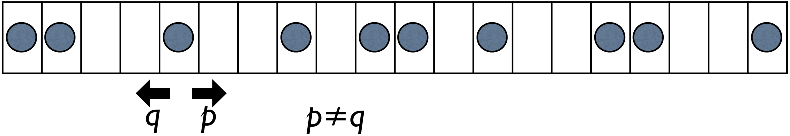
Asymmetric Simple Exclusion Process: Bethe Ansatz & Limit Theorems[1]

Craig A. Tracy (UC Davis) & Harold Widom (UC Santa Cruz)

Random Interfaces and Integrable Probability
GGI, Florence
June 22–26, 2015

[1]Work done between 2006–2008.

ASEP on Integer Lattice



- Each particle has an alarm clock -- exponential distribution with parameter one
- When alarm rings particle jumps to right with probability p and to the left with probability q
- Jumps are suppressed if neighbor is occupied

The short explanation of why Bethe Ansatz

- The generator L of the Markov process ASEP is a *similarity* (not unitary!) transformation of the XXZ quantum spin system.
- This observation goes back at least to Gwa & Spohn (1992).
- Apply Bethe Ansatz to L

$P_Y(X; t)$ for N-particle ASEP

- $X \in \mathbb{Z}^N$

$$X_i^{\pm} = \{x_1, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_N\}$$

The "free equation" on $\mathbb{Z}^N \times \mathbb{R}$ is

$$\frac{du}{dt}(X) = \sum_{i=1}^{N} \left(pu(X_i^-; t) + qu(X_i^+; t) - u(X; t) \right)$$

$P_Y(X;t)$ for N-particle ASEP

- $X \in \mathbb{Z}^N$
$$X_i^{\pm} = \{x_1, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_N\}$$
The "free equation" on $\mathbb{Z}^N \times \mathbb{R}$ is
$$\frac{du}{dt}(X) = \sum_{i=1}^{N} \left(pu(X_i^-;t) + qu(X_i^+;t) - u(X;t) \right)$$

- The boundary conditions are
$$pu(x_1, \ldots, x_i, x_i, \ldots, x_N; t) + qu(x_1, \ldots, x_i + 1, x_i + 1, \ldots, x_N)$$
$$= u(x_1, \ldots, x_i, x_i + 1, \ldots, x_N, \quad i = 1, 2, \ldots, N-1$$
This boundary condition comes when particle at x_i is neighbor to particle at $x_{i+1} = x_i + 1$

$P_Y(X;t)$ for N-particle ASEP

- $X \in \mathbb{Z}^N$
$$X_i^{\pm} = \{x_1, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_N\}$$
The "free equation" on $\mathbb{Z}^N \times \mathbb{R}$ is
$$\frac{du}{dt}(X) = \sum_{i=1}^{N} \left(pu(X_i^-;t) + qu(X_i^+;t) - u(X;t)\right)$$

- The boundary conditions are
$$pu(x_1, \ldots, x_i, x_i, \ldots, x_N; t) + qu(x_1, \ldots, x_i + 1, x_i + 1, \ldots, x_N) \\ = u(x_1, \ldots, x_i, x_i + 1, \ldots, x_N, \;\; i = 1, 2, \ldots, N-1$$
This boundary condition comes when particle at x_i is neighbor to particle at $x_{i+1} = x_i + 1$

- Check that no new boundary conditions are needed, e.g. when 3 or more particles are all adjacent.

$P_Y(X;t)$ for N-particle ASEP

- $X \in \mathbb{Z}^N$
$$X_i^{\pm} = \{x_1, \ldots, x_{i-1}, x_i \pm 1, x_{i+1}, \ldots, x_N\}$$
The "free equation" on $\mathbb{Z}^N \times \mathbb{R}$ is
$$\frac{du}{dt}(X) = \sum_{i=1}^{N} \left(pu(X_i^-;t) + qu(X_i^+;t) - u(X;t)\right)$$
- The boundary conditions are
$$pu(x_1, \ldots, x_i, x_i, \ldots, x_N; t) + qu(x_1, \ldots, x_i + 1, x_i + 1, \ldots, x_N) \\ = u(x_1, \ldots, x_i, x_i + 1, \ldots, x_N, \;\; i = 1, 2, \ldots, N-1$$
This boundary condition comes when particle at x_i is neighbor to particle at $x_{i+1} = x_i + 1$
- Check that no new boundary conditions are needed, e.g. when 3 or more particles are all adjacent.
- Require initial condition $u(X;0) = \delta_{X,Y}$ in physical region.

- Look for solutions of the form (Bethe's second idea)

$$u(X;t) = \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \sum_{\sigma \in \mathbb{S}_N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} \; e^{t \sum_i \varepsilon(\xi_i)} \; d\xi_1 \cdots d\xi_N$$

$\mathbb{S}_N$ is the permutation group.

- Look for solutions of the form (Bethe's second idea)

$$u(X;t) = \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \sum_{\sigma \in \mathbb{S}_N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} \, \mathrm{e}^{t \sum_i \varepsilon(\xi_i)} \, d\xi_1 \cdots d\xi_N$$

$\mathbb{S}_N$ is the permutation group.

- Find boundary conditions are satisfied if the A_σ satisfy

$$A_\sigma(\xi) = \prod \{S(\xi_\beta, \xi_\alpha) : \{\beta, \alpha\} \text{ is an inversion in } \sigma\}$$

The inversions in $\sigma = (3,1,4,2)$ are $\{3,1\}$, $\{3,2\}$, $\{4,2\}$. Thus $A_{\mathrm{id}} = 1$.

- Look for solutions of the form (Bethe's second idea)

$$u(X;t) = \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \sum_{\sigma \in \mathbb{S}_N} A_\sigma(\xi) \prod_i \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1} \; \mathrm{e}^{t \sum_i \varepsilon(\xi_i)} \; d\xi_1 \cdots d\xi_N$$

  $\mathbb{S}_N$ is the permutation group.

- Find boundary conditions are satisfied if the A_σ satisfy

$$A_\sigma(\xi) = \prod \{ S(\xi_\beta, \xi_\alpha) : \{\beta, \alpha\} \text{ is an inversion in } \sigma \}$$

  The inversions in $\sigma = (3,1,4,2)$ are $\{3,1\}$, $\{3,2\}$, $\{4,2\}$. Thus $A_{\mathrm{id}} = 1$.

- **Final step: Show $u(X;t)$ satisfies the initial condition.** As before, the term corresponding to the identity permutation gives $\delta_{X,Y}$. We must show the sum of the $N! - 1$ other terms sum to zero in the physical region! This turns out to be quite involved. It will be the case if r is chosen so that all singularities coming from the A_σ lie *outside* the contour $\mathcal{C}_r$ (we assume $p \neq 0$). Our original article had an error. See the erratum.

Let $\chi_N = \{(x_1, \ldots, x_N) \in \mathbb{Z}^N : x_1 < \cdots < x_N\}$

Semigroup $\mathrm{e}^{\mathbf{tL}}$:

$$\mathrm{e}^{tL}\, e_Y = \sum_{X \in \chi_N} P_Y(X; t) e_X, \quad Y \in \chi_N,$$

$$P_Y(X;t) = \int_{\mathcal{C}_r} \cdots \int_{\mathcal{C}_r} \sum_{\sigma \in \mathbb{S}_N} A_\sigma(\xi) \prod_{i=1}^{N} \xi_{\sigma(i)}^{x_i - y_{\sigma(i)} - 1}\ \mathrm{e}^{t \sum_i \varepsilon(\xi_i)}\ d\xi_1 \cdots d\xi_N$$

$\mathbb{S}_N$ is the permutation group, $\varepsilon(\xi) = p/\xi + q\xi - 1$, and

$$A_\sigma(\xi) = \prod \{S(\xi_\beta, \xi_\alpha) : \{\beta, \alpha\} \text{ is an inversion in } \sigma\}$$

$$S(\xi, \xi') = -\frac{p + q\xi\xi' - \xi}{p + q\xi\xi' - \xi'}$$

Choose $r \ll 1$ so that all poles from A_σ lie outside of $\mathcal{C}_r$.
Each $d\xi$ carries a factor $(2\pi i)^{-1}$.

Alternative form for A_σ

Set

$$f(\xi, \xi') = p + q\xi\xi' - \xi$$

then

$$A_\sigma = \operatorname{sgn}(\sigma) \frac{\prod_{i<j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)})}{\prod_{i<j} f(\xi_i, \xi_j)}$$

Alternative form for A_σ

Set

$$f(\xi, \xi') = p + q\xi\xi' - \xi$$

then

$$A_\sigma = \mathrm{sgn}(\sigma)\, \frac{\prod_{i<j} f(\xi_{\sigma(i)}, \xi_{\sigma(j)})}{\prod_{i<j} f(\xi_i, \xi_j)}$$

Marginal Distribution for $x_1(t)$

Take $p \neq 0$

$$\mathbb{P}(x_1(t) = x) : \quad = \sum_{x<x_2<\cdots<x_N} P_Y(\{x, x_2, \ldots, x_N\}; t)$$

$$= \int_{\mathcal{C}_r^N} \sum_{\sigma \in \mathbb{S}_N} A_\sigma(\xi) \frac{\xi_{\sigma(2)}\xi_{\sigma(3)}^2 \cdots \xi_{\sigma(N)}^{N-1}}{(1-\xi_{\sigma(2)}\cdots\xi_{\sigma(N)})(1-\xi_{\sigma(3)}\cdots\xi_{\sigma(N)})\cdots(1-\xi_{\sigma(N)})} \times$$

$$\prod_i \xi_i^{x-y_i-1}\, \mathrm{e}^{t\sum_i \varepsilon(\xi_i)}\; d\xi_1 \cdots d\xi_N$$

First Combinatorial Identity

$$\sum_{\sigma\in\mathbb{S}_N}\mathrm{sgn}(\sigma)\left\{\frac{\left(\prod_{i<j}f(\xi_{\sigma(i)},\xi_{\sigma(j)})\right)\ \xi_{\sigma(2)}\xi_{\sigma(3)}^2\cdots\xi_{\sigma(N)}^{N-1}}{(1-\xi_{\sigma(1)}\cdots\xi_{\sigma(N)})\cdots(1-\xi_{\sigma(N-1)}\xi_{\sigma(N)})(1-\xi_{\sigma(N)})}\right\}$$

$$=p^{N(N-1)/2}\frac{\prod_{i<j}(\xi_j-\xi_i)}{\prod_j(1-\xi_j)}$$

Thus we have, $p\neq 0$,

$$\mathbb{P}(x_1(t)=x)=$$

$$p^{N(N-1)/2}\int_{\mathcal{C}_r^N}\prod_{i<j}\frac{\xi_j-\xi_i}{f(\xi_i,\xi_j)}\,\frac{1-\xi_1\cdots\xi_N}{\prod_i(1-\xi_i)}\,\prod_i\left(\xi_i^{x-y_i-1}\,\mathrm{e}^{t\varepsilon(\xi_i)}\right)\,d\xi_1\cdots d\xi_N$$

$\longrightarrow$ A single N-dimensional integral!

Back Story to Proof of Identity

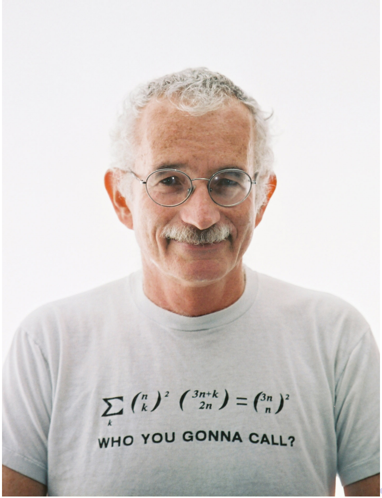
- Led to conjecture identity from special cases discovered using MATHEMATICA.

Back Story to Proof of Identity

- Led to conjecture identity from special cases discovered using MATHEMATICA.
- But how to prove identity?

Back Story to Proof of Identity

- Led to conjecture identity from special cases discovered using MATHEMATICA.
- But how to prove identity?



So we called Doron . . .

He suggested that Problem VII.47 of Pólya & Szegö, an identity of Issai Schur, had a similar look about it and might be proved in a similar way. Doron was right.

So we called Doron . . .

He suggested that Problem VII.47 of Pólya & Szegö, an identity of Issai Schur, had a similar look about it and might be proved in a similar way. Doron was right.

Idea of proof:

- Use induction on N

So we called Doron ...

He suggested that Problem VII.47 of Pólya & Szegö, an identity of Issai Schur, had a similar look about it and might be proved in a similar way. Doron was right.

Idea of proof:

- Use induction on N
- Call left-hand side $\varphi_N(\xi_1, \ldots, \xi_N)$ and sum over all permutations such that $\sigma(1) = k$. This gives an expression involving φ_{N-1}.

So we called Doron . . .

He suggested that Problem VII.47 of Pólya & Szegö, an identity of Issai Schur, had a similar look about it and might be proved in a similar way. Doron was right.

Idea of proof:

- Use induction on N
- Call left-hand side $\varphi_N(\xi_1, \ldots, \xi_N)$ and sum over all permutations such that $\sigma(1) = k$. This gives an expression involving φ_{N-1}.
- Use induction hypothesis to get a simpler identity to prove.

So we called Doron . . .

He suggested that Problem VII.47 of Pólya & Szegö, an identity of Issai Schur, had a similar look about it and might be proved in a similar way. Doron was right.

Idea of proof:

- Use induction on N
- Call left-hand side $\varphi_N(\xi_1, \ldots, \xi_N)$ and sum over all permutations such that $\sigma(1) = k$. This gives an expression involving φ_{N-1}.
- Use induction hypothesis to get a simpler identity to prove.
- Construct a function $f = f(z)$ so (1) integral over a large circle gives zero and (2) sum of residues of poles enclosed by the contour give the identity.

So we called Doron ...

He suggested that Problem VII.47 of Pólya & Szegö, an identity of Issai Schur, had a similar look about it and might be proved in a similar way. Doron was right.

Idea of proof:

- Use induction on N
- Call left-hand side $\varphi_N(\xi_1, \ldots, \xi_N)$ and sum over all permutations such that $\sigma(1) = k$. This gives an expression involving φ_{N-1}.
- Use induction hypothesis to get a simpler identity to prove.
- Construct a function $f = f(z)$ so (1) integral over a large circle gives zero and (2) sum of residues of poles enclosed by the contour give the identity.

Nice simplification but how do we take $N \to \infty$?

Large Contour Expansion

- Expand the contours C_r to large contours $\mathcal{C}_R$, $R \gg 1$ in order to let $N \to \infty$.

Large Contour Expansion

- Expand the contours C_r to large contours $\mathcal{C}_R$, $R \gg 1$ in order to let $N \to \infty$.
- In deforming the contours to large contours we encounter two types of poles:

Large Contour Expansion

- Expand the contours C_r to large contours $\mathcal{C}_R$, $R \gg 1$ in order to let $N \to \infty$.
- In deforming the contours to large contours we encounter two types of poles:
  - Poles coming from zeros of denominators at $\xi_k = 1$.
  - Poles coming from zeros of $f(\xi_i, \xi_j)$.
  - Remarkably the residues from the poles of the second type are zero.
  - Can then let $N \to \infty$, $Y = \{y_1, y_2, \ldots, \}$, $y_1 < y_2 < \cdots \longrightarrow +\infty$

Large Contour Expansion

- Expand the contours C_r to large contours $\mathcal{C}_R$, $R \gg 1$ in order to let $N \to \infty$.
- In deforming the contours to large contours we encounter two types of poles:
  - Poles coming from zeros of denominators at $\xi_k = 1$.
  - Poles coming from zeros of $f(\xi_i, \xi_j)$.
  - Remarkably the residues from the poles of the second type are zero.
  - Can then let $N \to \infty$, $Y = \{y_1, y_2, \ldots,\}$, $y_1 < y_2 < \cdots \longrightarrow +\infty$
- $$\mathbb{P}_Y(x_1(t) = x) = \sum_S \frac{p^{\sigma(S)-|S|}}{q^{\sigma(S)-|S|(|S|+1)/2}} \int_{\mathcal{C}_R^{|S|}} I(x, Y_S, \xi)\, d^{|S|}\xi$$

  where all the poles of the integrand lie *inside* $\mathcal{C}_R$. The sum runs over all nonempty, finite subsets S of $\mathbb{Z}^+$. Here $\sigma(S) = \sum_{i \in S} i$

$$I(x, Y, \xi) = \prod_{i<j} \frac{\xi_j - \xi_i}{f(\xi_i, \xi_j)} \frac{1 - \xi_1 \cdots \xi_N}{(1-\xi_1)\cdots(1-\xi_N)} \prod_i \left(\xi_i^{x-y_i-1}\, e^{t\varepsilon(\xi_i)}\right)$$

Position of the mth particle

- Want marginal distribution

$$\mathbb{P}_Y\left(x_m(t) = x\right)$$

where $x_m(t)$ is the position of the mth particle from the left at time t.

Position of the mth particle

- Want marginal distribution
$$\mathbb{P}_Y\left(x_m(t) = x\right)$$
where $x_m(t)$ is the position of the mth particle from the left at time t.
- For step initial condition, $Y = \mathbb{Z}^+$, the distribution of the "current"
$$\mathcal{I}(x,t) = \#\ \text{of particles}\ \ \leq x\ \ \text{at time}\ \ t$$
is related to the position of the mth particle by
$$\mathbb{P}_{\mathbb{Z}^+}\left(\mathcal{I}(x,t) \leq m\right) = 1 - \mathbb{P}_{\mathbb{Z}^+}\left(x_{m+1}(t) \leq x\right)$$
and the current fluctuations can be related to the height fluctuations.

Position of the mth particle

- Want marginal distribution
$$\mathbb{P}_Y\left(x_m(t) = x\right)$$
where $x_m(t)$ is the position of the mth particle from the left at time t.
- For step initial condition, $Y = \mathbb{Z}^+$, the distribution of the "current"
$$\mathcal{I}(x,t) = \#\ \text{of particles}\ \ \leq x\ \ \text{at time}\ \ t$$
is related to the position of the mth particle by
$$\mathbb{P}_{\mathbb{Z}^+}\left(\mathcal{I}(x,t) \leq m\right) = 1 - \mathbb{P}_{\mathbb{Z}^+}\left(x_{m+1}(t) \leq x\right)$$
and the current fluctuations can be related to the height fluctuations.
- $$\mathbb{P}_Y(x_m(t) = x) = \sum_{x_1 < \cdots < x_{m-1} < x < x_{m+1} < \cdots < x_N} P_Y(X;t)$$
Problem with doing this sum—need combination of small contours and large contours.

Position of the mth particle

- Want marginal distribution
$$\mathbb{P}_Y\left(x_m(t)=x\right)$$
where $x_m(t)$ is the position of the mth particle from the left at time t.
- For step initial condition, $Y=\mathbb{Z}^+$, the distribution of the "current"
$$\mathcal{I}(x,t)=\#\ \text{of particles}\ \ \leq x\ \ \text{at time}\ \ t$$
is related to the position of the mth particle by
$$\mathbb{P}_{\mathbb{Z}^+}\left(\mathcal{I}(x,t)\leq m\right)=1-\mathbb{P}_{\mathbb{Z}^+}\left(x_{m+1}(t)\leq x\right)$$
and the current fluctuations can be related to the height fluctuations.
- $$\mathbb{P}_Y(x_m(t)=x)=\sum_{x_1<\cdots<x_{m-1}<x<x_{m+1}<\cdots<x_N}P_Y(X;t)$$
Problem with doing this sum—need combination of small contours and large contours.
- This requires new combinatorial identities

Combinatorial Identity #2

$$[N] = \frac{p^N - q^N}{p - q}, \quad [N]! = [N][N-1]\cdots[1], \quad \begin{bmatrix} N \\ m \end{bmatrix} = \frac{[N]!}{[m]![N-m]!}$$

Identity:

$$\sum_{|S|=m} \prod_{\substack{i \in S \\ j \in S^c}} \frac{f(\xi_i, \xi_j)}{\xi_j - \xi_i} \left(1 - \prod_{j \in S^c} \xi_j \right) = q^m \begin{bmatrix} N-1 \\ m \end{bmatrix} \left(1 - \prod_{j=1}^{N} \xi_j \right)$$

The sum runs over all subsets of $\{1, \ldots, N\}$ with cardinality m and S^c denotes the complement of S in $\{1, \ldots, N\}$.

Case of Step Initial Condition $Y = \mathbb{Z}^+$

$$\mathbb{P}_{\mathbb{Z}^+}(x_m(t) \le x) = (-1)^{m+1} q^{m(m-1)/2} \times \sum_{k \ge m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix} p^{(k-m)(k-m+1)/2} q^{k(k+1)/2} \times \int_{\mathcal{C}_R^k} J_k(x,\xi)\, d\xi_1 \cdots d\xi_k$$

where

$$J_k(x,\xi) = \prod_{i \ne j} \frac{\xi_j - \xi_i}{f(\xi_i,\xi_j)} \frac{1}{\prod_i (1-\xi_i)(q\xi_i - p)} \prod_i \xi_i^{x-1} \, e^{t\varepsilon(\xi_i)}$$

Have a somewhat more complicated formula for arbitrary initial Y.

Case of Step Initial Condition $Y = \mathbb{Z}^+$

$$\mathbb{P}_{\mathbb{Z}^+}(x_m(t) \le x) = (-1)^{m+1} q^{m(m-1)/2} \times \sum_{k \ge m} \frac{1}{k!} \begin{bmatrix} k-1 \\ k-m \end{bmatrix} p^{(k-m)(k-m+1)/2} q^{k(k+1)/2} \times \int_{\mathcal{C}_R^k} J_k(x,\xi)\, d\xi_1 \cdots d\xi_k$$

where

$$J_k(x,\xi) = \prod_{i \ne j} \frac{\xi_j - \xi_i}{f(\xi_i, \xi_j)} \frac{1}{\prod_i (1-\xi_i)(q\xi_i - p)} \prod_i \xi_i^{x-1} \, e^{t\varepsilon(\xi_i)}$$

Have a somewhat more complicated formula for arbitrary initial Y.

But how does one analyze this for large t?

- In the integrand for J_k

$$\prod_{i \neq j} \frac{\xi_j - \xi_i}{f(\xi_i, \xi_j)} = \det\left(\frac{1}{f(\xi_i, \xi_j)}\right) \prod_i w(\xi_i)$$

- In the integrand for J_k

$$\prod_{i\neq j} \frac{\xi_j - \xi_i}{f(\xi_i, \xi_j)} = \det\left(\frac{1}{f(\xi_i, \xi_j)}\right) \prod_i w(\xi_i)$$

- Thus recognize integral as a coefficient in the Fredholm expansion of $\det(I - \lambda K)$ where K is an integral operator acting on $L^2(\mathcal{C}_R)$

$$K(\xi, \xi') = \frac{\xi^x e^{t\varepsilon(\xi)}}{f(\xi, \xi')}$$

- In the integrand for J_k

$$\prod_{i\neq j}\frac{\xi_j-\xi_i}{f(\xi_i,\xi_j)}=\det\left(\frac{1}{f(\xi_i,\xi_j)}\right)\prod_i w(\xi_i)$$

- Thus recognize integral as a coefficient in the Fredholm expansion of $\det(I-\lambda K)$ where K is an integral operator acting on $L^2(\mathcal{C}_R)$

$$K(\xi,\xi')=\frac{\xi^x e^{t\varepsilon(\xi)}}{f(\xi,\xi')}$$

- Can do sum over k to get

$$\mathbb{P}_{\mathbb{Z}^+}(x_m(t)\le x)=\int_{\mathcal{C}}\frac{\det(I-\lambda K)}{(\lambda;\tau)_m}\frac{d\lambda}{\lambda}$$

where

$$\tau=\frac{p}{q},\ \ (\lambda,\tau)_m=(1-\lambda)(1-\lambda\tau)\cdots(1-\lambda\tau^{m-1})$$

and $\mathcal{C}$ is a circle centered at the origin containing all the singularities of the integrand.

Deformation of Fredholm Determinant: Final Simplification of $\mathbb{P}_{\mathbb{Z}^+}(x_m(t) \leq x)$

Though we have reduced the problem to a single contour integral involving a Fredholm determinant, this determinant is difficult to analyze asymptotically.

Develop a **deformation theory**

$$K \longrightarrow J$$

so that Fredholm determinants remain equal. This final representation is amenable to asymptotic analysis much along the lines as encountered in random matrix theory and determinantal processes.

Lemma 1. *Suppose* $s \longrightarrow \Gamma_s$ *is a deformation of closed curves and a kernel* $L(\eta,\eta')$ *us holomorphic in a neighborhood of* $\Gamma_s \times \Gamma_s \subset \mathbb{C}^2$ *for each* s*. Then the Fredholm determinant acting on* Γ_s *is independent of* s*.*

Proof: Deform contours and apply Cauchy's theorem.

Lemma 2. *Suppose* $L_1(\eta,\eta')$ *and* $L_2(\eta,\eta')$ *are two kernels acting on a simple closed contour* Γ*, that* $L_1(\eta,\eta')$ *extends analytically to* η *inside* Γ *or to* η' *inside* Γ*, and* $L_2(\eta,\eta')$ *extends analytical to* η *inside* Γ *and to* η' *inside* Γ *Then the Fredholm determinants of* $L_1(\eta,\eta') + L_2(\eta,\eta')$*and* $L_1(\eta,\eta')$ *are equal.*

Proof: Assume $L_1(\eta, \eta')$ extends analytically to η' inside Γ.

$$\mathrm{tr}(L_2) = \int_\Gamma L_2(\eta, \eta)\, d\eta = 0.$$

by Cauchy. Thus $\mathrm{tr}(L_1 + L_2) = \mathrm{tr}(L_1)$.

$$\mathrm{tr}\left((L_1 + L_2)^2\right) = \mathrm{tr}(L_1^2) + 2\mathrm{tr}(L_1 L_2) + \mathrm{tr}(L_2^2)$$

By Cauchy again last two terms are zero. Thus

$$\mathrm{tr}\left((L_1 + L_2)^2\right) = \mathrm{tr}(L_1^2)$$

Argument extends to all powers.

In kernel $K(\xi,\xi')$ make the substitution

$$\xi = \frac{1-\tau\eta}{1-\eta},\ \ \xi' = \frac{1-\tau\eta'}{1-\eta'}$$

kernel becomes

$$K_2(\eta,\eta') = \frac{\varphi(\eta')}{\eta'-\tau\eta},\ \varphi(\eta) = \left(\frac{1-\tau\eta}{1-\eta}\right)^x \mathrm{e}^{\left[\frac{1}{1-\eta}-\frac{1}{1-\tau\eta}\right]t}$$

acting on a small circle centered at $\eta = 1$. Define

$$K_1(\eta,\eta') = \frac{\varphi(\tau\eta)}{\eta'-\tau\eta}$$

Then an application of the two Lemmas shows that the Fredholm determinant of $K(\xi,\xi')$ acting on $\mathcal{C}_R$ has the same Fredholm determinant as $K_1(\eta,\eta') - K_2(\eta,\eta')$ acting on Γ.

Then further (!) analysis gives

$$\begin{aligned}
\det(I - \lambda K) &= \det(I - \lambda K_1) \det\left(I + \lambda K_2 (I - \lambda k_1)^{-1}\right) \\
&= \prod_{k=0}^{\infty} (1 - \lambda \tau^k) \det(I + \mu J)
\end{aligned}$$

where J is a “nice” kernel:

$$J(\eta, \eta') = \int \frac{\varphi_\infty(\zeta)}{\varphi_\infty(\eta')} \frac{\zeta^m}{(\eta')^{m+1}} \frac{f(\mu, \zeta/\eta')}{\zeta - \eta} \, d\zeta$$

$$\varphi_\infty(\eta) = (1 - \eta)^{-x} \, e^{\frac{\eta}{1-\eta} t}$$

$$f(\mu, z) = \sum_{k=-\infty}^{\infty} \frac{\tau^k}{1 - \tau^k \mu} \, z^k$$

Theorem (TW).

$$\mathbb{P}\left(x_m(t/\gamma) \leq x\right) = \int \prod_{k=0}^{\infty}(1-\mu\tau^k)\det(I+\mu J)\,\frac{d\mu}{\mu}$$

where μ runs over a circle of fixed radius larger than τ but not equal to any τ^{-k} with $k \geq 0$.

This final expression is suitable for a saddle point analysis.

Theorem (TW).

$$\mathbb{P}\left(x_m(t/\gamma) \le x\right) = \int \prod_{k=0}^{\infty}(1-\mu\tau^k)\,\det(I+\mu J)\,\frac{d\mu}{\mu}$$

where μ runs over a circle of fixed radius larger than τ but not equal to any τ^{-k} with $k \ge 0$.

This final expression is suitable for a saddle point analysis.

Theorem (TW). Let $m = [\sigma t]$, $\gamma = q - p$ fixed, then

$$\lim_{t\to\infty} \mathbb{P}_{\mathbb{Z}^+}\left(x_m(t/\gamma) \le c_1(\sigma)t + c_2(\sigma)\,s\,t^{1/3}\right) = F_2(s)$$

uniformly for σ in compact subsets of $(0,1)$ where $c_1(\sigma) = -1 + 2\sqrt{\sigma}$, $c_2(\sigma) = \sigma^{-1/6}(1-\sqrt{\sigma})^{2/3}$.