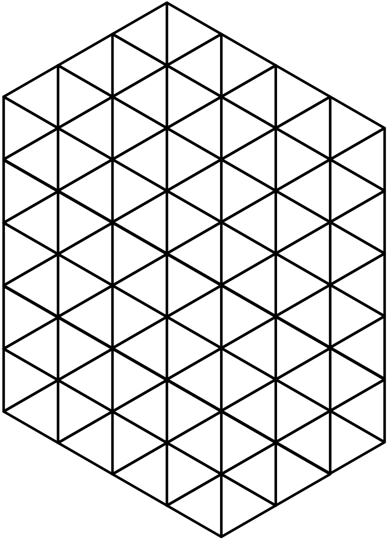
A factorisation theorem for the number of rhombus tilings of a hexagon with triangular holes

Mihai Ciucu and Christian Krattenthaler

Indiana University; Universität Wien

Rhombus tilings



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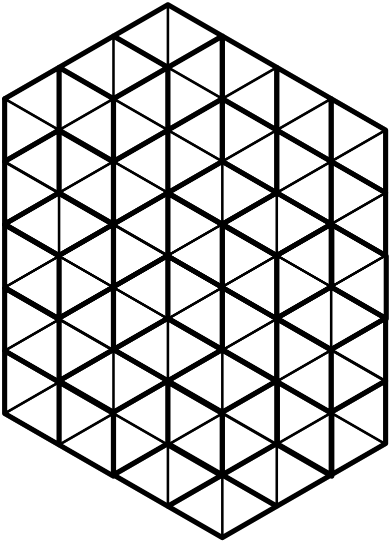
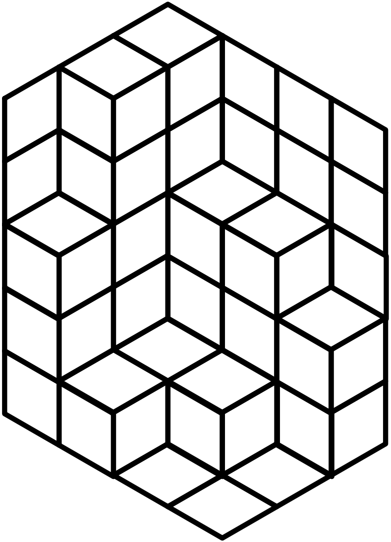
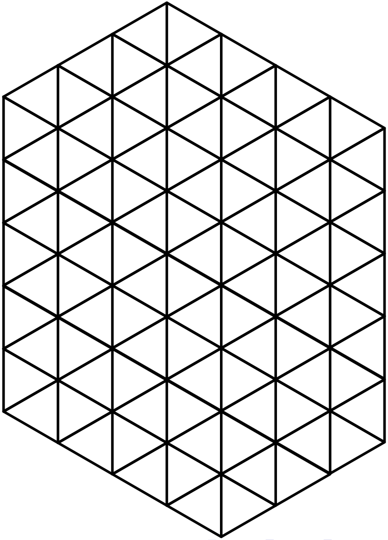


Prelude

Rhombus tilings — Dimer configurations





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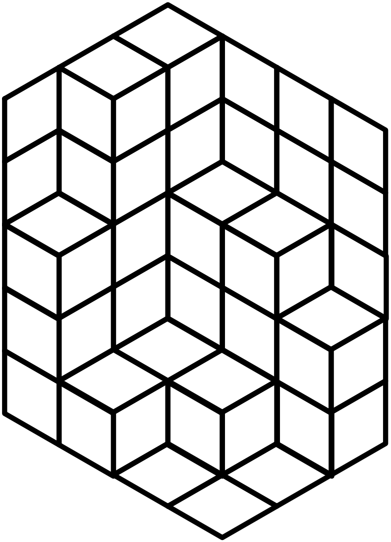

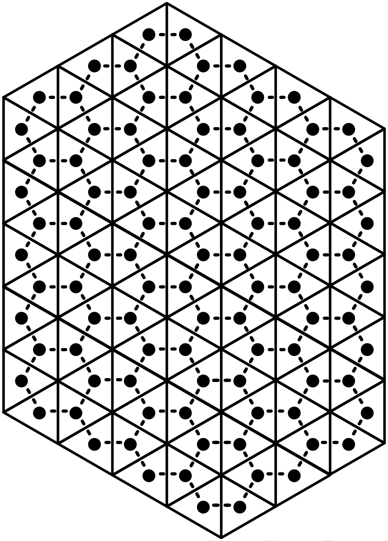

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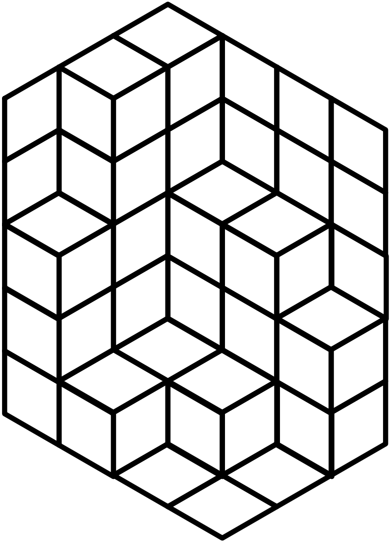

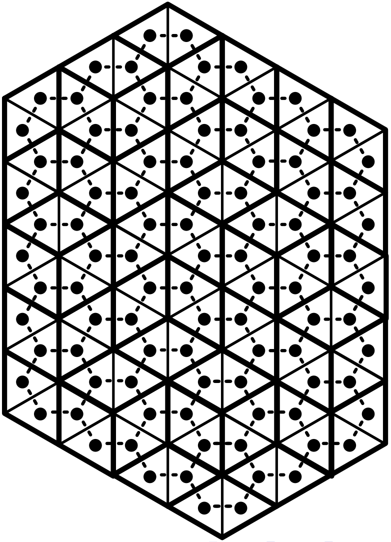

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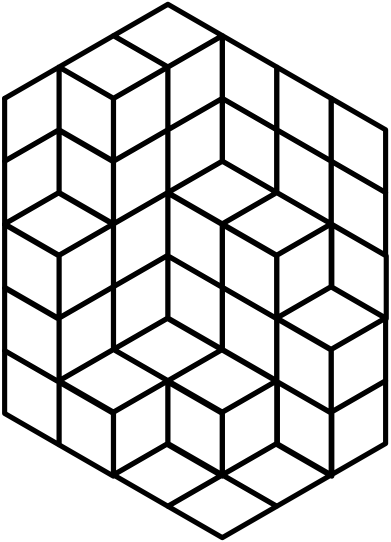

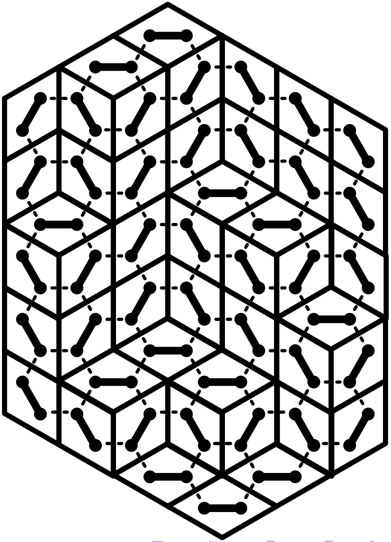

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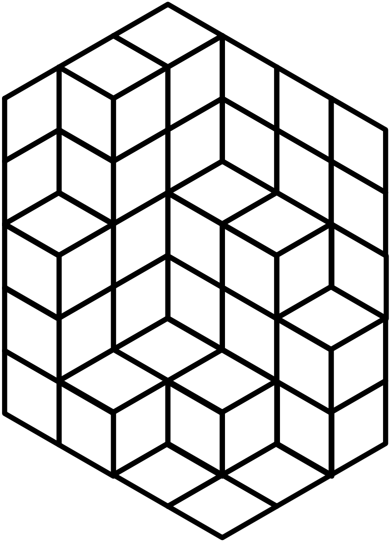

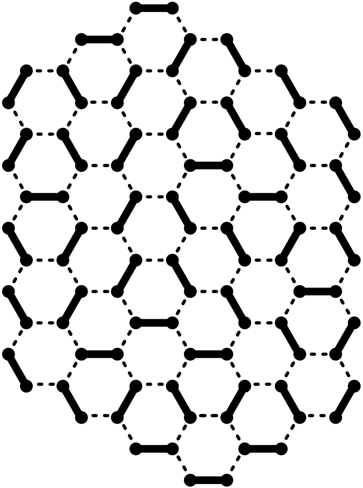

Science Fiction (Mihai Ciucu)

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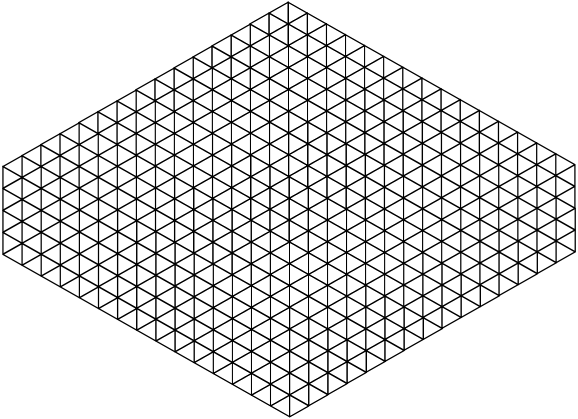

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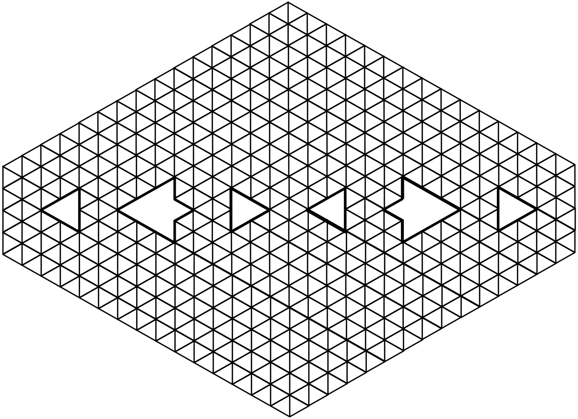

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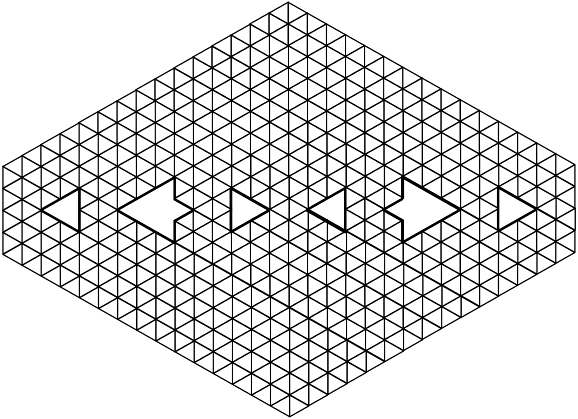
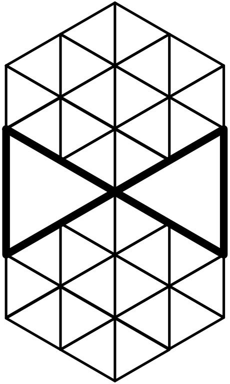


Let R be that region. Then

$$\mathrm{M}(R) \stackrel{?}{=} \mathrm{M}^{hs}(R) \cdot \mathrm{M}^{vs}(R),$$

where $\mathrm{M}(R)$ denotes the number of rhombus tilings of R.

A small problem



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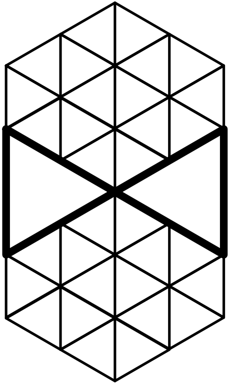

For this region R, we have $\mathrm{M}(R) = 6 \times 6 = 36$, $\mathrm{M}^{hs}(R) = 6$, and $\mathrm{M}^{vs}(R) = 4 \times 4 = 16$. But,

$$36 \neq 6 \times 16.$$

Evidence?

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It is true for the case without holes!

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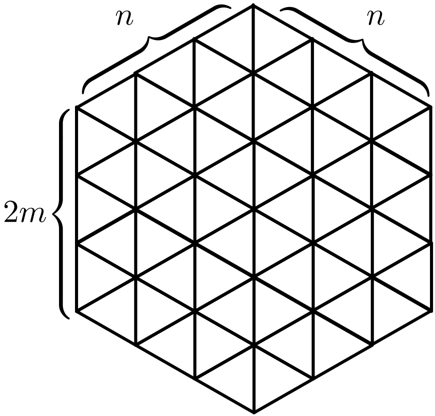
Actually, this is "trivial" and "well-known".

Evidence?



Once and for all, let us fix $H_{n,2m}$ to be the hexagon with side lengths $n, n, 2m, n, n, 2m$.

Evidence?

MacMahon showed that ("plane partitions" in a given box)

$$\mathrm{M}(H_{n,2m}) = \prod_{i=1}^{n} \prod_{j=1}^{n} \prod_{k=1}^{2m} \frac{i+j+k-1}{i+j+k-2}.$$

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Proctor showed that ("transpose-complementary plane partitions" in a given box)

$$\mathrm{M}^{hs}(H_{n,2m}) = \prod_{1 \le i < j \le n} \frac{2m+2n+1-i-j}{2n+1-i-j}.$$

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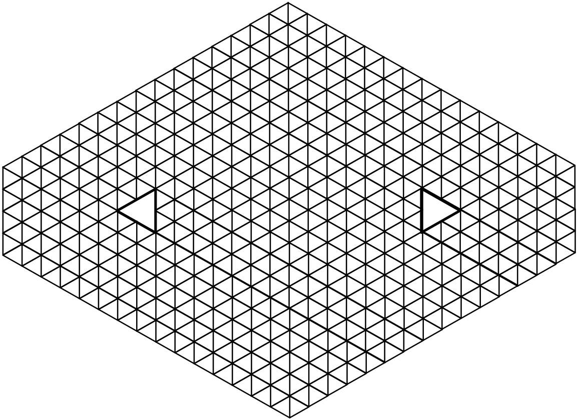
$$\mathrm{M}^{hs}(H_{n,2m}) = \prod_{1\le i<j\le n} \frac{2m+2n+1-i-j}{2n+1-i-j}.$$

Andrews showed that ("symmetric plane partitions" in a given box)

$$\mathrm{M}^{vs}(H_{n,2m}) = \prod_{i=1}^{n} \frac{2m+2i-1}{2i-1} \prod_{1\le i<j\le n} \frac{2m+i+j-1}{i+j-1}.$$

Evidence?



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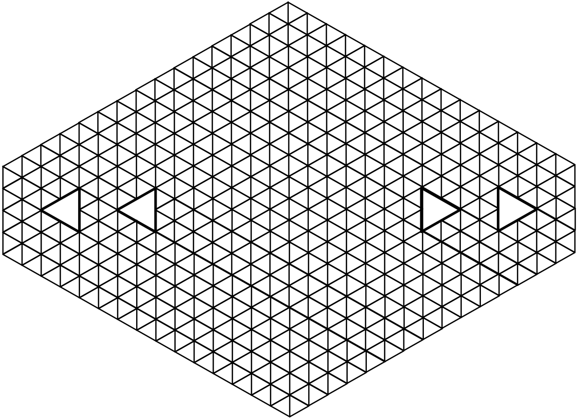

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Maybe introducing weights helps in seeing what one can do ?

Interlude: without holes

It is well-known that the number of rhombus tilings of the hexagon $H_{n,2m}$ is the same as the number of semistandard tableaux of rectangular shape $((2m)^n)$ with entries between 1 and $2n$.
This observation connects $\mathrm{M}(H_{n,2m})$ with Schur functions. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n)$, the Schur function s_λ is given by

$$
\begin{aligned}
s_\lambda(x_1, \dots, x_N) &= \frac{\det_{1\le i,j\le N}\left(x_i^{\lambda_j+N-j}\right)}{\det_{1\le i,j\le N}\left(x_i^{N-j}\right)} \\
&= \sum_T \prod_{i=1}^N x_i^{\#(\text{occurrences of } i \text{ in } T)},
\end{aligned}
$$

where the sum is over all semistandard tableaux of shape λ with entries between 1 and N.

Hence:

$$s_\lambda(\underbrace{1,\dots,1}_{2n}) = \mathrm{M}(H_{n,2m}).$$

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$$s_\lambda(\underbrace{1,\dots,1}_{2n}) = \mathrm{M}(H_{n,2m}).$$

So, let us consider the Schur function, when not all variables are specialised to 1.

Interlude: without holes

Computer experiments lead one to:

Theorem

For any non-negative integers m and n, we have

$$s_{((2m)^n)}(x_1, x_1^{-1}, x_2, x_2^{-1}, \ldots, x_n, x_n^{-1})$$

Computer experiments lead one to:

Theorem

For any non-negative integers m and n, we have

$$s_{((2m)^n)}(x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}) \\ = (-1)^{mn}\, so_{(m^n)}(x_1, x_2, \dots, x_n)\; so_{(m^n)}(-x_1, -x_2, \dots, -x_n).$$

Here,

$$so_\lambda(x_1, x_2, \dots, x_N) = \frac{\det\limits_{1\le i,j\le N}\left(x_i^{\lambda_j+N-j+\frac{1}{2}} - x_i^{-(\lambda_j+N-j+\frac{1}{2})}\right)}{\det\limits_{1\le i,j\le N}\left(x_i^{N-j+\frac{1}{2}} - x_i^{-(N-j+\frac{1}{2})}\right)}$$

is an irreducible character of $SO_{2N+1}(\mathbb{C})$.

Interlude: without holes

The odd orthogonal character is "expected", since all existing proofs for the enumeration of symmetric plane partitions use — in one form or another, directly or indirectly — the summation

$$so_{(m^n)}(x_1, x_2, \dots, x_n) = (x_1 x_2 \cdots x_n)^{-m} \cdot \sum_{\nu : \nu_1 \le 2m} s_\nu(x_1, \dots, x_n),$$

and, in particular, one obtains

$$\mathrm{M}^{vs}(H_{n,2m}) = so_{(m^n)}(\underbrace{1, \dots, 1}_{n}).$$

However, the appearance of $so_{(m^n)}(-x_1,-x_2,\dots,-x_n)$ is "unwanted". What one would actually like to see in place of this is a *symplectic character* of rectangular shape, because this is what goes into all proofs of the enumeration of transpose-complementary plane partitions (in one form or another).

Nevertheless, by substituting $x_i=-q^{i-1}$ in the Weyl character formula, both determinants can be evaluated in closed form, and subsequently the limit $q\to 1$ can be performed. The result is that, indeed,

$$(-1)^{mn}\, so_{(m^n)}(\underbrace{-1,\dots,-1}_{n}) = \mathrm{M}^{hs}(H_{n,2m}).$$

Proof of the theorem. By the definition of the Schur function, we have

$$s_{((2m)^n)}(x_1, x_1^{-1}, x_2, x_2^{-1}, \dots, x_n, x_n^{-1}) \\ = \frac{\det\limits_{1\le i,j\le 2n} \begin{pmatrix} x_i^{2m\chi(j\le n)+2n-j} & 1\le i\le n \\ x_{i-n}^{-(2m\chi(j\le n)+2n-t)} & n+1\le i\le 2n \end{pmatrix}}{\text{denominator}}.$$

Now do a Laplace expansion with respect to the first n rows. This leads to a huge sum.

Interlude: without holes

For the odd orthogonal character(s), one also starts with the Weyl character formula

$$so_\lambda(x_1, x_2, \dots, x_N) = \frac{\det\limits_{1\le i,j\le N}\left(x_i^{\lambda_j+N-j+\frac{1}{2}} - x_i^{-(\lambda_j+N-j+\frac{1}{2})}\right)}{\text{denominator}}.$$

Here, each entry in the determinant is a sum of two monomials. We use linearity of the determinant in the rows to expand the determinant. Also here, this leads to a huge sum.

Interlude: without holes

In the end, one has to prove identities such as

$$\sum_{\substack{A\subseteq[2N]\\|A|=N}} V(A)V\left(A^{-1}\right)V(A^c)V\left((A^c)^{-1}\right)R\left(A,A^{-1}\right)R\left(A^c,(A^c)^{-1}\right)$$

$$=\sum_{A\subseteq[2N]} V(A)V\left(A^{-1}\right)V(A^c)V\left((A^c)^{-1}\right)R\left(A,(A^c)^{-1}\right)R\left(A^c,A^{-1}\right),$$

where A^c denotes the complement of A in $[2N]$. Here,

$$R\left(A,B^{-1}\right):=\prod_{a\in A}\prod_{b\in B}(x_a-x_b^{-1}),\quad V(A):=\prod_{\substack{a,b\in A\\a<b}}(x_a-x_b),$$

and $V\left(A^{-1}\right):=\prod_{\substack{a,b\in A\\a<b}}(x_a^{-1}-x_b^{-1})$, which can be accomplished by induction.

Ciucu's Matchings Factorisation Theorem

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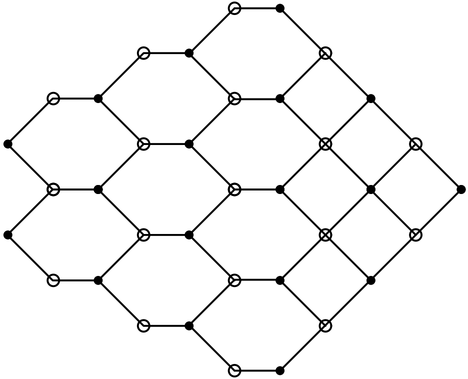
Consider a symmetric bipartite graph G.



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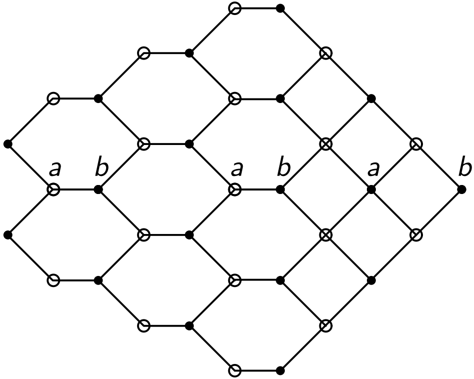

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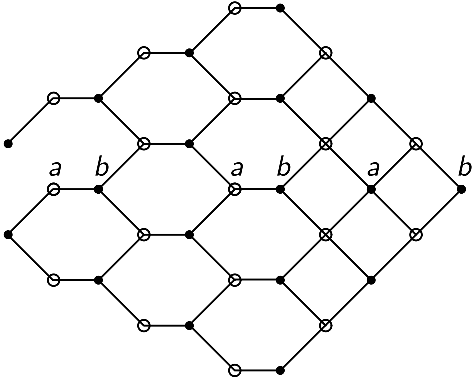

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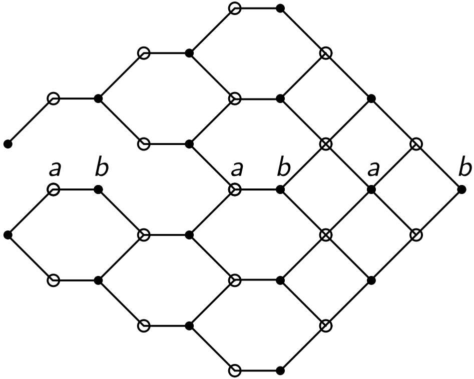

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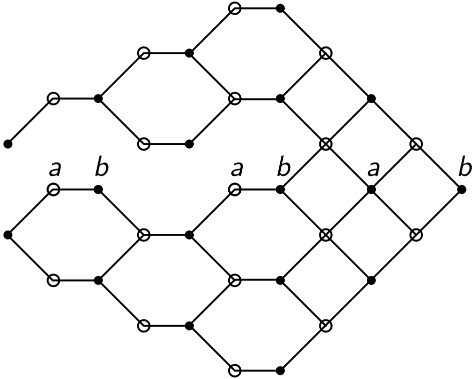

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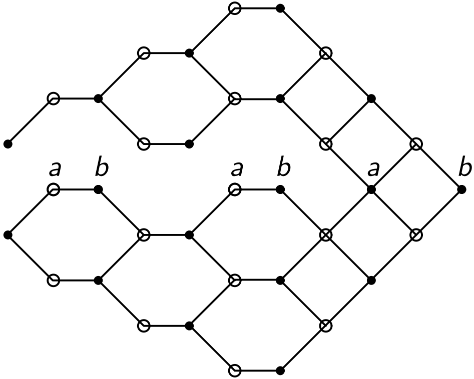

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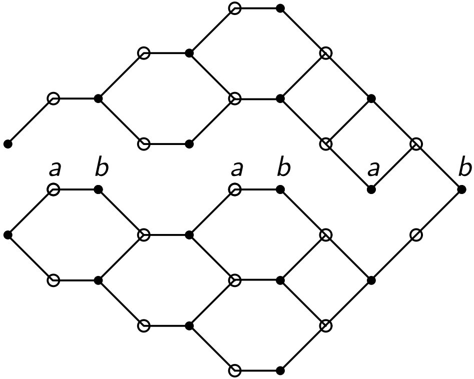

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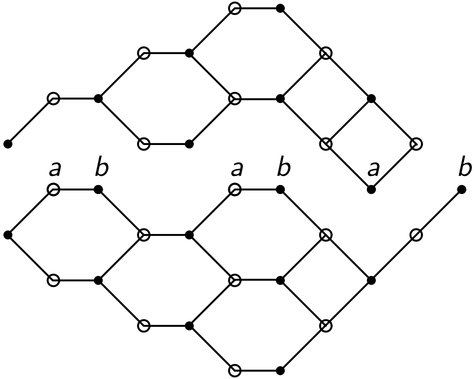

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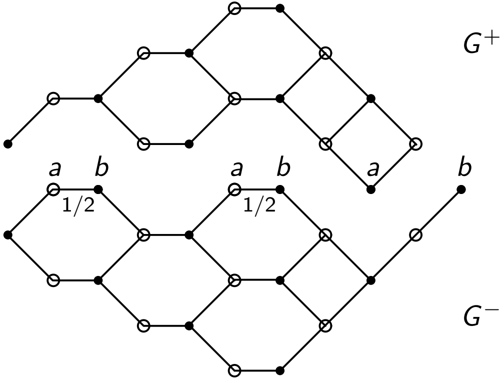

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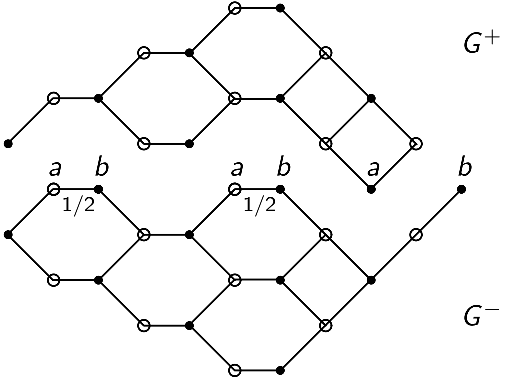
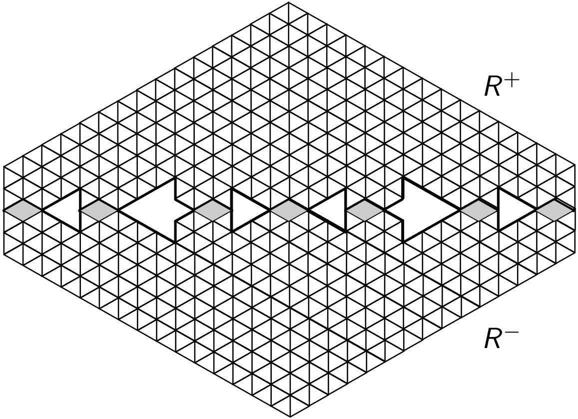


Then

$$\mathrm{M}(G) = 2^{\#(\text{edges on symm. axis})} \cdot \mathrm{M}(G^+) \cdot \mathrm{M}_{\text{weighted}}(G^-).$$

If we translate this to our situation:



we obtain

$$\mathrm{M}(R) = 2^{\#(\text{rhombi on symm. axis})} \cdot \mathrm{M}(R^+) \cdot \mathrm{M}_{\text{weighted}}(R^-).$$

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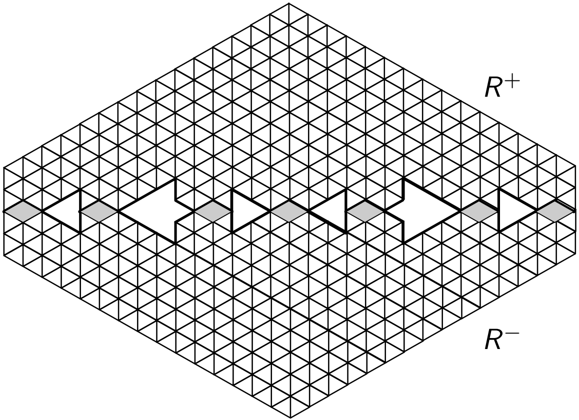


we obtain

$$\mathrm{M}(R) = 2^{\#(\text{rhombi on symm. axis})} \cdot \mathrm{M}(R^+) \cdot \mathrm{M}_{\text{weighted}}(R^-).$$

We "want"

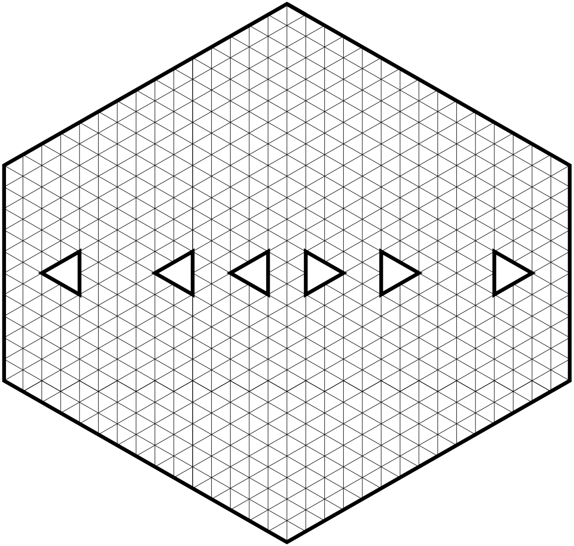
$$\mathrm{M}(R) \overset{?}{=} \mathrm{M}^{hs}(R) \cdot \mathrm{M}^{vs}(R).$$

The "actual" problem

So, it "only" remains to prove

$$\mathrm{M}^{vs}(R) = 2^{\#(\text{rhombi on symm. axis})} \cdot \mathrm{M}_{\text{weighted}}(R^-).$$

The theorem



The hexagon with holes $H_{15,10}(2,5,7)$

The theorem

Theorem

For all positive integers n, m, l and non-negative integers $k_1, k_2, \dots k_l$ with $0 < k_1 < k_2 < \cdots < k_l \le n/2$, we have

$$\mathrm{M}\left(H_{n,2m}(k_1, k_2, \dots, k_l)\right) \\ = \mathrm{M}^{hs}\left(H_{n,2m}(k_1, k_2, \dots, k_l)\right) \mathrm{M}^{vs}\left(H_{n,2m}(k_1, k_2, \dots, k_l)\right).$$

Sketch of proof

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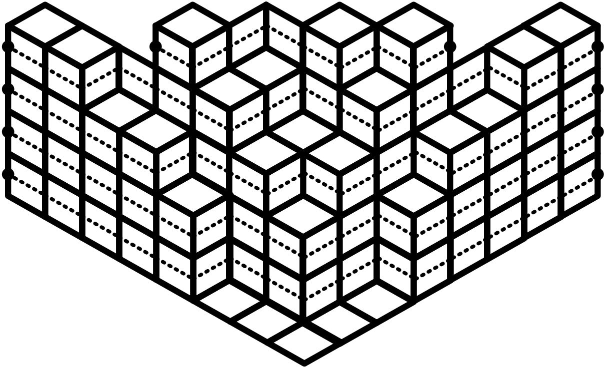
First step. Use non-intersecting lattice paths to get a determinant for $\mathrm{M}_{\text{weighted}}\left(H^{-}_{n,2m}(k_1,k_2,\dots,k_l)\right)$ and a Pfaffian for $\mathrm{M}^{vs}\left(H_{n,2m}(k_1,k_2,\dots,k_l)\right)$.

Sketch of proof



A tiling of $H^{-}_{n,2m}(k_1, k_2, \dots, k_l)$

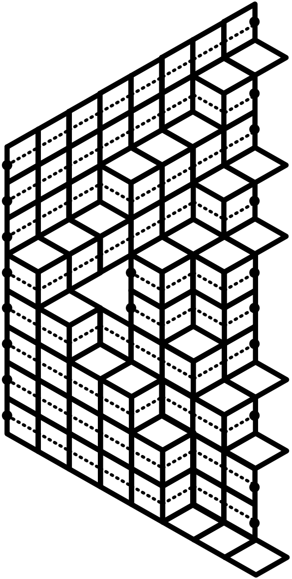
By the Karlin–McGregor, Lindström, Gessel–Viennot, Fisher, John–Sachs, Gronau–Just–Schade–Scheffler–Wojciechowski Theorem on non-intersecting lattice paths, we obtain a determinant.

Proposition

$\mathrm{M}_{weighted}\left(H^{-}_{n,2m}(k_1,k_2,\dots,k_l)\right)$ is given by $\det(N)$, where N is the matrix with rows and columns indexed by $\{1,2,\dots,m,1^+,2^+,\dots,l^+\}$, and entries given by

$$N_{i,j}=\begin{cases}\binom{2n}{n+j-i}+\binom{2n}{n-i-j+1}, & \text{if } 1\le i,j\le m,\\ \binom{2n-2k_t}{n-k_t-i+1}+\binom{2n-2k_t}{n-k_t-i}, & \text{if } 1\le i\le m \text{ and } j=t^+,\\ \binom{2n-2k_t}{n-k_t-j+1}+\binom{2n-2k_t}{n-k_t-j}, & \text{if } i=t^+ \text{ and } 1\le j\le m\ ,\\ \binom{2n-2k_t-2k_{\hat t}}{n-k_t-k_{\hat t}}+\binom{2n-2k_t-2k_{\hat t}}{n-k_t-k_{\hat t}-1}, & \text{if } i=t^+,\ j=\hat t^+,\\ & \qquad \text{and } 1\le t,\hat t\le l.\end{cases}$$

Sketch of proof



The left half of a vertically symmetric tiling

Sketch of proof

Theorem (Okada, Stembridge)

Let $\{u_1, u_2, \ldots, u_p\}$ and $I = \{I_1, I_2, \ldots\}$ be finite sets of lattice points in the integer lattice $\mathbb{Z}^2$, with p even. Let $\mathfrak{S}_p$ be the symmetric group on $\{1, 2, \ldots, p\}$, set $\mathbf{u}_\pi = (u_{\pi(1)}, u_{\pi(2)}, \ldots, u_{\pi(p)})$, and denote by $\mathcal{P}^{nonint}(\mathbf{u}_\pi \to I)$ the number of families $(P_1, P_2, \ldots, P_p)$ of non-intersecting lattice paths, with P_k running from $u_{\pi(k)}$ to I_{j_k}, $k = 1, 2, \ldots, p$, for some indices $j_1, j_2, \ldots, j_p$ satisfying $j_1 < j_2 < \cdots < j_p$.
Then we have

$$\sum_{\pi \in \mathfrak{S}_p} (\operatorname{sgn} \pi) \cdot \mathcal{P}^{nonint}(\mathbf{u}_\pi \to I) = \operatorname{Pf}(Q),$$

Sketch of proof

with the matrix $Q = (Q_{i,j})_{1 \le i,j \le p}$ *given by*

$$Q_{i,j} = \sum_{1 \le u < v} \big(\mathcal{P}(u_i \to I_u) \cdot \mathcal{P}(u_j \to I_v) - \mathcal{P}(u_j \to I_u) \cdot \mathcal{P}(u_i \to I_v)\big),$$

where $\mathcal{P}(A \to E)$ *denotes the number of lattice paths from* A *to* E*.*

Sketch of proof

Proposition

$\mathrm{M}^{vs}\left(H_{n,2m}(k_1,k_2,\dots,k_l)\right)$ *is given by*

$$(-1)^{\binom{l}{2}}\,\mathrm{Pf}(M),$$

where M *is the skew-symmetric matrix with rows and columns indexed by*

$$\{-m+1,-m+2,\dots,m,1^-,2^-,\dots,l^-,1^+,2^+,\dots,l^+\},$$

and entries given by

Sketch of proof

$$M_{i,j} = \begin{cases} \sum_{r=i-j+1}^{j-i} \binom{2n}{n+r}, & \textit{if } -m+1 \le i < j \le m, \\ \sum_{r=i+1}^{-i} \binom{2n-2k_t}{n-k_t+r}, & \textit{if } -m+1 \le i \le m \textit{ and } j = t^-, \\ \sum_{r=i}^{-i+1} \binom{2n-2k_t}{n-k_t+r}, & \textit{if } -m+1 \le i \le m \textit{ and } j = t^+, \\ 0, & \textit{if } i = t^-,\ j = \hat{t}^-,\ \textit{and } 1 \le t < \hat{t} \le l, \\ \binom{2n-2k_t-2k_{\hat{t}}}{n-k_t-k_{\hat{t}}} & \\ \quad + \binom{2n-2k_t-2k_{\hat{t}}}{n-k_t-k_{\hat{t}}+1}, & \textit{if } i = t^-,\ j = \hat{t}^+,\ \textit{and } 1 \le t, \hat{t} \le l, \\ 0, & \textit{if } i = t^+,\ j = \hat{t}^+,\ \textit{and } 1 \le t < \hat{t} \le l, \end{cases}$$

where sums have to be interpreted according to

$$\sum_{r=M}^{N-1} \mathrm{Expr}(k) = \begin{cases} \sum_{r=M}^{N-1} \mathrm{Expr}(k) & N > M \\ 0 & N = M \\ -\sum_{k=N}^{M-1} \mathrm{Expr}(k) & N < M. \end{cases}$$

Second step.

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Lemma

For a positive integer m and a non-negative integer l, let A be a matrix of the form

$$A = \begin{pmatrix} X & Y \\ -Y^t & Z \end{pmatrix},$$

where $X = (x_{j-i})_{-m+1 \le i,j \le m}$ and $Z = (z_{i,j})_{i,j \in \{1^-,\dots,l^-,1^+,\dots,l^+\}}$ are skew-symmetric, and $Y = (y_{i,j})_{-m+1 \le i \le m,\, j \in \{1^-,\dots,l^-,1^+,\dots,l^+\}}$ is a $2m \times 2l$ matrix. Suppose in addition that $y_{i,t^-} = -y_{-i,t^-}$ and $y_{i,t^+} = -y_{-i+2,t^+}$, for all i with $-m+1 \le i \le m$ for which both sides of an equality are defined, and $1 \le t \le l$, and that $z_{i,j} = 0$ for all $i,j \in \{1^-,\dots,l^-\}$. Then

$$\operatorname{Pf}(A) = (-1)^{\binom{l}{2}} \det(B),$$

Sketch of proof

where

$$B = \begin{pmatrix} \bar{X} & \bar{Y}_1 \\ \bar{Y}_2 & \bar{Z} \end{pmatrix},$$

with

$$\begin{aligned} \bar{X} &= (\bar{x}_{i,j})_{1\le i,j\le m}, \\ \bar{Y}_1 &= (y_{-i+1,j})_{1\le i\le m,\, j\in\{1^+,\dots,l^+\}}, \\ \bar{Y}_2 &= (-y_{i,j})_{i\in\{1^-,\dots,l^-\},\, 1\le j\le m}, \\ \bar{Z} &= (z_{i,j})_{i\in\{1^-,\dots,l^-\},\, j\in\{1^+,\dots,l^+\}}, \end{aligned}$$

and the entries of $\bar{X}$ are defined by

$$\bar{x}_{i,j} = x_{|j-i|+1} + x_{|j-i|+3} + \cdots + x_{i+j-1}.$$

Sketch of proof

By the lemma, the Pfaffian for $\mathrm{M}^{vs}\left(H_{n,2m}(k_1,k_2,\dots,k_l)\right)$ can be converted into a determinant, of the same size as the determinant we obtained for $\mathrm{M}_{\text{weighted}}\left(H^-_{n,2m}(k_1,k_2,\dots,k_l)\right)$.

Sketch of proof

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Third step. Alas, it is not the same determinant. However, further row and column operations do indeed convert one determinant into the other. $\square$

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