Appearance of determinants for stochastic growth models

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0. Free fermion and non free fermion models

From discussions yesterday after the talk by Sanjay Ramassamy

- Non free fermion models are more interesting than free fermion models.
- There are nontrivial aspects for free fermion models.
- Finding free fermion properties in apparently non free fermion models is interesting.

XXZ spin chain

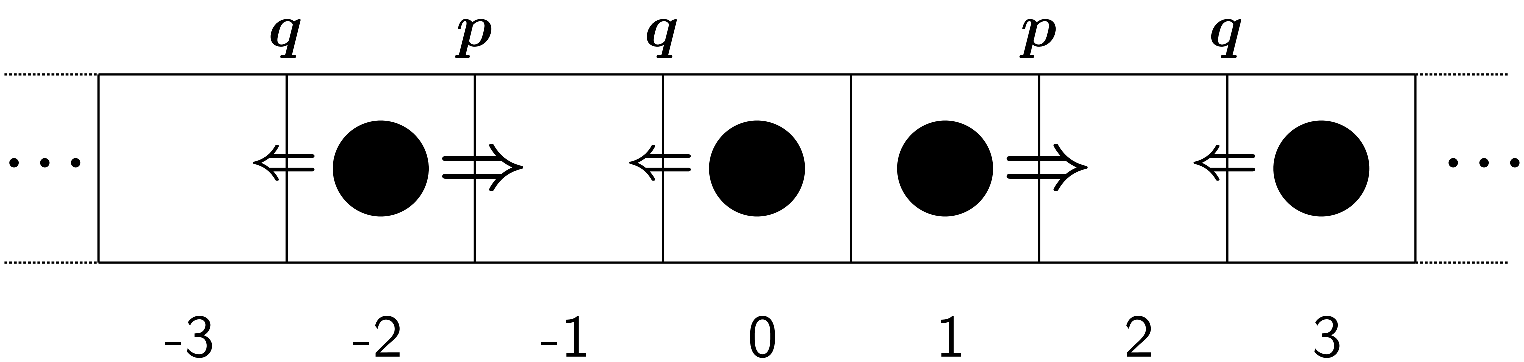
Hamiltonian for XXZ spin chain

$$H_{\mathrm{XXZ}} = \frac{1}{2}\sum_j [\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta(\sigma_j^z \sigma_{j+1}^z - 1)]$$

- It is well known that $\Delta = 0$ case becomes free fermion by Jordan-Wigner transformation. (An analogous statement applies also the six vertex model.)
- Usually $\Delta \neq 0$ case is not associated with free fermion.
- Jimbo et al found some free fermion like objects for $|\Delta| < 1$.

ASEP

ASEP (asymmetric simple exclusion process)



(Transpose) generator of ASEP

$$ {}^{t}L_{\mathrm{ASEP}} = \sum_{j} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -q & p & 0 \\ 0 & q & -p & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{j,j+1} $$

With $Q = \sqrt{q/p}$, $\Delta = (Q + Q^{-1})/2$ and

$$V = \prod_j Q^{jn_j}$$

where $n_j = \frac{1}{2}(1 - \sigma_j^z)$ they are related by

$$V\ {}^t L_{\mathrm{ASEP}} V^{-1}/\sqrt{pq} = H_{\mathrm{XXZ}}$$

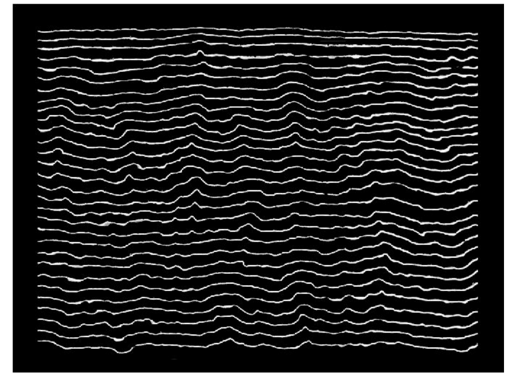
- ASEP is related by a similarity transformation to XXZ with $\Delta > 1$. (Ferromagnetic case. But note boundary conditions and different physical contexts.)
- TASEP ($p = 0$ or $q = 0$, $\Delta \to \infty$) on $\mathbb{Z}$ is not Ising but related to the Schur process (free fermion).
- For general ASEP (again on $\mathbb{Z}$) a generating function for the current can be written as a Fredholm determinant (Tracy).

Plan

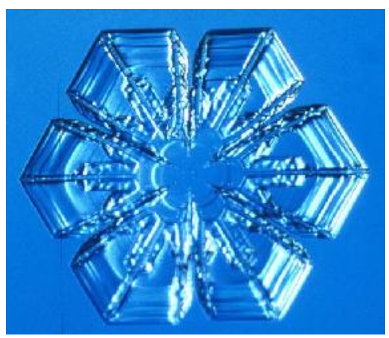
1. TASEP
2. Random matrix theory (and TASEP)
3. KPZ equation (and ASEP)
4. O'Connell-Yor polymer (with T. Imamura, arXiv:1506.05548)

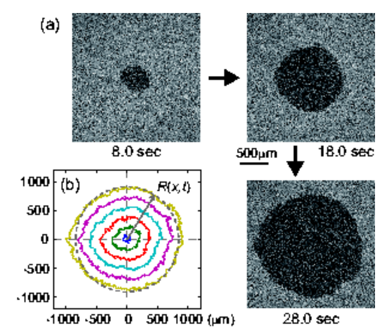
1. Surface growth and TASEP formula

- Paper combustion, bacteria colony, crystal growth, etc
- Non-equilibrium statistical mechanics
- Stochastic interacting particle systems



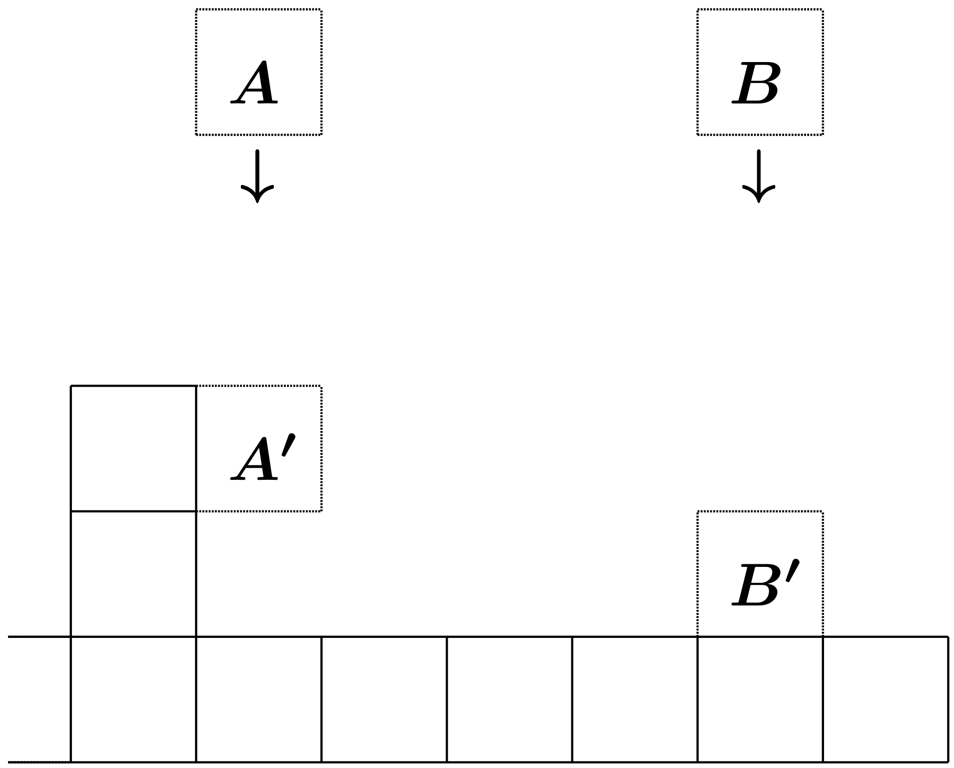
- Connections to integrable systems, representation theory, etc





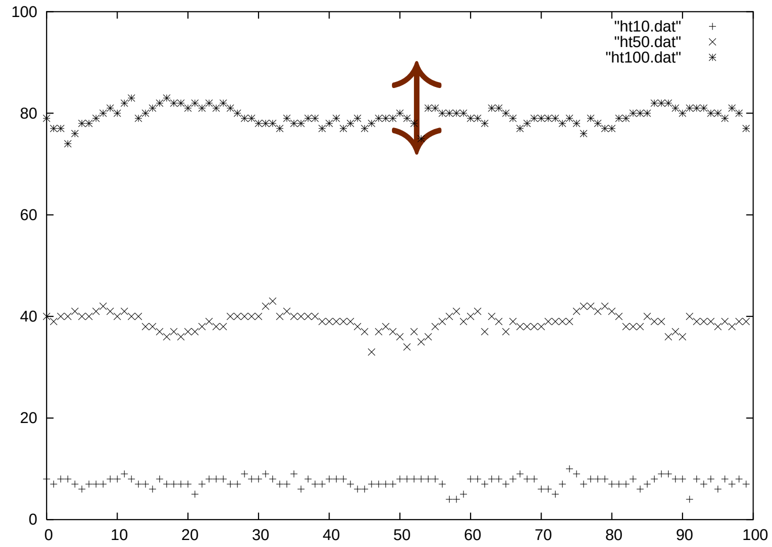
Simulation models

Ex: ballistic deposition



Height fluctuation

$$O(t^{\beta}), \beta = 1/3$$

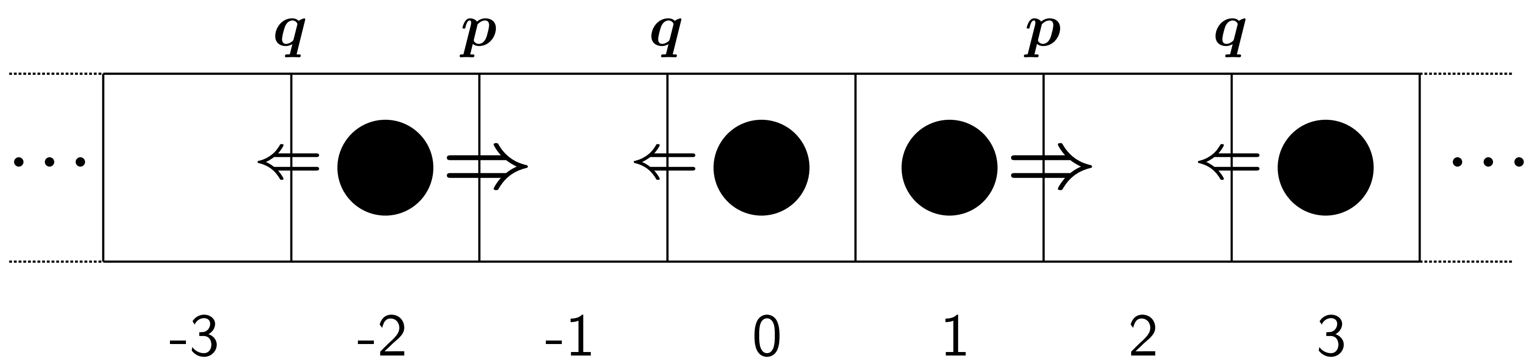


Flat

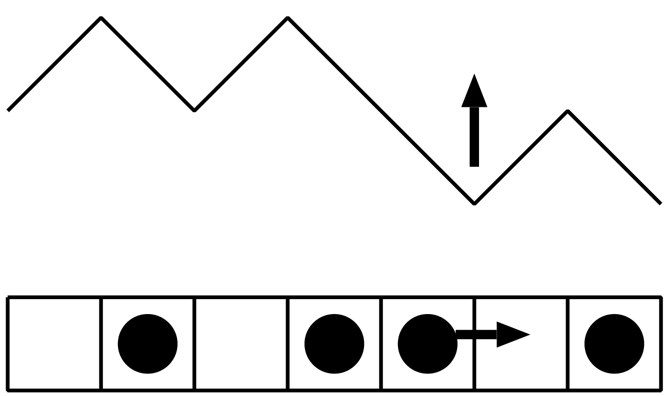
Universality: exponent and height distribution

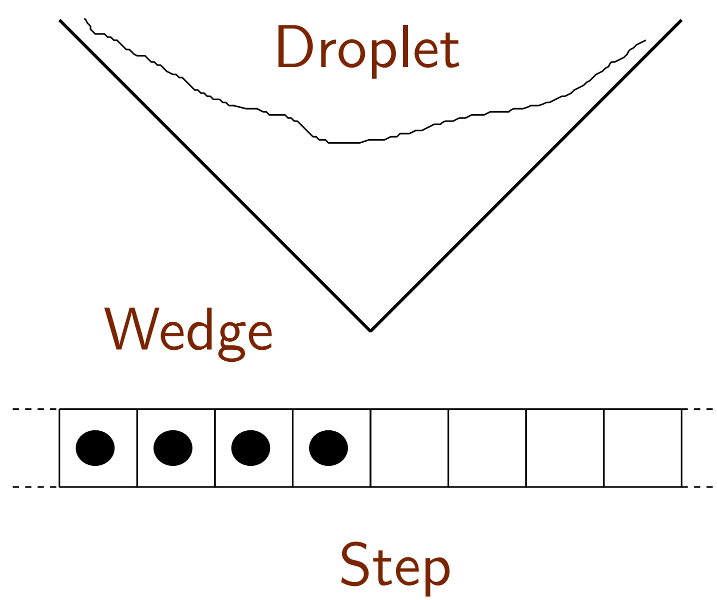
Totally ASEP ($q = 0$)

ASEP (asymmetric simple exclusion process)



Mapping to a surface growth model (single step model)





TASEP: LUE formula and Schur measure

- 2000 Johansson

  Formula for height (current) distribution for finite t (step i.c.)

$$\mathbb{P}\left[\frac{h(0,t)-t/4}{-2^{-4/3}t^{1/3}} \le s\right] = \frac{1}{Z}\int_{[0,s]^N} \prod_{i<j}(x_j - x_i)^2 \prod_i e^{-x_i} \prod_i dx_i$$

- The proof is based on Robinson-Schensted-Knuth (RSK) correspondence. For a discrete TASEP with parameters $a = (a_1, \cdots, a_N), b = (b_1, \cdots, b_M)$ associated with the Schur measure for a partition λ

$$\frac{1}{Z} s_\lambda(a) s_\lambda(b)$$

  The Schur function s_λ can be written as a single determinant (Jacobi-Trudi identity).

Long time limit: Tracy-Widom distribution

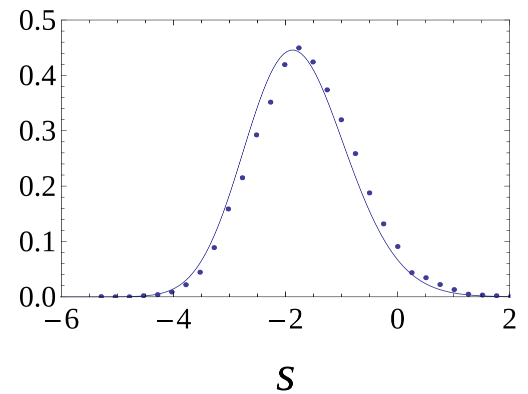
$$\lim_{t\to\infty} \mathbb{P}\left[\frac{h(0,t) - t/4}{-2^{-4/3}t^{1/3}} \leq s\right] = F_2(s)$$

where $F_2(s)$ is the GUE Tracy-Widom distribution

$$F_2(s) = \det(1 - P_s K_{\mathrm{Ai}} P_s)_{L^2(\mathbb{R})}$$

where P_s: projection onto the interval $[s, \infty)$
and K_{Ai} is the Airy kernel

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{d}\lambda \mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)$$



2. Random matrix theory (and TASEP)

GUE (Gaussian unitary ensemble): For a matrix H: $N \times N$ hermitian matrix

$$P(H)dH \propto e^{-\mathrm{Tr}H^2}dH$$

Each independent matrix element is independent Gaussian.

Joint eigenvalue density

$$\frac{1}{Z}\prod_{i<j}(x_j - x_i)^2 \prod_i e^{-x_i^2}$$

This is written in the form of a product of two determinants using

$$\prod_{i<j}(x_j - x_i) = \det(x_i^{j-1})_{i,j=1}^N$$

From this follows

- All m point correlation functions can be written as determinants using the "correlation kernel" $K(x, y)$.

- The largest eigenvalue distribution

$$\mathbb{P}\left[x_{\max} \leq s\right] = \frac{1}{Z} \int_{(-\infty, s]^N} \prod_{i<j} (x_j - x_i)^2 \prod_i e^{-x_i^2} \prod_i dx_i$$

can be written as a Fredholm determinant using the same kernel $K(x, y)$.

In the limit of large matrix dimension, we get

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{x_{\max} - \sqrt{2N}}{2^{-1/2}N^{-1/6}} \le s\right] = F_2(s) = \det(1 - P_s K_2 P_s)_{L^2(\mathbb{R})}$$

where P_s: projection onto $[s, \infty)$ and K_2 is the Airy kernel

$$K_2(x, y) = \int_0^\infty d\lambda \mathrm{Ai}(x + \lambda)\mathrm{Ai}(y + \lambda)$$

$F_2(s)$ is known as the **GUE Tracy-Widom distribution**

Determinantal process

- The point process whose correlation functions are written in the form of determinants are called a determinantal process.
- Eigenvalues of the GUE is determinantal.
- This is based on the fact that the joint eigenvalue density can be written as a product of two determinants. The Fredholm determinant expression for the largest eigenvalue comes also from this.
- Once we have a measure in the form of a product of two determinants, there is an associated determinantal process and the Fredholm determinant appears naturally.
- TASEP is associated with Schur measure, hence determinatal.

Dyson's Brownian motion

In GUE, one can replace the Gaussian random variables by Brownian motions. The eigenvalues are now stochastic process, satisfying SDE

$$dX_i = dB_i + \sum_{j \neq i} \frac{dt}{X_i - X_j}$$

known as the Dyson's Brownian motion.

Warren's Brownian motion in Gelfand-Tsetlin cone

Let $Y(t)$ be the Dyson's BM with m particles starting from the origin and let $X(t)$ be a process with $(m+1)$ components which are interlaced with those of Y, i.e.,

$$X_1(t) \leq Y_1(t) \leq X_2(t) \leq \ldots \leq Y_m(t) \leq X_{m+1}(t)$$

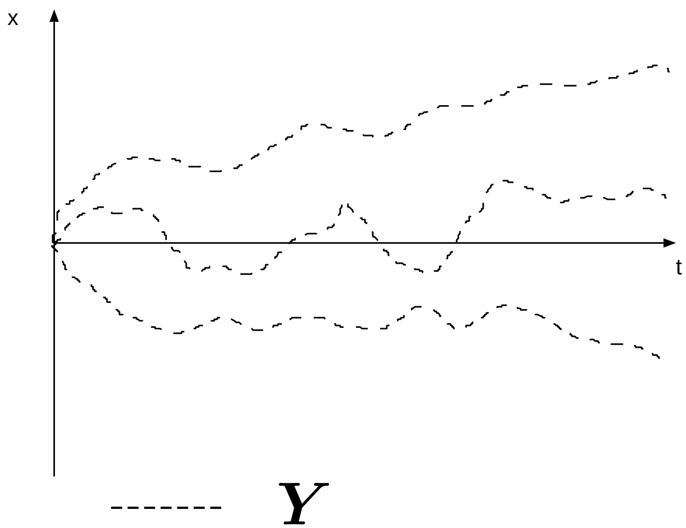
and satisfies

$$X_i(t) = x_i + \gamma_i(t) + \{L_i^-(t) - L_i^+(t)\}.$$

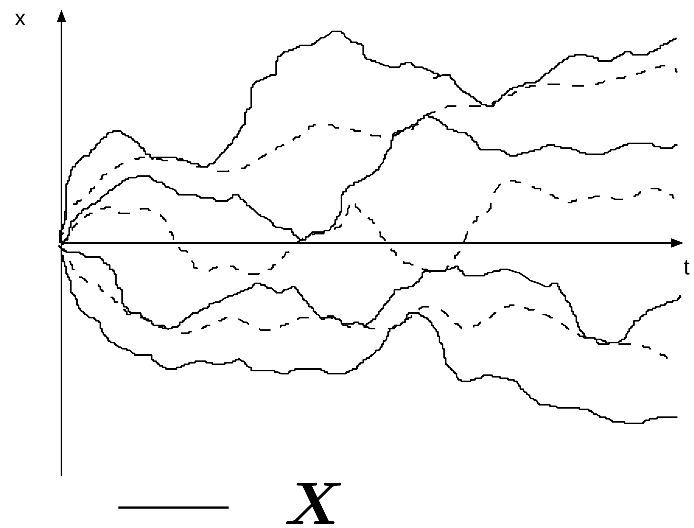
Here $\gamma_i, 1 \leq i \leq m$ are indep. BM and $L_i^{\pm}$ are local times.

Warren showed that the process X is distributed as a Dyson's BM with $(m+1)$ particles.

$m = 3$ Dyson BM



$m = 3, 4$ Dyson BM



Warren's Brownian motion in Gelfand-Tsetlin cone

- Repeating the same procedure for $m = 1, 2, \ldots, n-1$, one can construct a process $X_i^j, 1 \le j \le n, 1 \le i \le j$ in Gelfand-Tsetlin cone

- The marginal $X_i^i, 1 \le i \le n$ is the diffusion limit of TASEP (reflective BMs). One can understand how the random matrix expression for TASEP appears.

$$
\begin{array}{ccccccccc}
 & & & & x_1^1 & & & & \\
 & & & x_1^2 & & x_2^2 & & & \\
 & & x_1^3 & & x_2^3 & & x_3^3 & & \\
 & \iddots & & & \vdots & & & \ddots & \\
x_1^n & x_2^n & x_3^n & & \cdots & & & x_{n-1}^n & x_n^n
\end{array}
$$

3. KPZ equation

$h(x,t)$: height at position $x \in \mathbb{R}$ and at time $t \geq 0$

1986 Kardar Parisi Zhang

$$\partial_t h(x,t) = \frac{1}{2}\lambda(\partial_x h(x,t))^2 + \nu\partial_x^2 h(x,t) + \sqrt{D}\eta(x,t)$$

where η is the Gaussian noise with mean 0 and covariance

$$\langle \eta(x,t)\eta(x',t')\rangle = \delta(x-x')\delta(t-t')$$

By a simple scaling we can and will do set $\nu = \frac{1}{2}, \lambda = D = 1$.

The KPZ equation now looks like

$$\partial_t h(x,t) = \frac{1}{2}(\partial_x h(x,t))^2 + \frac{1}{2}\partial_x^2 h(x,t) + \eta(x,t)$$

If we set

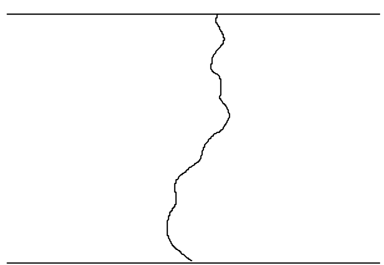
Cole-Hopf transformation

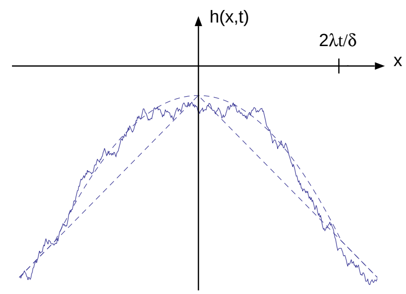
$$Z(x,t) = \exp\left(h(x,t)\right)$$

this quantity (formally) satisfies

$$\frac{\partial}{\partial t} Z(x,t) = \frac{1}{2}\frac{\partial^2 Z(x,t)}{\partial x^2} + \eta(x,t) Z(x,t)$$

This can be interpreted as a (random) partition function for a directed polymer in random environment η.





The polymer from the origin: $Z(x,0) = \delta(x) = \lim_{\delta\to 0} c_\delta e^{-|x|/\delta}$ corresponds to narrow wedge for KPZ.

The formula for KPZ equation

Thm (2010 TS Spohn, Amir Corwin Quastel)

For the initial condition $Z(x,0) = \delta(x)$ (narrow wedge for KPZ)

$$\mathbb{E}\left[e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}}\right] = \det(1 - K_{s,t})_{L^2(\mathbb{R}_+)}$$

where $\gamma_t = (t/2)^{1/3}$ and $K_{s,t}$ is

$$K_{s,t}(x,y) = \int_{-\infty}^{\infty} \mathrm{d}\lambda \frac{\mathrm{Ai}(x+\lambda)\mathrm{Ai}(y+\lambda)}{e^{\gamma_t(s-\lambda)}+1}$$

- As $t \to \infty$, one gets the Tracy-Widom distribution.
- The final result is written as a Fredholm determinant, but this was obtained without using a measure in the form of a product of two determinants

Derivation of the formula by replica approach

Dotsenko, Le Doussal, Calabrese

Feynmann-Kac expression for the partition function,

$$Z(x,t) = \mathbb{E}_x \left(e^{\int_0^t \eta(b(s),t-s)ds} Z(b(t),0) \right)$$

Because η is a Gaussian variable, one can take the average over the noise η to see that the replica partition function can be written as (for narrow wedge case)

$$\langle Z^N(x,t) \rangle = \langle x | e^{-H_N t} | 0 \rangle$$

where H_N is the Hamiltonian of the (attractive) δ-Bose gas,

$$H_N = -\frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} - \frac{1}{2} \sum_{j \neq k}^{N} \delta(x_j - x_k).$$

We are interested not only in the average $\langle h \rangle$ but the full distribution of h. We expand the quantity of our interest as

$$\langle e^{-e^{h(0,t)+\frac{t}{24}-\gamma_t s}} \rangle = \sum_{N=0}^{\infty} \frac{\left(-e^{-\gamma_t s}\right)^N}{N!} \left\langle Z^N(0,t) \right\rangle e^{N \frac{\gamma_t^3}{12}}$$

Using the integrability (Bethe ansatz) of the δ-Bose gas, one gets explicit expressions for the moment $\langle Z^n \rangle$ and see that the generating function can be written as a Fredholm determinant. But for the KPZ, $\langle Z^N \rangle \sim e^{N^3}$!

One should consider regularized discrete models.

ASEP and more

- One can find an analogous formula for ASEP by using (stochastic) duality ($\sim$ replica, related to $U_q(sl_2)$ symmetry).
- This can be proved rigorously (no problem about the moment divergence).
- This approach can be generalized to q-TASEP and further to the higher-spin stochastic vertex model (Corwin, Petrov next week).
- This is fairy computational. The Fredholm determinant appears by rearranging contributions from poles of Bethe wave functions.

4. O'Connell-Yor polymer

2001 O'Connell Yor

Semi-discrete directed polymer in random media

$B_i, 1 \le i \le N$: independent Brownian motions

Energy of the polymer π

$$E[\pi] = B_1(t_1) + B_2(t_1, t_2) + \cdots + B_N(t_{N-1}, t)$$

Partition function

$$Z_N(t) = \int_{0<t_1<\cdots<t_{N-1}<t} e^{\beta E[\pi]} dt_1 \cdots dt_{N-1}$$

$\beta = 1/k_B T$: inverse temperature

In a limit, this becomes the polymer related to KPZ equation.

Zero-temperature limit (free fermion)

In the $T \to 0$ (or $\beta \to \infty$) limit

$$f_N(t) := \lim_{\beta\to\infty} \log Z_N(t)/\beta = \max_{0<s_1<\cdots<s_{N-1}<t} E[\pi]$$

2001 Baryshnikov Connection to random matrix theory

$$\text{Prob}\,(f_N(1) \le s) = \int_{(-\infty,s]^N} \prod_{j=1}^{N} dx_j \cdot P_{\text{GUE}}(x_1,\cdots,x_N),$$

$$P_{\text{GUE}}(x_1,\cdots,x_N) = \prod_{j=1}^{N} \frac{e^{-x_j^2/2}}{j!\sqrt{2\pi}} \cdot \prod_{1\le j<k\le N} (x_k - x_j)^2$$

where $P_{\text{GUE}}(x_1,\cdots,x_N)$ is the probability density function of the eigenvalues in the Gaussian Unitary Ensemble (GUE)

Whittaker measure: non free fermion

O'Connell discovered that the OY polymer is related to the quantum version of the Toda lattice, with Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{N-1} e^{x_i - x_{i-1}}$$

and as a generalization of Schur measure appears a measure written as a product of the two Whittaker functions (which is the eigenfunction of the Toda Hamiltonian):

$$\frac{1}{Z} \Psi_0(\beta x_1, \cdots, \beta x_N) \Psi_\mu(\beta x_1, \cdots, \beta x_N)$$

A determinant formula for Ψ is not known.

From this connection one can find a formala

$$\text{Prob}\left(\frac{1}{\beta}\log Z_N(t) \le s\right) = \int_{(-\infty,s]^N} \prod_{j=1}^{N} dx_j \cdot m_t(x_1,\cdots,x_N)$$

where $m_t(x_1,\cdots,x_N)\prod_{j=1}^N dx_j$ is given by

$$m_t(x_1,\cdots,x_N) = \Psi_0(\beta x_1,\cdots,\beta x_N)$$
$$\times \int_{(i\mathbb{R})^N} d\lambda \cdot \Psi_{-\lambda}(\beta x_1,\cdots,\beta x_N) e^{\sum_{j=1}^N \lambda_j^2 t/2} s_N(\lambda)$$

where $s_N(\lambda)$ is the Sklyanin measure

$$s_N(\lambda) = \frac{1}{(2\pi i)^N N!}\prod_{i<j}\Gamma(\lambda_i - \lambda_j)$$

Doing asymptotics using this expression has not been possible.

Macdonald measure and Fredholm determinant formula

Borodin, Corwin (2011) introduced the Macdonald measure

$$\frac{1}{Z} P_\lambda(a) Q_\lambda(b)$$

Here $P_\lambda(a), Q_\lambda(b)$ are the Macdonald polynomials, which are also not known to be a determinant.

By using this, they found a formula for OY polymer

$$\mathbb{E}[e^{-\frac{e^{-\beta u} Z_N(t)}{\beta^{2(N-1)}}}] = \det(1+L)_{L^2(C_0)}$$

where the kernel $L(v, v'; t)$ is written as

$$\frac{1}{2\pi i} \int_{i\mathbb{R}+\delta} dw \frac{\pi/\beta}{\sin(v'-w)/\beta} \frac{w^N e^{w(t^2/2-u)}}{v'^N e^{v'(t^2/2-u)}} \frac{1}{w-v} \frac{\Gamma(1+v'/\beta)^N}{\Gamma(1+w/\beta)^N}$$

By using this expression, one can study asymptotics.

Our new formula for finite β

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t)$$

$$W(x_1, \cdots, x_N; t) = \prod_{j=1}^{N} \frac{1}{j!} \prod_{1\le j<k\le N} (x_k - x_j) \cdot \det\left(\psi_{k-1}(x_j; t)\right)$$

where $f_F(x) = 1/(e^{\beta x} + 1)$ is Fermi distribution function and

$$\psi_k(x; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dw e^{-iwx - w^2 t/2} \frac{(iw)^k}{\Gamma\left(1 + iw/\beta\right)^N}$$

A formula in terms of a determinantal measure W for finite temperature polymer.

From this one gets the Fredholm determinant by using standard techniques of random matrix theory and does asymptotics.

Proof of the formula

We start from a formula by O'Connell

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{(i\mathbb{R}-\epsilon)^N} \prod_{j=1}^{N} \frac{d\lambda_j}{\beta} e^{-u\lambda_j+\lambda_j^2 t/2} \Gamma\left(-\frac{\lambda_j}{\beta}\right)^N s_N\left(\frac{\lambda}{\beta}\right)$$

where $\epsilon > 0$.

This is a formula which is obtained by using Whittaker measure.

In this sense, we have not really found a determinant structure for the OY polymer itself.

There is a direct route from the above to the Fredholm determinant (2013 Borodin, Corwin, Remnik).

Here we generalize Warren's arguments.

An intermediate formula

$$\mathbb{E}\left(e^{-\frac{e^{-\beta u}Z_N(t)}{\beta^{2(N-1)}}}\right) = \int_{\mathbb{R}^N} \prod_{\ell=1}^{N} dx_\ell f_F(x_\ell - u) \cdot \det\left(F_{jk}(x_j;t)\right)_{j,k=1}^{N}$$

with $(0 < \epsilon < \beta)$

$$F_{jk}(x;t) = \int_{i\mathbb{R}-\epsilon} \frac{d\lambda}{2\pi i} \frac{e^{-\lambda x + \lambda^2 t/2}}{\Gamma\left(\frac{\lambda}{\beta}+1\right)^N} \left(\frac{\pi}{\beta}\cot\frac{\pi\lambda}{\beta}\right)^{j-1} \lambda^{k-1}$$

For a proof start from

$$\prod_{1\le i<j\le N} \sin(x_i - x_j) = \prod_{j=1}^{N} \sin^{N-1} x_j \cdot \prod_{1\le k<\ell\le N} (\cot x_\ell - \cot x_k)$$

$$= \prod_{j=1}^{N} \sin^{N-1} x_j \cdot \det\left(\cot^{\ell-1} x_k\right)_{k,\ell=1}^{N}$$

and use

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

$$\int_{-\infty}^{\infty} dx \frac{e^{ax}}{1+e^x} = \frac{\pi}{\sin \pi a} \quad \text{for } 0 < \text{Re } a < 1$$

Now it is sufficient to prove the relation

$$\int_{\mathbb{R}^N} \prod_{\ell=1}^{N} dt_\ell f_F(t_\ell - u) \cdot \det\left(F_{jk}(t_j; t)\right)_{j,k=1}^{N}$$

$$= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j f_F(x_j - u) \cdot W(x_1, \cdots, x_N; t).$$

A determinantal measure on $\mathbb{R}^{N(N+1)/2}$

For $\underline{x}_k := (x_i^{(j)}, 1 \le i \le j \le k) \in \mathbb{R}^{k(k+1)/2}$, we define a measure $R_u(\underline{x}_N; t) d\underline{x}_N$ with R_u given by

$$\prod_{\ell=1}^{N} \frac{1}{\ell!} \det\left(f_i(x_j^{(\ell)} - x_{i-1}^{(\ell-1)})\right)_{i,j=1}^{\ell} \cdot \det\left(F_{1i}(x_j^{(N)}; t)\right)_{i,j=1}^{N}$$

where $x_0^{(\ell-1)} = u$, $\underline{x}_N = \prod_{j=1}^{N} \prod_{i=1}^{j} dx_i^{(j)}$,

$$f_i(x) = \begin{cases} f_F(x) := 1/(e^{\beta x} + 1) & i = 1, \\ f_B(x) := 1/(e^{\beta x} - 1) & i \ge 2. \end{cases}$$

and $F_{1i}(x; t)$ is given by $F_{ji}(x; t)$ with $j = 1$ in the previous slide.

Two ways of integrations

$$\int_{\mathbb{R}^{N(N+1)/2}} d\underline{x}_N R_u(\underline{x}_N;t)$$

$$= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_1^{(j)} f_F\left(x_1^{(j)} - u\right) \cdot \det\left(F_{jk}\left(x_1^{(N-j+1)};t\right)\right)_{j,k=1}^{N}$$

$$\int_{\mathbb{R}^{N(N+1)/2}} d\underline{x}_N R_u(\underline{x}_N;t)$$

$$= \int_{\mathbb{R}^N} \prod_{j=1}^{N} dx_j^{(N)} f_F\left(x_j^{(N)} - u\right) \cdot W\left(x_1^{(N)}, \cdots, x_N^{(N)};t\right)$$

Lemma

1. For $\beta > 0$ and $a \in \mathbb{C}$ with $-\beta < \text{Re } a < 0$, we have

$$\int_{-\infty}^{\infty} e^{-ax} f_B(x) dx = \frac{\pi}{\beta} \cot \frac{\pi}{\beta} a.$$

2. Let $G_0(x) = f_F(x)$ and
$G_j(x) = \int_{-\infty}^{\infty} dy f_B(x-y) G_{j-1}(y), \; j = 1, 2, \cdots.$
Then we have for $m = 0, 1, 2, \cdots$

$$G_m(x) = f_F(x) \left(\frac{x^m}{m!} + p_{m-1}(x) \right),$$

where $p_{-1}(x) = 0$ and $p_k(x)(k = 0, 1, 2, \cdots)$ is some kth order polynomial.

Dynamics of X_i^N

The density for the positions of $X_i^N, 1 \leq i \leq N$ satisfies

$$\begin{aligned}
&\frac{\partial}{\partial t} W(x_1, \cdots, x_N; t) \\
&= \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} W(x_1, \cdots, x_N; t) \\
&\quad - \sum_{i=1}^{N} \left(\sum_{j \neq i} \frac{1}{x_i - x_j} \right) \frac{\partial}{\partial x_i} W(x_1, \cdots, x_N; t)
\end{aligned}$$

which is the equation for the Dyson's Brownian motion.

Dynamics of X_i^i's

The transition density of X_i^i's

$$G(x_1, \cdots, x_N; t) = \det \left(F_{jk}\left(x_k; t\right)\right)_{j,k=1}^{N}$$

satisfy

$$\frac{\partial}{\partial t} G(x_1, \cdots, x_N; t) = \frac{1}{2} \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \cdot G(x_1, \cdots, x_N; t)$$

$$-\frac{\beta^2}{\pi^2} \int_{-\infty}^{\infty} dx_{j+1} \frac{e^{-\frac{\beta}{2}(x_{j+1}-x_j)}}{e^{\beta(x_{j+1}-x_j)} - 1} G(x_1, \cdots, x_N; t) = 0$$

As $\beta \to \infty$, the latter becomes

$$\partial_{x_i} G(x_1, \cdots, x_N; t)|_{x_{i+1}=x_i+0} = 0$$

which represents reflective interaction like TASEP.

Summary

- We have seen how the determinantal ($\sim$ free fermonic) structures in stochastic growth models. The point is "whether a product of two determinants appear and if so how".
- The generator of TASEP does not become a free fermion by Jordan-Wigern transformation. But it is associated with the Schur measure (a product of determinants) and hence determinantal.
- For ASEP and KPZ equation, one can find a Fredholm determinant formula by duality (or replica).

- The finite temperature O'Connell-Yor polymer is associated with the Whittaker measure (not a product of determinants) but we have given a formula using a measure in the form of a product of determinants.
- The proof is by generalizing Warren's process on Gelfand-Tsetlin cone. There are interesting generalizations of Dyson's Brownian motion and reflective Brownian motions.
- We started from a formula which is obtained from Whittaker measure. In this sense we have not found a determinantal structure for the OY polymer model itself. We should try to find a better understanding.