periodic weights

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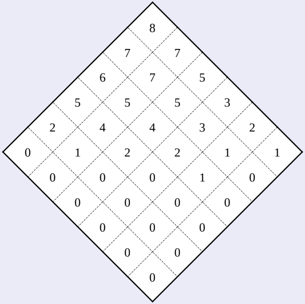
GGI
June 15, 2015

The Model

The Model

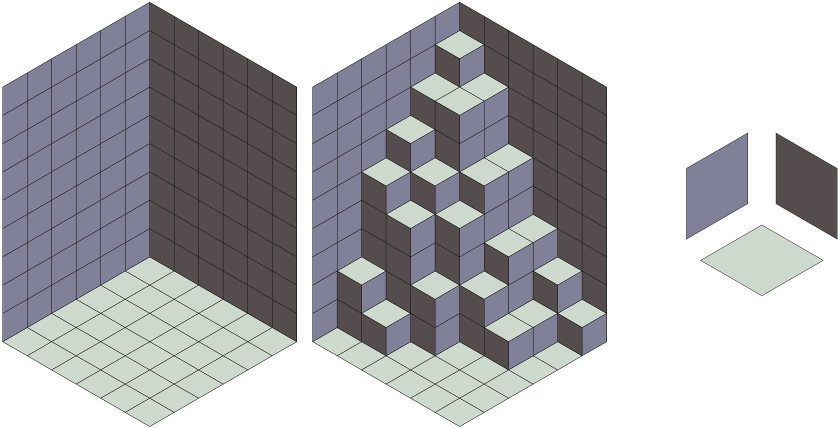
Plane partitions

Definition



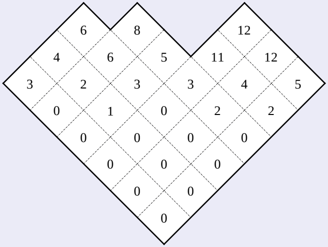
Notation: $\pi = \{\pi_{i,j}\}_{i>0,j>0}$, confined to an $M \times N$ rectangle (here 6×6).

Plane partitions as stacks of cubes



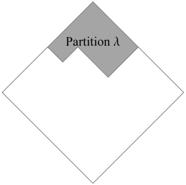
Skew plane partitions

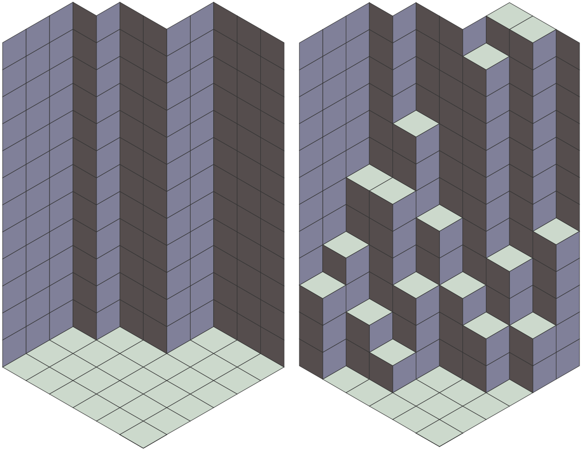
Definition



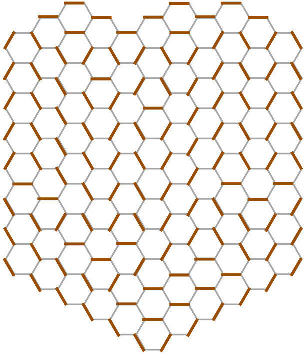
Notation: $\pi_\lambda = \{\pi_{i,j}\}$, defined for all pairs (i,j) not in λ.

Skew plane partitions as stacks of cubes

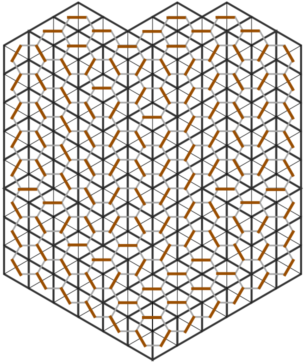




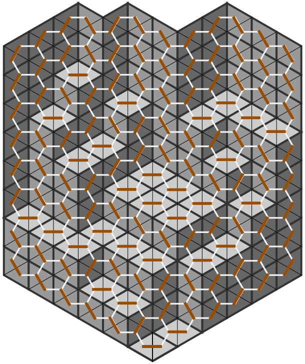
Plane partitions as dimer configurations



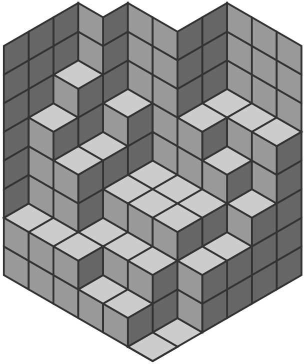
Plane partitions as dimer configurations



Plane partitions as dimer configurations



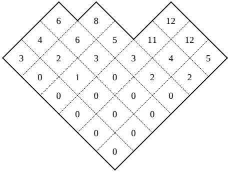
Plane partitions as dimer configurations

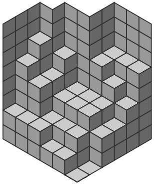


The Model

Homogeneous weights

The probability measure





- Consider the system consisting of all skew plane partitions with boundary λ, confined to the $N \times M$ box, with the distribution

$$Prob(\pi) \propto q^{|\pi|} = q^{volume},$$

for some $q \in (0,1)$, where $|\pi| = \sum \pi_{i,j}$ is the total volume.

- $q^{volume} \leftrightarrow$ homogeneous weights.

An instance of the Schur-process

- Recall the Schur-process introduced by Okounkov-Reshetikhin (2003):
  - A measure on sequences of Young diagrams $\{\lambda(i)\}_i$.
  - Position-dependent transition weights between two Young diagrams:

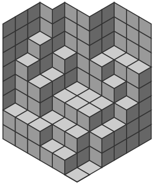
$$S^{(t)}(\lambda(t), \lambda(t-1)).$$

  - Schur-process:

$$Prob(\{\lambda(i)\}_i) \propto \prod_t S^{(t)}(\lambda(t), \lambda(t-1)).$$

- q^{volume} on skew plane partitions is a special case of the Schur-process.
- Okounkov-Reshetikhin showed that this is a determinantal process and computed the correlation kernel.

The correlation functions



- The positions of the horizontal tiles completely determine the (skew) plane partition.
- To understand the fluctuations, study the local correlation functions of the positions of the horizontal tiles.
- Denote by $\rho((t_1, h_1), \ldots, (t_n, h_n))$ the probability that there are horizontal tiles at positions (t_i, h_i), $i = 1, \ldots, n$.

The discrete correlation kernel

Theorem (Okounkov-Reshetikhin 2003)

- $\rho((t_1, h_1), \ldots, (t_n, h_n)) = \det(K((t_i, h_i), (t_j, h_j))_{i,j=1}^n$.
- *The correlation kernel* $K((t_1, h_1), (t_2, h_2))$ *is given by:*

$$\frac{1}{(2\pi i)^2} \iint \frac{\Phi_-(z, t_1)\Phi_+(w, t_2)}{\Phi_+(z, t_1)\Phi_-(w, t_2)} \frac{\sqrt{zw}}{z - w} z^{-h_1 + b_{\lambda_q}(t_1) - 1/2} w^{h_2 - b_{\lambda_q}(t_2) - 1/2} \frac{dzdw}{zw},$$

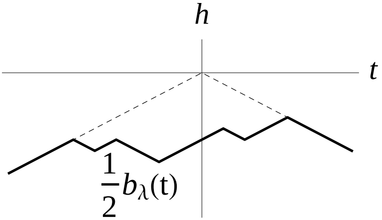
where

$$\Phi_\pm(z, t) = \prod_{\substack{m \gtrless t,\, m \in \mathbb{Z} + \frac{1}{2} \\ \text{slope at } m \text{ is } \mp 1}} (1 \mp z^{\pm 1} q^{\pm m}),$$

and b_{λ_q} *encodes the "back wall".*

The scaling limit

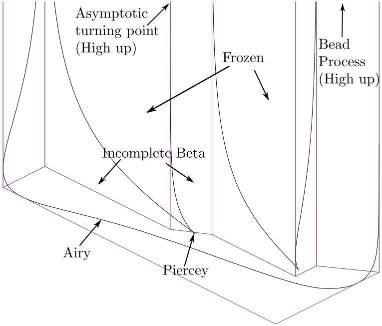
- The thermodynamic limit of the system is when $q \to 1^-$.
- Write q as $q = e^{-r}$, and study the limit $r \to 0^+$.
- Question: How should the parameters N, M and λ change in the limit?
- Answer for N, M: The typical size of a plane partition not restricted to a finite box is $\frac{1}{r}$, so one should study the limit when N and M grow at the rate $\frac{1}{r}$, and scale the system by r in all directions.
- Answer for back wall: Let $b_{\lambda_q}(t)$ be the functions giving the back walls.



Study the limit when after rescaling $b_{\lambda_q}(\tau)$ converges to a 1-Lipschitz function.

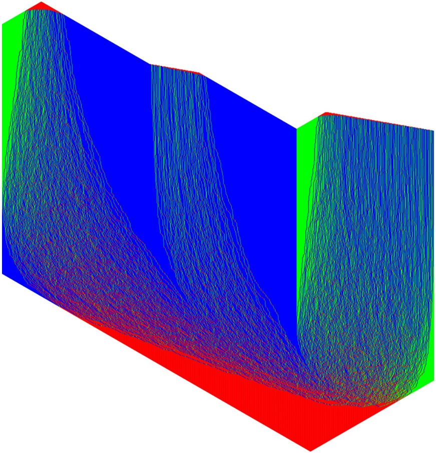
M. - Skew plane partitions with arbitrary piecewise linear back walls

Most general case studied: back wall is a piecewise linear curve of slopes in $[-1, 1]$.



Earlier works on limit shapes by Nienhuis (plane partitions), Kenyon (plane partitions), Okounkov and Reshetikhin (slopes ± 1), Boutilier, M., Reshetikhin and Tingley (slopes in $(-1, 1)$).

M. - Skew plane partitions with arbitrary piecewise linear back walls



The Model

Inhomogeneous weights

The probability distribution

- Recall the Schur-Process representation: write a skew plane partition π as a sequence $\{\pi^i\}_i$ of its diagonal slices.
- Given $\{q_i\}_{i\in\mathbb{Z}}$, $q_i > 0$, consider the system consisting of all skew plane partitions with boundary λ, with the distribution

$$Prob(\pi) \propto \prod_{i\in\mathbb{Z}} q_i^{|\pi^i|},$$

where $|\pi^i|$ is the total volume of the i-th slice of π.

Discrete correlation kernel for non-homogeneous weights

Theorem (Okounkov-Reshetikhin 2003)

- $\rho((t_1, h_1), \ldots, (t_n, h_n)) = \det(K((t_i, h_i), (t_j, h_j))_{i,j=1}^n$.
- *The correlation kernel $K((t_1, h_1), (t_2, h_2))$ is given by:*

$$\frac{1}{(2\pi i)^2} \iint \frac{\Phi_-(z, t_1)\Phi_+(w, t_2)}{\Phi_+(z, t_1)\Phi_-(w, t_2)} \frac{\sqrt{zw}}{z - w} z^{-h_1 + b_{\lambda_q}(t_1) - 1/2} w^{h_2 - b_{\lambda_q}(t_2) - 1/2} \frac{dzdw}{zw},$$

where

$$\Phi_\pm(z, t) = \prod_{\substack{m \gtrless t, m \in \mathbb{Z} + \frac{1}{2} \\ \text{slope at } m \text{ is } \mp 1}} (1 \mp z^{\pm 1} a^{\pm 1} q_0^{\pm 1} \ldots q_{m-\frac{1}{2}}^{\pm 1}),$$

and b_{λ_q} encodes the "back wall".

Only change is q^m is replaced with $aq_0 \ldots q_{m-\frac{1}{2}}$.

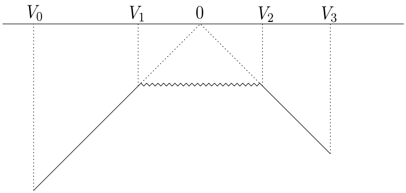
Periodic weights

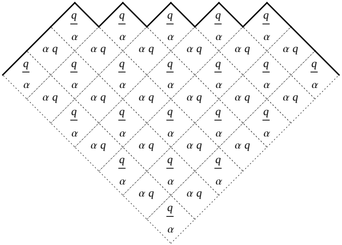
- Consider weights with

$$q_0 = q_{2k} \text{ and } q_1 = q_{2k+1} \; \forall k \in \mathbb{Z}.$$

- What scaling limit should we study?
- Nothing new, if you take $q_0 \to 1^-$ and $q_1 \to 1^-$.
- More interesting: $\alpha \geq 1$, $q_0 = \alpha q$, $q_1 = \alpha^{-1} q$ and $q \to 1^-$.
- Obstacle: partition function may be infinite.

Periodic weights





Note, that we have $q = e^{-r}$ and we are scaling by r, thus there are $(V_2 - V_1)/r$ microscopic linear sections between V_1 and V_2. Hence in order for the measure to be well defined, we must have

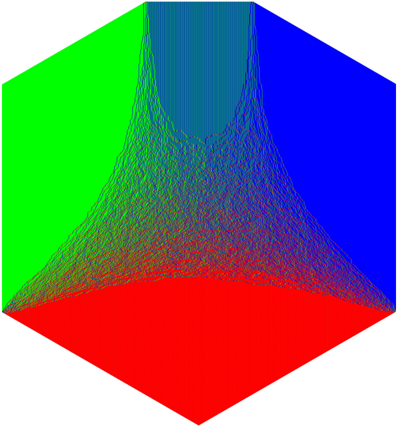
$$q^{(V_2 - V_1)/r}\alpha < 1,$$

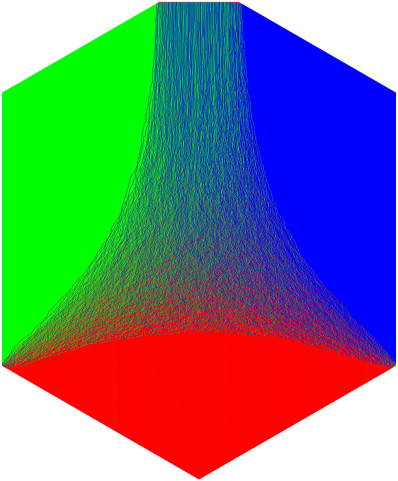
or equivalently

$$e^{-(V_2 - V_1)}\alpha < 1.$$

Unbounded floor: Frozen boundary

Unbounded floor: A sample





Bulk correlations

Theorem (M.)

The correlation functions of the system near a point (χ, τ) in the bulk are given by

$$K^{\alpha}_{\chi,\tau}(t_1, t_2, \Delta h) = \int_{\gamma} (1 - e^{-\tau}\alpha^{\frac{1}{2}} z)^{\frac{\Delta t + c}{2}} (1 - e^{-\tau}\alpha^{-\frac{1}{2}} z)^{\frac{\Delta t - c}{2}} z^{-\Delta h - \frac{\Delta t}{2}} \frac{dz}{2i\pi z},$$

where $\Delta t = t_1 - t_2$, $c = 0$ if Δt is even, $c = 1$ if Δt is odd and t_1 is even, $c = -1$ otherwise.

- When $\alpha = 1$ we recover the incomplete beta kernel, which is the correlation kernel in the bulk for the q^{volume} measure:

$$K^{\alpha}_{\chi,\tau}(t_1, t_2, \Delta h) = \int_{\gamma} (1 - e^{-\tau} z)^{\Delta t} z^{-\Delta h - \frac{\Delta t}{2}} \frac{dz}{2i\pi z}.$$

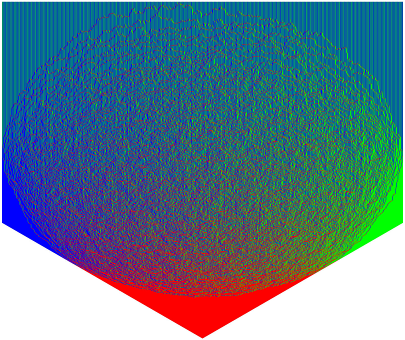
- The correlation functions in the bulk are not $\mathbb{Z} \times \mathbb{Z}$ invariant as in the homogeneous case. The local process is $\mathbb{Z} \times 2\mathbb{Z}$ translation invariant.
- The process is a special case of a family of processes studied by Borodin('07).

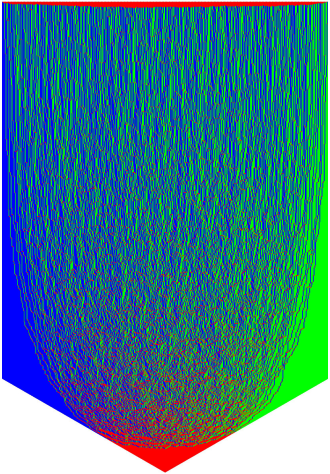
Triangular floor: Frozen boundary

Triangular floor: Turning points

In the limit $\alpha \to 1$ the two-periodic model converges to the homogeneous model. The turning points converge to the turning points at infinity studied by Boutilier, M., Reshetikhin, Tingley.

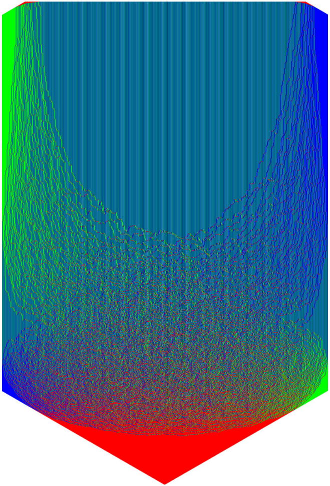
Triangular floor: A sample

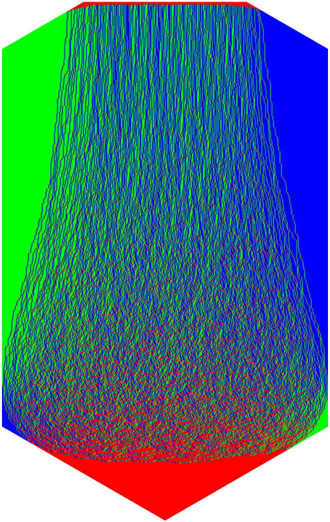




Bounded floor: Frozen boundary

Bounded floor: A sample

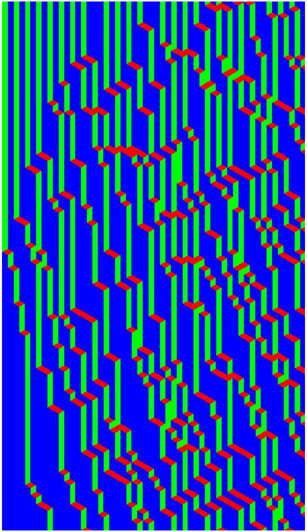




Turning points

- Informal arguments were given by Okounkov-Reshetikhin that the local point process at turning points should be the GUE-minors process. Rigorous results have been obtained by Johansson-Nordenstam, Gorin-Panova.
- There are two turning points near each vertical boundary section.
- The fact that there are two turning points implies that locally you do not have the interlacing property from slice to slice.
- $\chi_1 - \chi_2$ converges to zero when α converges to 1.

Turning points



Turning point correlations

Theorem (M.)

Let (τ,χ) be a turning point and let $t_i = \lfloor \frac{\tau}{r} \rfloor - \hat{t}_i$, and $h_i = \lfloor \frac{\chi}{r} \rfloor + \frac{\tilde{h}_i}{r^{\frac{1}{2}}}$. If $\lfloor \frac{\tau}{r} \rfloor$ is odd, then the correlation functions near a turning point (τ,χ) of the system with periodic weights are given by

$$\lim_{r\to 0} r^{-\frac{1}{2}} K_{\lambda,\bar{q}}((t_1,h_1),(t_2,h_2)) = \frac{1}{(2\pi \mathrm{i})^2} \iint e^{\frac{\sigma^2}{2}(\zeta^2-\omega^2)} \frac{e^{\tilde{h}_2\omega}}{e^{\tilde{h}_1\zeta}} \frac{\omega^{\lfloor \frac{\hat{t}_2+e}{2} \rfloor}}{\zeta^{\lfloor \frac{\hat{t}_1+e}{2} \rfloor}} \frac{d\zeta \; d\omega}{\zeta-\omega},$$

where e is 1 when $\chi = \chi_{top}$ and 2 when $\chi = \chi_{bottom}$. When $\lfloor \frac{\tau}{r} \rfloor$ is even, e is replaced by $2-e$.

Remark: If we restrict the process to horizontal lozenges of only even or only odd distances from the edge, then the correlation kernel coincides with the correlation kernel of the GUE-minors process, so we have two GUE-minors processes non-trivially correlated.

Intermediate regime

- Question: What happens when $\alpha \to 1$?
- Consider two-periodic weights q_t given by

$$q_t = \begin{cases} e^{-r+\gamma r^{1/2}}, & t \text{ is even} \\ e^{-r-\gamma r^{1/2}}, & t \text{ is odd} \end{cases}, \tag{1}$$

  where $\gamma > 0$ is an arbitrary constant. This is an intermediate regime between the homogeneous weights and the inhomogeneous weights considered earlier.
- The macroscopic limit shape and correlations in the bulk are the same as in the homogeneous case.
- Periodicity disappears in the limit and we have a $\mathbb{Z} \times \mathbb{Z}$ translation invariant ergodic Gibbs measure in the bulk. However, the local point process at turning points is different from the homogeneous one. In particular, while we only have one turning point near each edge, we still do not have the GUE minors process, but rather a one-parameter deformation of it.

Turning point correlations in the intermediate regime

Theorem (M.)

Let (τ, χ) be a turning point and let $t_i = \lfloor \frac{\tau}{r} \rfloor - \hat{t}_i$, and$h_i = \lfloor \frac{\chi}{r} \rfloor + \frac{\tilde{h}_i}{r^{\frac{1}{2}}}$. If $\lfloor \frac{\tau}{r} \rfloor$ is odd, then the correlation functions near a turning point (τ, χ) of the system with periodic weights (1) are given by

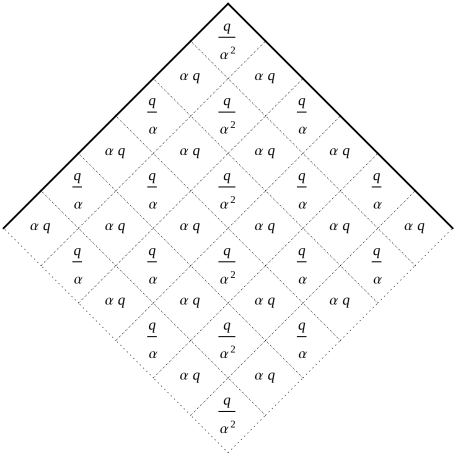
$$\lim_{r \to 0} r^{-\frac{1}{2}} K_{\lambda, \bar{q}}((t_1, h_1), (t_2, h_2))$$

$$= \frac{1}{(2\pi\mathrm{i})^2} \iint e^{\frac{S''_{\tau,\chi}(z_\tau, \chi)}{2}(\zeta^2 - \omega^2)} \frac{e^{\tilde{h}_2 \omega}}{e^{\tilde{h}_1 \zeta}} \frac{\omega^{\lfloor \frac{\hat{t}_2+1}{2} \rfloor}}{\zeta^{\lfloor \frac{\hat{t}_1+1}{2} \rfloor}} \frac{(\omega - \gamma)^{\lfloor \frac{\hat{t}_2+2}{2} \rfloor}}{(\zeta - \gamma)^{\lfloor \frac{\hat{t}_1+2}{2} \rfloor}} \frac{d\zeta \; d\omega}{\zeta - \omega}.$$

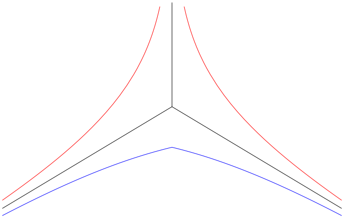
Almost periodic weights

Almost periodic weights, or introducing creases.

Almost periodic weights



Almost periodic: The frozen boundary



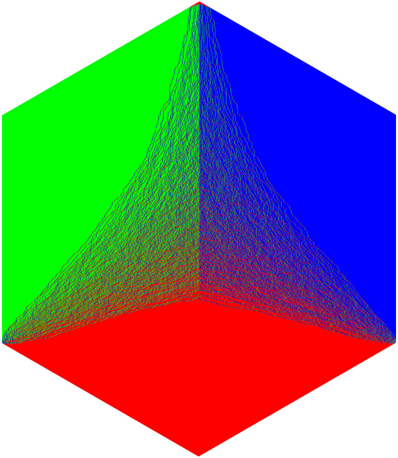
The frozen boundary is the union of the two curves

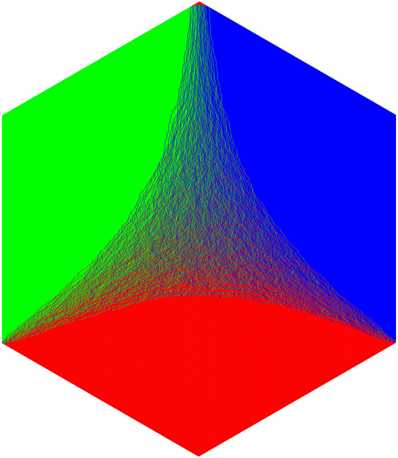
$$\chi(\tau) = -\ln\left(1 \pm e^{-\frac{|\tau|}{2}}\right) - \ln\left(1 \pm \alpha^{-1} e^{-\frac{|\tau|}{2}}\right) - \frac{1}{2}|\tau|.$$

The frozen boundary is not differentiable at $\tau = 0$. We have

$$\lim_{\tau \to 0^{\pm}} \frac{d\chi}{d\tau} = \pm \frac{1}{4}(\alpha^{-1} - 1).$$

Almost periodic: A sample





Other regimes

Other Regimes

Two-periodic done differently

Recall, that we got the following "deformation" of the incomplete beta kernel in the bulk for two-periodic weights:

$$K^{\alpha}_{\chi,\tau}(t_1, t_2, \Delta h) = \int_{\gamma} (1 - e^{-\tau}\alpha^{\frac{1}{2}} z)^{\frac{\Delta t + c}{2}} (1 - e^{-\tau}\alpha^{-\frac{1}{2}} z)^{\frac{\Delta t - c}{2}} z^{-\Delta h - \frac{\Delta t}{2}} \frac{dz}{2i\pi z}.$$

If you take not the weights but the parameters of the Schur-process to be two-periodic (considered by Dan Betea), you get a system which has this kernel in the bulk, but α depends on the macroscopic position in the bulk.

Arbitrary semi-frozen regions

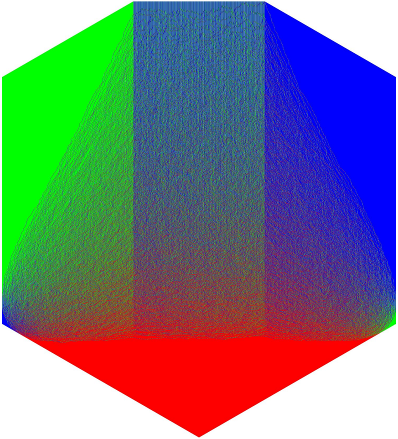
Conjecture

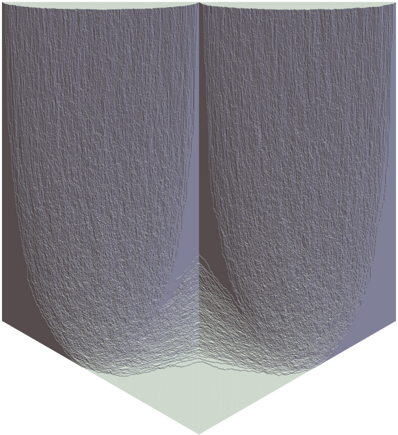
Given any $p \in \mathbb{N}$ and any sequence $(s_1, s_2, \ldots, s_p) \in \{-1, 1\}^p$, there exist (almost) p-periodic weights, such that skew-plane partitions develop a semi-frozen region with profile $(s_1, s_2, \ldots, s_p)$ in the scaling limit.

Some simulations

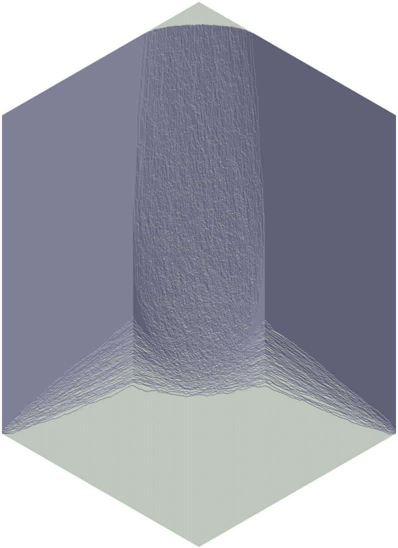
Some simulations

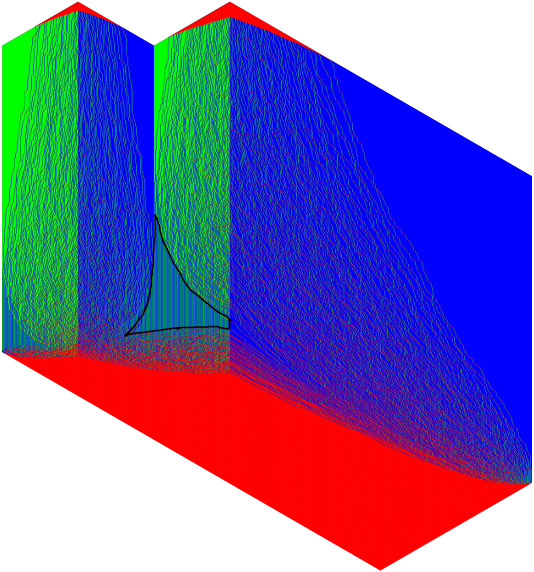
Trapezoid





Rocket





Thank you for your attention.