Integrability of Limit Shape Phenomena in Six Vertex Model

Ananth Sridhar

UC Berkeley Physics
asridhar@berkeley.edu

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- In the thermodynamic limit, the limiting average height function becomes deterministic and can be found by solving a certain boundary value problem.
- The six vertex model is quantum integrable in the sense that it admits commuting transfer matrices and can be solved by Bethe ansatz.
- What does the quantum integrability imply for the PDE governing the limiting height function?

Outline of Talk

- Quick Review of Six Vertex Model
- Thermodynamic Limit
- Integrability:
    - Transfer Matrices
    - Commuting Hamiltonians
- Examples
- Outlook

Configurations and Weights

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- Each vertex has a weight $v(s)$.
- The Boltzmann weight of s:

$$w(s) = \prod_{\text{vertex } v} v(s)$$

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- The partition function and the normalized free energy are:

$$Z^{\epsilon}_{\eta_1,\eta_2,T} = \sum_{\substack{s(0)=\eta_1 \\ s(1)=\eta_2}} w(s)$$

$$f^{\epsilon}_{\eta_1,\eta_2,T} = \epsilon^2 \log\left(Z_{\eta_1,\eta_2}\right)$$

Height Function

- A height function is a function on faces satisfying a gradient constraint:
  - $0 \leq h(x, y) - h(x + \epsilon, y) \leq 1$
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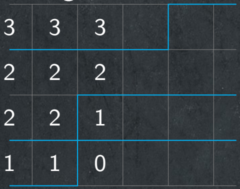

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- The normalized height function $\bar{h} = \epsilon h$. The average height function $\langle \bar{h} \rangle$ is the ensemble average of the normalized height function.

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- In this case, there exist limiting free energy and limiting height function:

$$f_{\eta_1,\eta_2,T} = \lim_{\epsilon\to 0} f^{\epsilon}_{\eta_1,\eta_2,T}$$

$$\langle h \rangle = \lim_{\epsilon\to 0} \langle h \rangle^{\epsilon}$$

Variational Principle

- The limiting free energy and average height function can be computed by variational principle.

$$f_{\eta_1,\eta_2,T} = \max_{h\in\mathcal{H}} \int_0^1 \int_0^T \sigma_w(\partial_t h, \partial_y h)\; dt\; dy$$

where σ is called the surface tension function, and $\mathcal{H}$ is the set of limiting height functions, $h : S_t \to \mathbb{R}$ satisfying: $h(0,0) = 0$, monotonicity, and Lipschitz continuity with constant 1.

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- Euler Lagrange equations:

$$\partial_{11}\sigma_w\, \partial_t^2 h + 2\, \partial_{12}\sigma_w\, \partial_t\partial_y h + \partial_{22}\sigma_w\, \partial_y^2 h = 0$$

Transfer Matrices

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- Define the transfer matrix $T_w : V \to V$ by its matrix elements:

$$\langle s_1 | T_w | s_2 \rangle = Z_{s_1, s_2, \epsilon}$$

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- Then:

$$Z_{\eta_1, \eta_2, t} = \langle \eta_1 | T_w^{\lfloor t/\epsilon \rfloor} | \eta_2 \rangle$$

Hamiltonian Formulation of Variational Principle

- Recast the variational problem in the Hamiltonian formulation by Legendre transform:

$$\mathcal{H}_w(\pi, t) = \max_s \pi s - \sigma_w(s, t)$$

The new variables are h and π, where π is conjugate to $\partial_t h$.

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$$f_{\eta_1,\eta_2,T} = \max_{\pi,h} S[\pi, h]$$

$$S[\pi, h] = \int_0^T \int_0^1 \pi\, \partial_t h - H_w\big(\pi, \partial_y h\big)\, dt\, dy$$

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These are equivalent to the Euler-Lagrange equations.

Commuting Transfer Matrices and Hamiltonians

- Recall $\Delta w = \frac{w_1^2 + w_2^2 - w_3^2}{2 w_1 w_2}$.
- Quantum Integrability: if w and $\widetilde{w}$ satisfy $\Delta w = \Delta \widetilde{w}$ then the transfer matrices commute:

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- Main result is semiclassical integrability: if $\Delta w = \Delta \widetilde{w}$ then the corresponding Hamiltonians Poisson commute:

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- Lemma 2: The Hessian of the surface tension σ_w of the six vertex model σ depends on w only via $\Delta(w)$.

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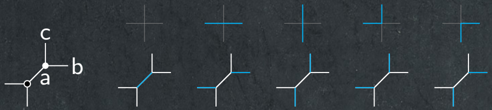
- Lemma: The hessian of σ is π^2, independent of weights.

Hexagonal Dimer Model

- The six vertex model with weights

$$w_1 = 0 \quad w_2 = a \quad w_3 = b \quad w_4 = c \quad w_5 = \sqrt{bc} \quad w_6 = \sqrt{bc}$$

Corresponds to the dimer model on the hexagonal lattice with edge weights (a, b, c).



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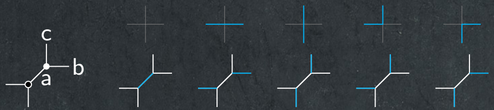

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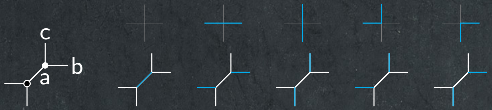

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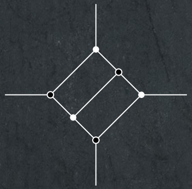
- The surface tension function σ can be calculated in closed form, and the Hamiltonians can be shown directly to commute.

Free Fermion Point

- More generally, when $\Delta w = 0$, the six vertex model is equivalent to the dimer model on the graph:



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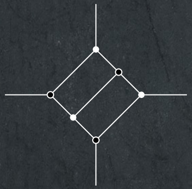

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- In the limit $\epsilon \to 0$, by large deviation principle:

$$f_{\eta_1,\eta_2,t,\widetilde{t}} = \max_{\eta} \; f_{\eta_1,\eta,t} + \widetilde{f}_{\eta,\eta_2,\widetilde{t}}$$

Generalities

- The commutation of the transfer matrices implies:

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- Recall that f is the Hamilton-Jacobi action.

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- Recall that f is the Hamilton-Jacobi action.
- Generally:
  If the Hamilton-Jacobi actions of H and $\widetilde{H}$ commute in the above sense, then does $\{H, \widetilde{H}\}$?
- Generally no, but under mild assumptions then yes,

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Integrability

- The existence of commuting transfer matrices underlies the solvability of the six vertex model by Bethe Ansatz.
- In the infinite dimensional setting, the Liouville integrability (the existence of many commuting Hamiltonians) is not enough to have the complete solvability.
- The existence of commuting hamiltonians is first step towards showing the integrability of the limit shape PDE.

End!