Simple approaches to arctic curves for Alternating Sign Matrices

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work in collaboration with F. Colomo

Random Interfaces and Integrable Probability
Statistical Mechanics, Integrability and Combinatorics
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June 25^{th} 2015

Alternating Sign Matrices

This talk approaches what is now one of the classical models for the interplay between *Statistical Mechanics, Integrability and Combinatorics*: the (bijectively related) models of 6 Vertex Model DWBC (6VM), Fully-Packed Loops (FPL) and Alternating-Sign Matrices (ASM).

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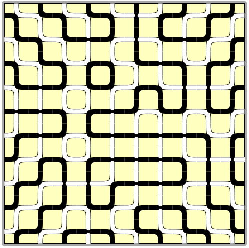
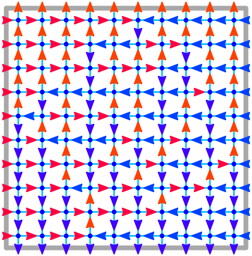
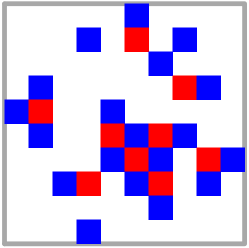


FPL



6VM



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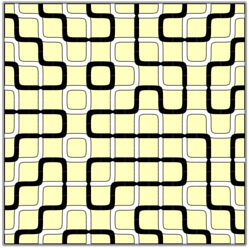
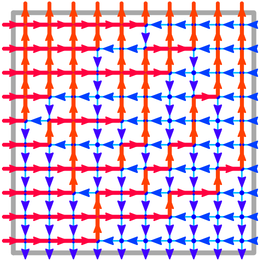
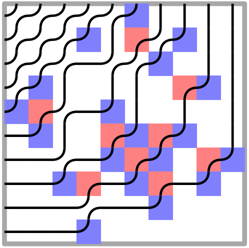


FPL



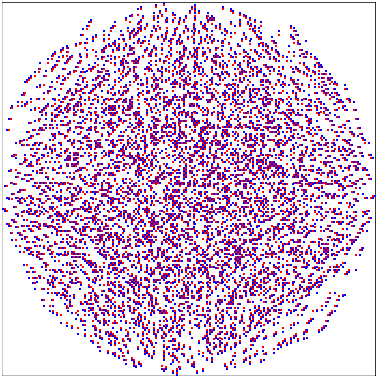
6VM



ASM

Asymptotic shape of ASM's

In large ASM's you see the emergence of a limit shape



The analytic determination of this curve is our subject today.

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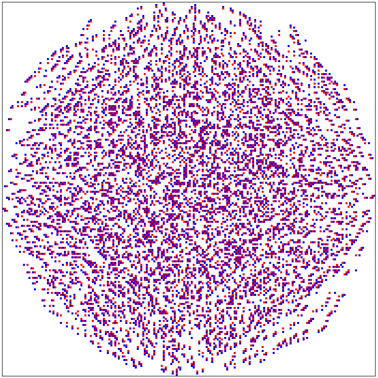

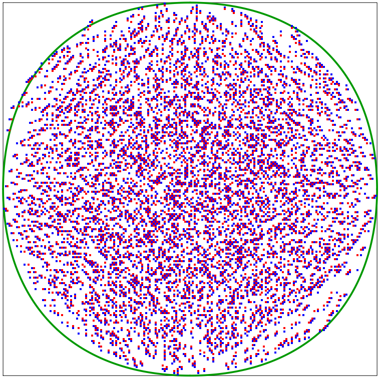

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Asymptotic shape of ASM's

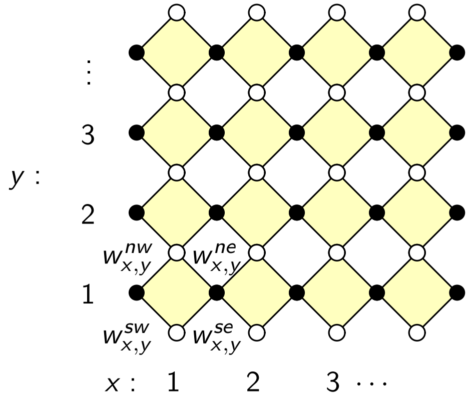
The role of Integrability has been quite important for the combinatorics of ASM's. For example, the enumeration of size-n ASM's is performed in a much easier way as a corollary of the evaluation of the partition function of the 6VM, with generic $2n$ spectral parameters (at the combinatorial point $\Delta = \frac{a^2+b^2-c^2}{2ab} = -\frac{1}{2}$)

Still, there is some need of extra technology in order to produce large-size asymptotics. Various tools are there, but the roadmap is less clear. In particular, it is not clear at which point of your calculations you shall quit your nice and neat exact formulas from integrable systems, and start using ϵ-and-δ estimations...

In this talk we will take a quite unusual path...

Arctic curves at free-fermion points

weighted "Domino Tilings of the Aztec Diamond"
(a planar-graph dimer-covering problem, thus a fermionic system. . .)



Domino Tilings of the Aztec Diamond ➽ ASM at $\omega = 2$

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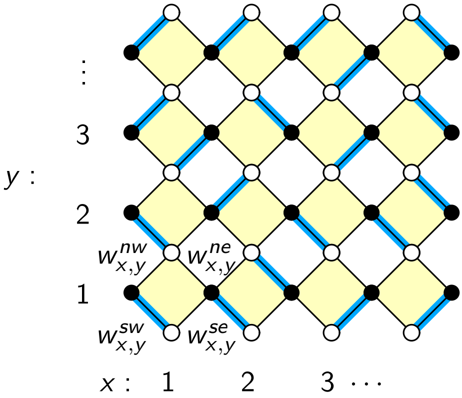

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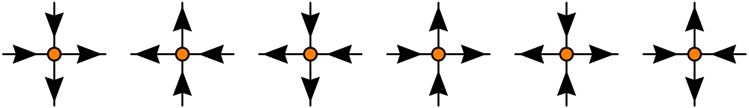
Consider the customary 6-Vertex Model weights...



$a(x,y)$ $a'(x,y)$ $b(x,y)$ $b'(x,y)$ $c(x,y)$ $c'(x,y)$

...and now consider the following map: *(note: $\Delta = 0$)*

$w^{sw}_{x,y}$ $w^{ne}_{x,y}$ $w^{se}_{x,y}$ $w^{nw}_{x,y}$ 1 $w^{se}_{x,y} w^{nw}_{x,y} + w^{sw}_{x,y} w^{ne}_{x,y}$

a a' b b' c c'

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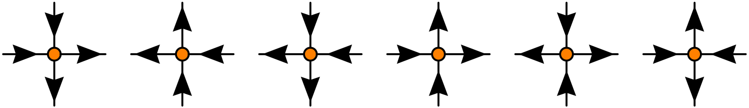

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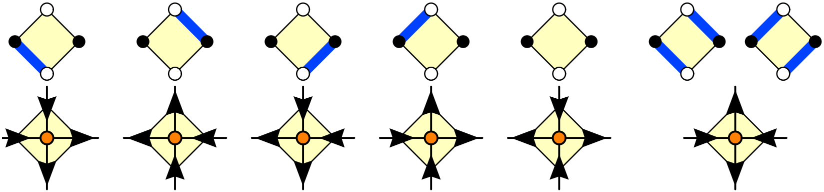

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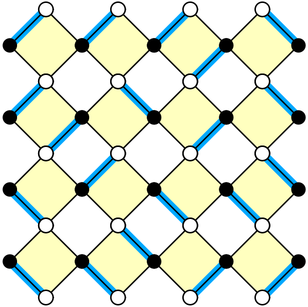
Schröder NILPs in Domino Tilings

The NILP construction for Domino Tilings of the Aztec Diamond is similar to the one for Lozenge Tilings on the triangular lattice, with Schröder paths ($\{\nearrow, \searrow, \xrightarrow{2}\}$) instead of Dyck paths ($\{\nearrow, \searrow\}$)



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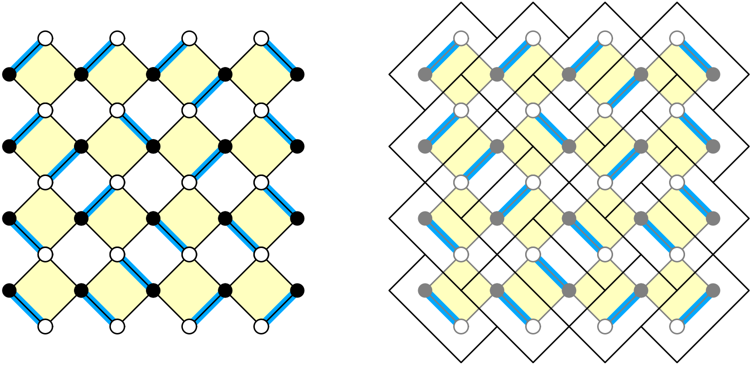

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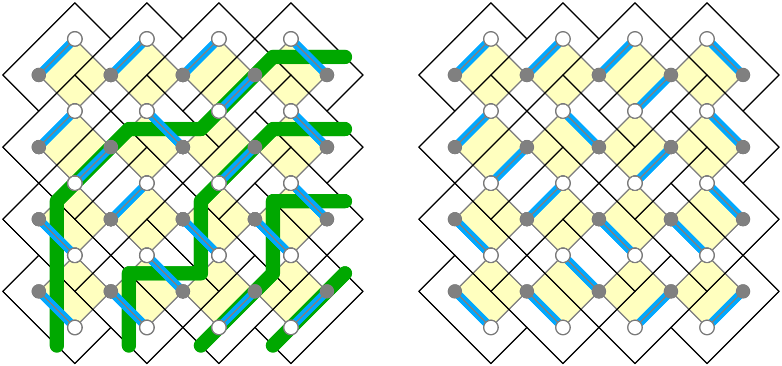
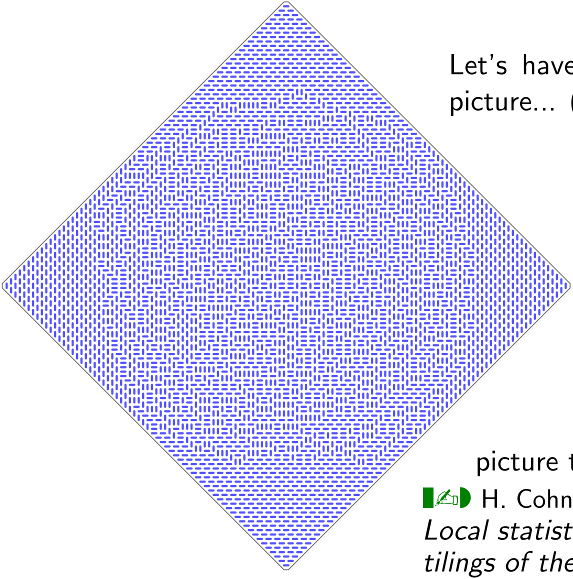


Domino Tilings of the Aztec Diamond: a bigger picture

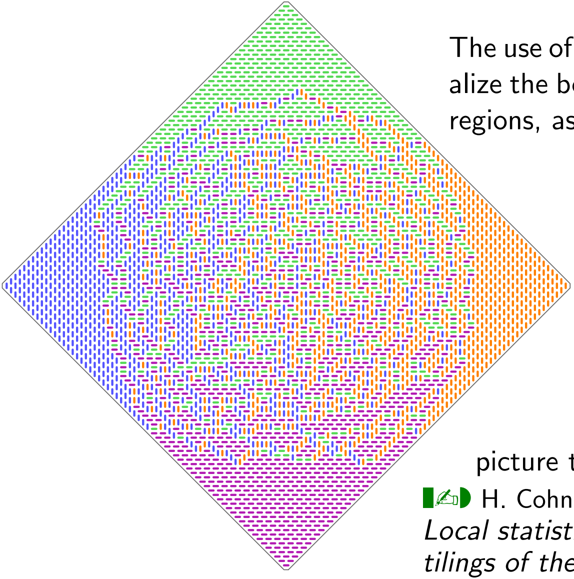


Let's have a look at a bigger picture... (here $L = 64$)

picture taken from:

H. Cohn, N. Elkies and J. Propp, *Local statistics for random domino tilings of the Aztec diamond*, 1995

Domino Tilings of the Aztec Diamond: a bigger picture



The use of colours allow to visualize the boundary of the frozen regions, as well as the NILP's

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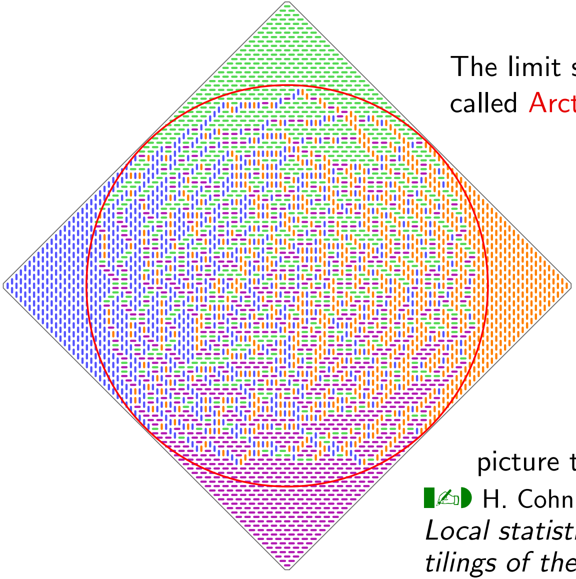

The limit shape, that they called Arctic curve, is a circle.

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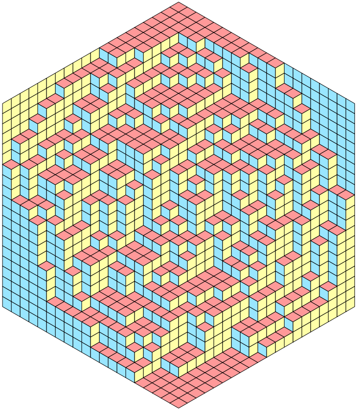
Other Arctic Circles

A similar feature, with Dyck paths, was already known to occur in lozenge tilings of a regular hexagon
(the MacMahon problem of "boxed plane partitions")

H. Cohn, M. Larsen and J. Propp, *The Shape of a Typical Boxed Plane Partition*, 1998



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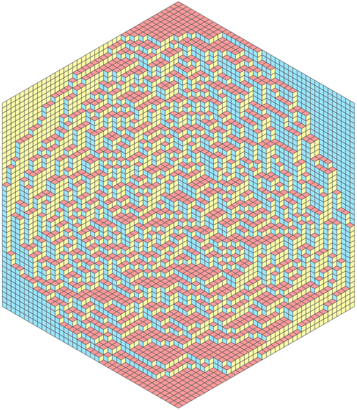

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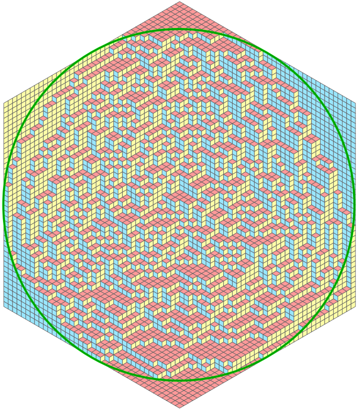
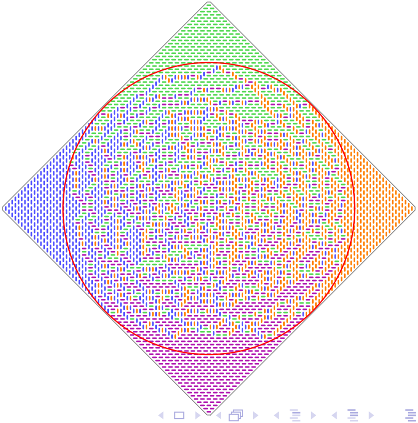




Dimer coverings of periodic planar bipartite graphs

So, we find similar features in dimer coverings of periodic planar bipartite graphs, for different unit tiles. A general unified theory indeed exists:

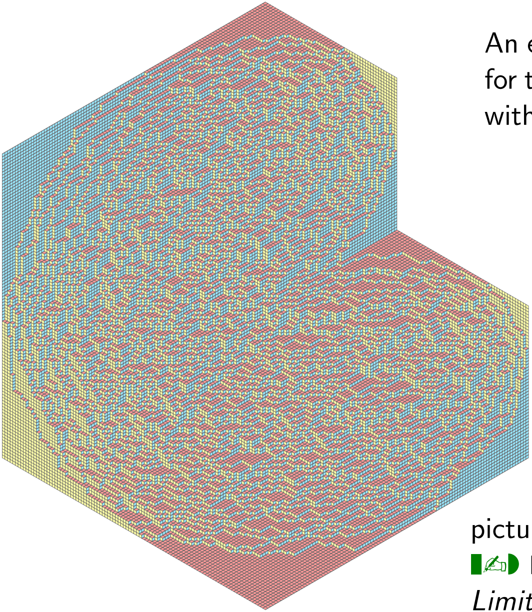
R. Kenyon, A. Okounkov, S. Sheffield, *Dimers and Amœbæ*, 2003

Within this class of models, lozenge tilings are by far the most studied case, even more than the square lattice.

This because the spectral curve associated to this lattice *(sic!)* is the simplest possible: $P(z, w) = z + w - 1$.

This study culminates into

R. Kenyon, A. Okounkov, *Limit shapes and the complex Burgers equation*, 2005



An example: the cardioid for the hexagonal domain with a frozen corner

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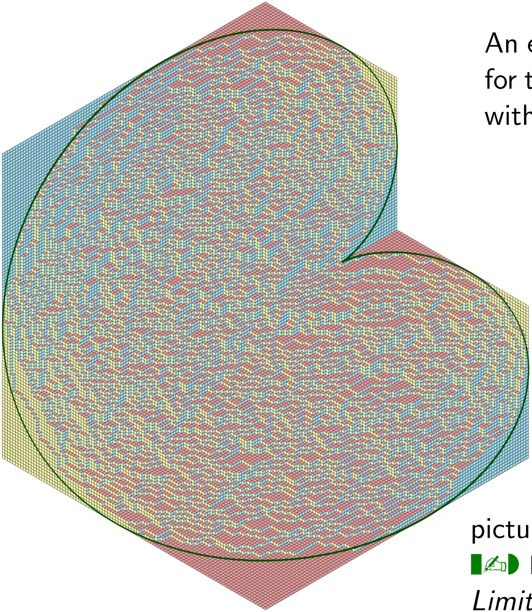

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The Colomo–Pronko formula

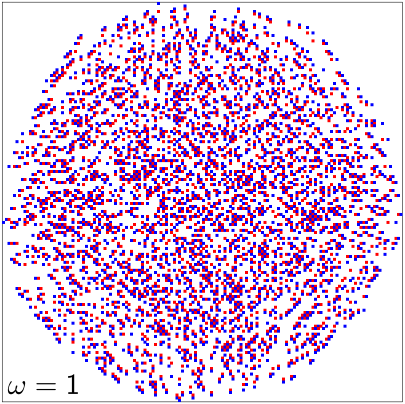
What about integrable systems out of fermionic points?

All of this is beautiful, but planar dimer coverings are fermionic...

As we know, ω-enumerations of ASM form a YB-integrable line, with a fermionic point at $\omega = 2$ (domino tilings of the Aztec Diamond)

Numerical simulations (thanks to CFTP!) seem to show that the arctic curve varies smoothly with ω, at least within certain ranges...

...but what is know theoretically?



from this point on, ASM pictures are produced with C code based on a version kindly provided by Ben Wieland

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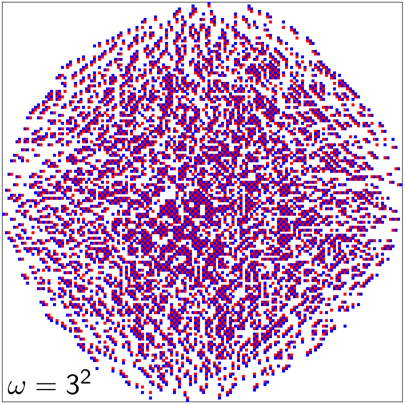

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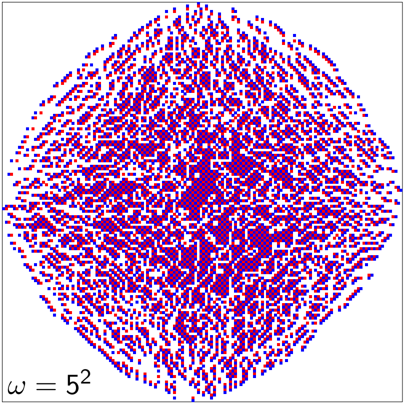

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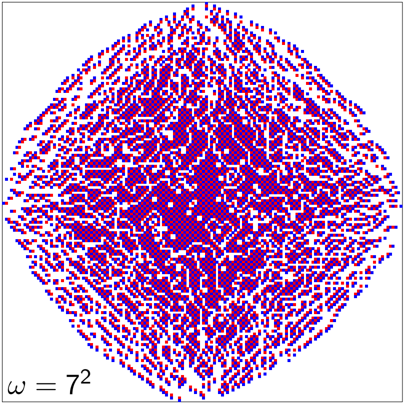

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The Colomo–Pronko formula

...but what is know theoretically?

...this was almost nothing up to recent times...
Then Colomo and Pronko came with a series of papers in which:

- they found explicitly the Arctic Curve for $\omega = 1$ ASM;
- they found a formula for the Arctic Curve at generic ω, in terms of the refined enumerations $A_\omega(n; r)$;
- they found the necessary asymptotic properties of a certain multi-contour integral, using methods of Random Matrices, first for $\omega \leq 4$,. . .
- . . . and then, together with P. Zinn-Justin, also for $\omega > 4$ (where the corresponding 6-Vertex Model is "antiferromagnetic");

F. Colomo and A.G. Pronko,
The arctic circle revisited, 2007

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The limit shape of large alternating sign matrices, 2008

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The arctic curve of the domain-wall six-vertex model, 2009

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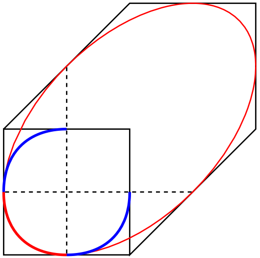
F. Colomo, A.G. Pronko and P. Zinn-Justin, *The arctic curve of the domain-wall six-vertex model in its anti-ferroelectric regime*, 2010

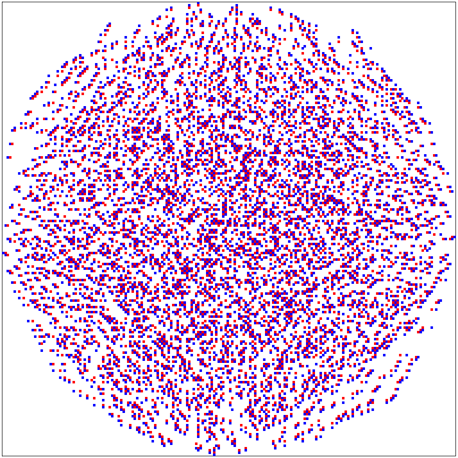
The Colomo–Pronko formula: $\omega = 1$

Picture and formula for $\omega = 1$:

The South-West arc satisfies
$x(1-x) + y(1-y) + xy = 1/4$
$x, y \in [0, 1/2]$

(just a "$+xy$" modification w.r.t. a circle)





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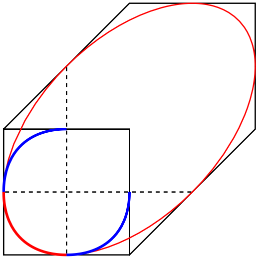
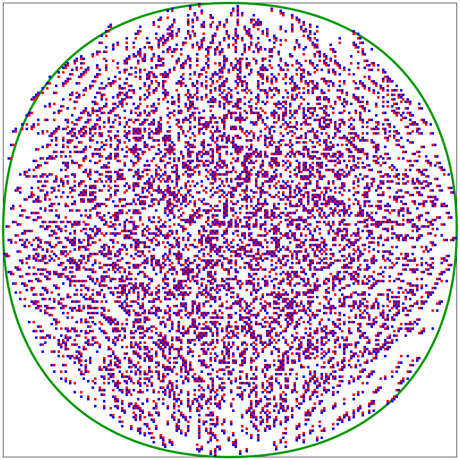




We call $A_\omega(n)$ the counting polynomial associated to ω-weighted ASM of size n:

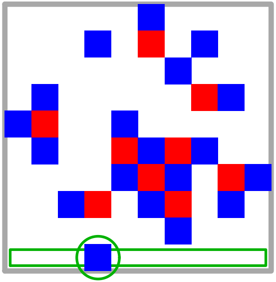
$$A_\omega(n) = \sum_{A \in \mathcal{A}_n} \omega^{\#\{-1 \text{ in } A\}}$$

Thus, e.g., $A_1(n) = \prod_{0 \le j \le n-1} \frac{(3j+1)!}{(n+j)!}$, the total number of size-n ASM

Call $A_\omega(n,r)$ the counting polynomial associated to ω-weighted ASM of size n, such that the only $+1$ in the bottom row is at the r-th column

Thus, e.g.,

$$\frac{A_1(n+1, r+1)}{A_1(n+1)} = \frac{\binom{n+r}{n}\binom{2n-r}{n}}{\binom{3n+1}{n}}$$



example at $n = 10,\ r = 4$

The Colomo–Pronko formula: generic ω

For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x,y)$, in parametric form $x = x(z)$, $y = y(z)$ on the interval $z \in [1, +\infty)$, is the solution of the system of equations

$$F(z;x,y) = 0\,; \qquad\qquad \frac{\partial}{\partial z} F(z;x,y) = 0\,.$$

The function $F(z;x,y)$, that depends on x and y linearly, is

$$F(z;x,y) = \frac{1}{z}(x-1) + \frac{\omega}{(z-1)(z-1+\omega)} y + \lim_{n\to\infty} \frac{1}{n} \frac{\partial}{\partial z} \ln \left(\sum_{r=1}^{n} A_\omega(n,r) z^{r-1} \right)\,.$$

$\mathcal{C}(x,y)$ is algebraic only at discrete special values of ω (including $0, 1, 2, 3$).

How are these results derived?

Call $h_n(z) = \sum_{r=1}^{n} A_\omega(n,r) z^{r-1}$

Define the Emptiness Formation Probability, $EFP(n;r,s)$: the probability that in the top-left $s \times r$ rectangle of the $n \times n$ ASM there are no ± 1 elements.

Clearly, $A(n;r) = EFP(n;r-1,1) - EFP(n;r,1)$.

But for $s \geq 2$ there is no evident relation...

...nonetheless, it can be determined that also $EFP(n;r,s)$ is related to $h_n(z)$, through a multi-contour integral formula

$$h_{n,s}(z_1,\dots,z_s) := \frac{1}{\boldsymbol{\Delta}(z)} \det \left(z_j^{k-1} (z_j - 1)^{s-k} h_{n-k+1}(z_j) \right)_{j,k}$$

$$EFP(n;r,s) = \oint_0 \frac{\mathrm{d}z_1}{2\pi i} \cdots \oint_0 \frac{\mathrm{d}z_s}{2\pi i} \prod_j \frac{((t^2 - 2t\Delta) z_j + 1)^{s-j}}{z_j^r (z_j - 1)^{s-j+1}}$$

$$\times \prod_{j<k} \frac{z_j - z_k}{t^2 z_j z_k - 2t\Delta z_j + 1} \; h_{n,s}(z_1,\dots,z_s)$$

How are these results derived?

Call $h_n(z) = \sum_{r=1}^{n} A_\omega(n,r) z^{r-1}$

Define the Emptiness Formation Probability, $EFP(n;r,s)$: the probability that in the top-left $s \times r$ rectangle of the $n \times n$ ASM there are no ± 1 elements.

Clearly, $A(n;r) = EFP(n;r-1,1) - EFP(n;r,1)$.

But for $s \geq 2$ there is no evident relation...

...nonetheless, it can be determined that also $EFP(n;r,s)$ is related to $h_n(z)$, through a multi-contour integral formula

For (r,s) crossing the Arctic Curve, $EFP(n;r,s)$ shows a 0-1 threshold transition, that you can study through saddle-point methods, helped by previous techniques developed for a certain Random Matrix Model (Triple Penner Model)

How to derive this?

...in a few words, something very complicated already for the square.

And something relying deeply on "miracles" of integrability methods, that have no guarantee to occur in other domains.

Furthermore, already for $\omega = 1$, the curve is not $\mathcal{C}_\infty$
at the points of contact with the boundary of the domain,
and is not even piecewise algebraic at generic ω
(differently from the curves in the Kenyon–Okounkov theory)

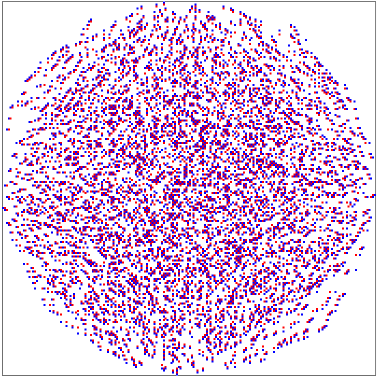
How can we hope for an analogue of Kenyon–Okounkov results on the whole YB-integrable line for ω?

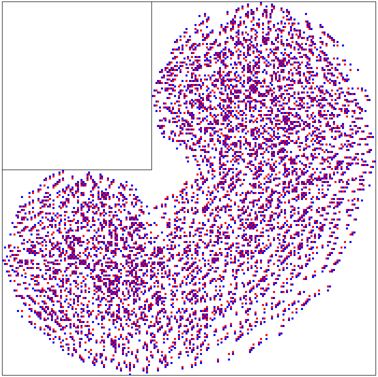
Staying less ambitious, can we determine in ASM something like the KO cardioid for the hexagon with a frozen corner?

Emptiness Formation: typical configurations

. . . It is instructive to observe a typical configuration in the ensemble pertinent to $EFP(n; r, s)$.
For (r, s) inside the arctic curve, we see the emergence of a new cardioid-like arctic curve (just like in Kenyon–Okounkov)

here $n = 200$, $(r, s) = (80, 90)$





A reminder on the basic theory of Plane Curves

J. Dennis Lawrence, *A catalog of special plane curves*, Dover, New York, 1972

A *curve* $\mathcal{C}$ will be represented either by the *Cartesian equation* $A(x, y) = 0$, or the *parametric equations* $x = f(t)$, $y = g(t)$. It is constituted by the concatenation of a finite number of *arcs*. An arc is a portion of the curve for which a "smooth" parametric presentation exists.

A curve is *algebraic* if the defining Cartesian equation $A(x, y) = 0$ is algebraic, otherwise it is *trascendental*.

A double point s.t. the two arcs passing through P have the same tangent is a *cusp*. A cusp is *of the first kind* if P is an endpoint of both arcs, and there is an arc of $\mathcal{C}$ on each side of the tangent, and *of the second kind* if P is an endpoint of both arcs, and the two arcs lie on the same side of the tangent,

A reminder on the basic theory of Plane Curves

The *envelope* $\mathcal{E}$ of a one-parameter family of curves $\{\mathcal{C}_z\}_{z\in I}$ is the curve, minimal under inclusion, that is tangent to every curve of the family.

If the equation of the family $\{\mathcal{C}_z\}$ is given in Cartesian coordinates by $U(z; x, y) = 0$, the non-singular points (x, y) of the envelope $\mathcal{E}$ are the solutions of the system of equations

$$U(z; x, y) = 0\,; \qquad\qquad \frac{\mathrm{d}}{\mathrm{d}z} U(z; x, y) = 0\,.$$

We call *geometric caustic* the envelope of a family of straight lines. In this case U is linear in x and y:

$$U(z; x, y) = x\,A(z) + y\,B(z) + C(z)$$

A reminder on the basic theory of Plane Curves

Caustics in optics are a special case of geometric caustics, in which the family of straight lines can be interpreted as the family of reflections of a beam of parallel rays from a curved mirror.

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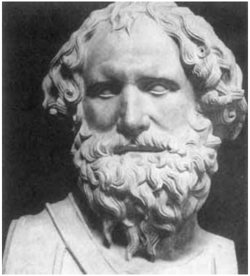

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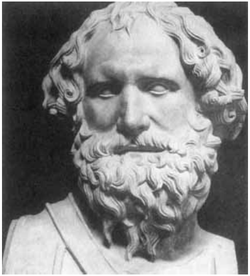

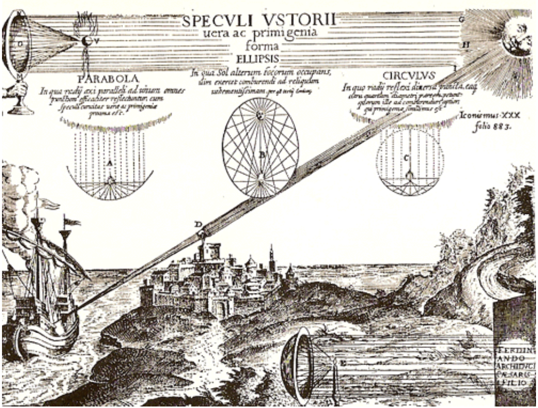

The Colomo–Pronko formula at generic ω – reloaded

For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x, y)$, in parametric form $x = x(z)$, $y = y(z)$ on the interval $z \in [1, +\infty)$, is the solution of the system of equations

$$F(z; x, y) = 0 \,; \qquad\qquad \frac{\partial}{\partial z} F(z; x, y) = 0 \,.$$

The function $F(z; x, y)$, that depends on x and y linearly, is

$$F(z; x, y) = \frac{1}{z}(x - 1) + \frac{\omega}{(z-1)(z-1+\omega)} y + \lim_{n\to\infty} \frac{1}{n} \frac{\partial}{\partial z} \ln \left(\sum_{r=1}^{n} A_\omega(n, r) z^{r-1} \right) .$$

$\mathcal{C}(x, y)$ is algebraic only at discrete special values of ω (including $0, 1, 2, 3$).

The Colomo–Pronko formula at generic ω – reloaded

For ω-weighted ASM on the square, the arctic curve $\mathcal{C}(x,y)$ is the geometric caustic of the family of lines, for z in the interval $z \in [1, +\infty)$,

$$
\begin{aligned}
F(z;x,y) = {} & \frac{1}{z}(x-1) + \frac{\omega}{(z-1)(z-1+\omega)} y \\
& + \lim_{n\to\infty} \frac{1}{n} \frac{\partial}{\partial z} \ln \left(\sum_{r=1}^{n} A_\omega(n,r) z^{r-1} \right) .
\end{aligned}
$$

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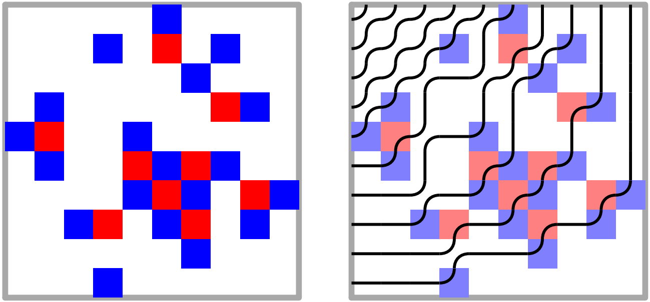
$$F(z; x, y) = \frac{1}{z}(x-1) + \frac{\omega}{(z-1)(z-1+\omega)} y + \lim_{n\to\infty} \frac{1}{n} \frac{\partial}{\partial z} \ln \left(\sum_{r=1}^{n} A_\omega(n, r) z^{r-1} \right) .$$

But this has not been derived geometrically!

The tangent method

A reminder on interacting NILP

Recall that an ASM can be seen (in 4 different ways) as a configuration of interacting non-intersecting lattice paths (NILP), which are in fact non-interacting when $\omega = 2$.



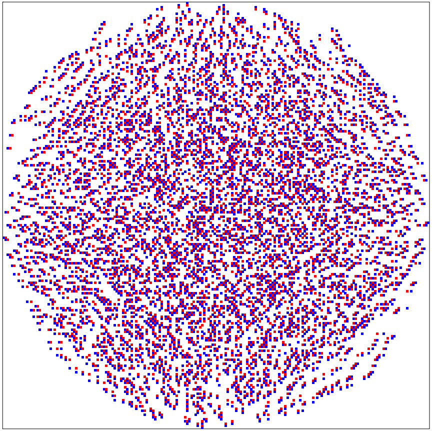
The refinement position is the point at which the most external path leaves the boundary

So we would like a more geometric strategy for attacking this sort of questions...

Hopefully, with some luck, this could also be more generally applicable to domains of different shape...

Let's have a deeper look to the domain with a frozen rectangle...

$n = 200$, no frozen region



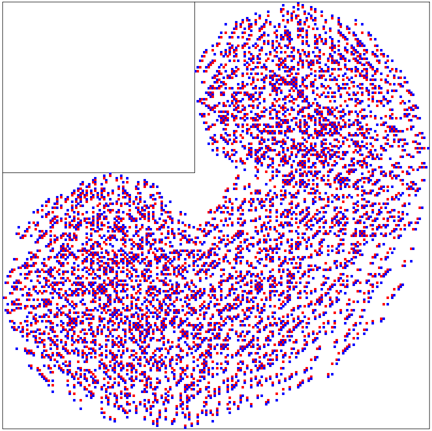
A quest for a new strategy...

So we would like a more geometric strategy for attacking this sort of questions...

Hopefully, with some luck, this could also be more generally applicable to domains of different shape...

Let's have a deeper look to the domain with a frozen rectangle...

$n = 200$, $(r, s) = (90, 80)$



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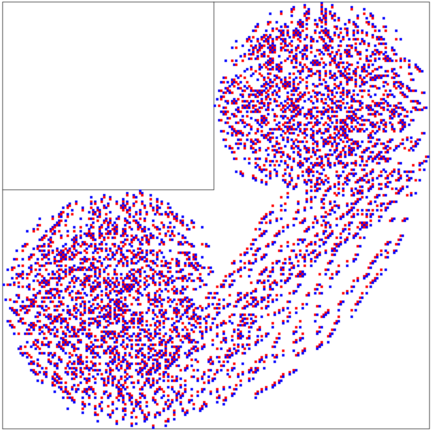
Let's have a deeper look to the domain with a frozen rectangle...

$n = 200$, $(r, s) = (99, 88)$



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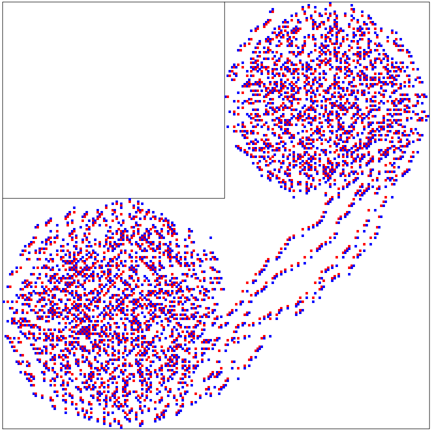
$n = 200$, $(r, s) = (104, 92)$



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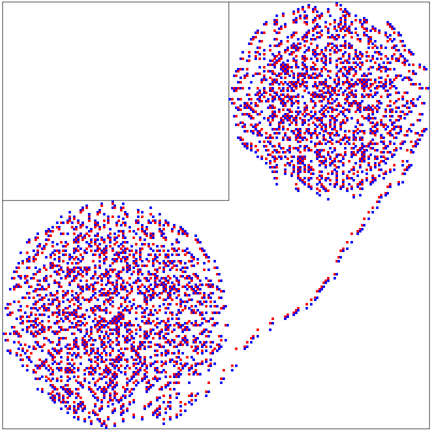
$n = 200$, $(r, s) = (106, 93)$



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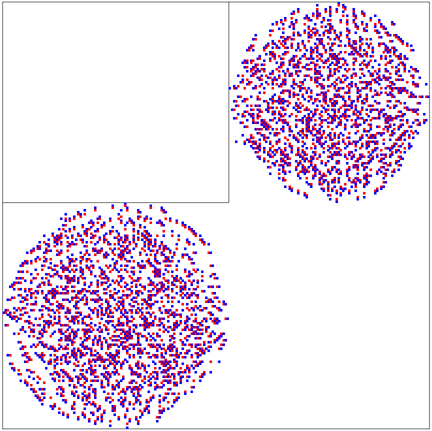
$n = 200$, $(r, s) = (106, 94)$



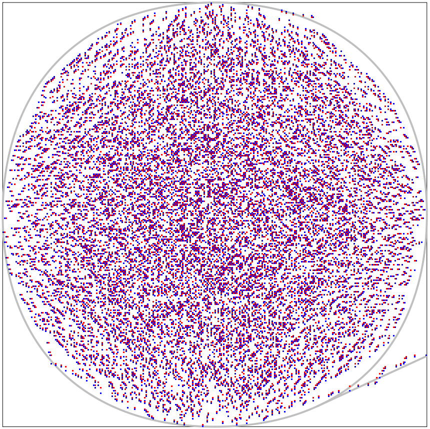
The structure of a typical refined ASM

...so this teaches us how does it look like a typical large *ASM*, of size n refined at r...

It must be like a typical ASM, plus a straight line connecting $(0, r)$ to the Arctic Curve, and tangent to the Arctic Curve

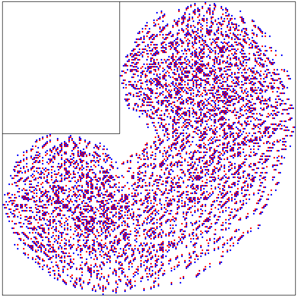
Indeed, this is what you see in a simulation...

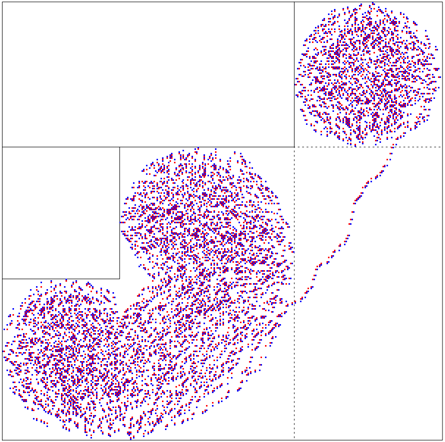
$n = 300, r = 250$



What about generic domains?

...our strategy has chances of working in general circumstances...





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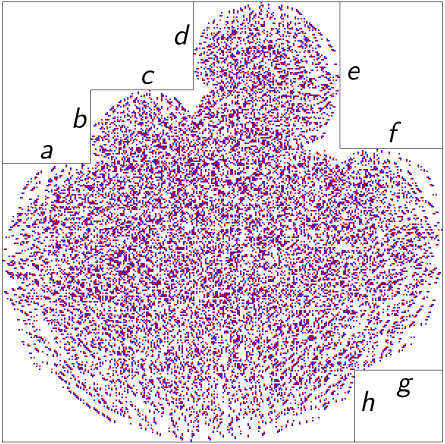
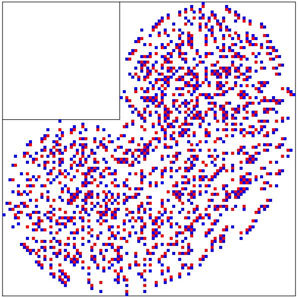
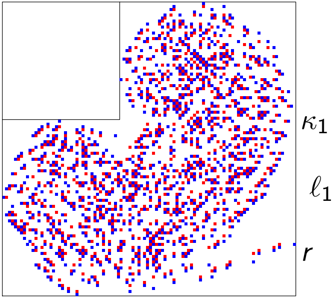


$$n = 300, \qquad (a, b, c, \ldots, h) = (60, 50, 70, 60, 100, 70, 60, 50)$$

The strategy: trying a precise statement



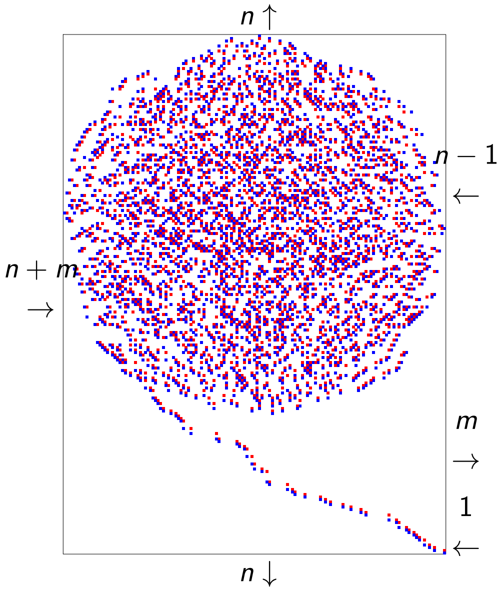


Principle 1: **Geometric Tangent Method**

Call Λ the domain shape, and $\mathcal{C}$ the corresponding Arctic Curve.

In the large n limit, a typical refined ASM on Λ, for having a $+1$ at position r along ℓ_1, shows the Arctic Curve $\mathcal{C}$ of unrefined ASM, plus a straight path from r to the tangent point on $\mathcal{C}$.

The Geometric Tangent Method in a picture



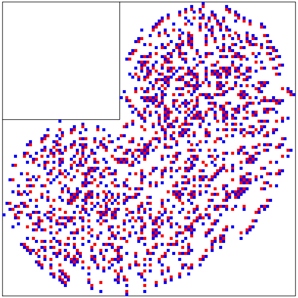
In this geometry, there is no reason for the isolated line to change direction at row n. Then:

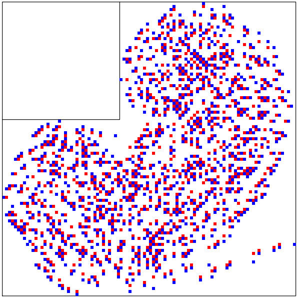
IF the arctic curve exists

IF it does not depend on m

THEN from the method we get a caustic parametrisation of the curve

The strategy: trying a precise statement



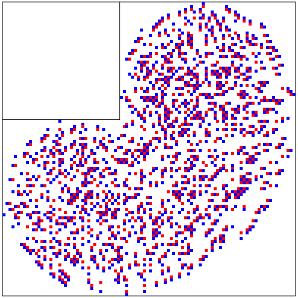


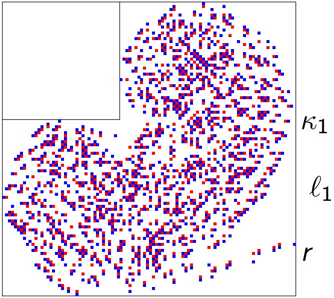
Principle 2: **Entropic Tangent Method**

Call Λ the domain shape, and $\mathcal{C}$ the corresponding Arctic Curve.

Call Λ' the domain Λ minus one row/column along the sides containing κ_1 and κ_2

The strategy: trying a precise statement



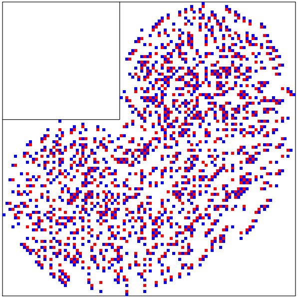


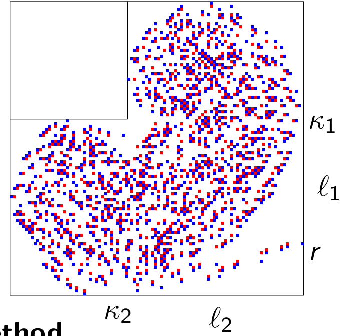
Principle 2: **Entropic Tangent Method**

Call $A(\Lambda)$ the number of ASM in Λ, and $A^{(1)}(\Lambda, r)$, $A^{(2)}(\Lambda, r)$ the refined ASM enumerations along ℓ_1 and ℓ_2

Say $X(n) \sim Y(n)$ if $\lim_{n\to\infty} \frac{1}{n} \ln \frac{Y(n)}{X(n)} = \mathcal{O}(\ln n)$.

The strategy: trying a precise statement





Principle 2: **Entropic Tangent Method**

Then

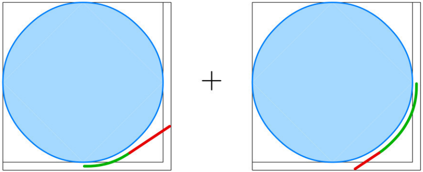
$$A^{(1)}(\Lambda, r) A^{(2)}(\Lambda, s) \sim A(\Lambda) A(\Lambda') \binom{r+s}{r}$$

If and only if the line $\big((0,r),(s,0)\big)$ is tangent to $\mathcal{C}$ (otherwise $LHS \ll RHS$)

The Entropic Tangent Method in a picture

$$A^{(1)}(\Lambda, r)A^{(2)}(\Lambda, s) \sim A(\Lambda)A(\Lambda')\binom{r+s}{r} \text{ iff } ((0,r),(s,0)) \text{ tangent to } \mathcal{C}.$$

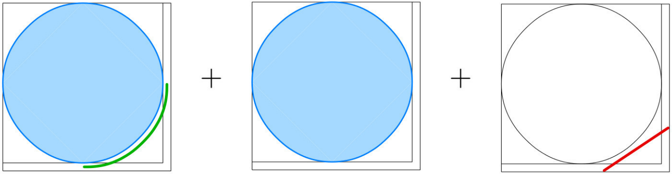
Free energy
(logarithm)
of...



$A^{(1)}(\Lambda, r)$ + $A^{(2)}(\Lambda, s)$

=

Free energy
(logarithm)
of...

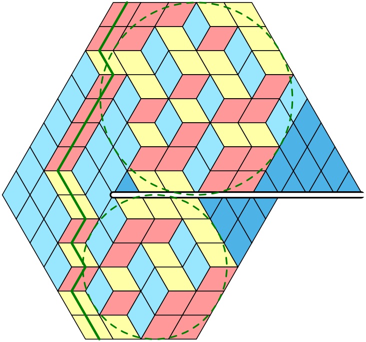


$A(\Lambda)$ + $A(\Lambda')$ + $\binom{r+s}{r}$

Does this really work?

I understand that this is not rigorous. . . but does it really work?

❶ Yes, both methods, for the Arctic Circle in lozenge tilings of the regular hexagon
(hint: use the formula for Semi-strict Gelfand Patterns
to deduce all the refined enumerations you need)



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❷ Yes, both methods, for the Colomo–Pronko $\omega = 1$ Arctic Curve

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(hint: use the formula for Semi-strict Gelfand Patterns
to deduce all the refined enumerations you need)

❷ Yes, both methods, for the Colomo–Pronko $\omega = 1$ Arctic Curve

❸ Yes, the "geometric method", for deriving the Colomo–Pronko "caustic theorem" at generic ω (I did not try the entropic method)

Well ok... what about something rigorous now?

We have also a third strategy, with a good and a bad news.

The bad news is that now you need the doubly-refined enumeration, $A^{(1,2)}(n; r, s)$

The good news is that this method can be made rigorous, and determines the arctic curve at size n, up to a $\mathcal{O}(\sqrt{n})$ band of uncertainty.

For simplicity, I discuss this new method only for the $\omega = 1$ square-domain case.

Prolog: Emptiness formation probability of anything...

For X a (deterministic or random) object (let's call it a probe), define $E_n(X)$ as the probability that $X \cap B = \varnothing$, where B is the set of positions of ± 1's in a random ASM of size n (i.e., positions of c-vertices in the 6VM)

Examples of X:

- $E_n^{\mathrm{point}}(r, s)$, a single cell at coordinate (r, s) (1-point function in the bulk);
- $E_n^{\mathrm{rect}}(r, s)$, a $r \times s$ rectangle in a corner of the domain (Colomo–Pronko EFP);
- $E_n^{\mathrm{line}}(r, s)$, a straigth segment from $(r, 0)$ to $(0, s)$;
- $E_n^{\mathrm{RW}}(r, s)$, a directed random walk from $(r, 0)$ to $(0, s)$;

Prolog: Emptiness formation probability of anything...

For X a (deterministic or random) object (let's call it a probe), define $E_n(X)$ as the probability that $X \cap B = \varnothing$, where B is the set of positions of ± 1's in a random ASM of size n (i.e., positions of c-vertices in the 6VM)

Examples of X:

- $E_n^{\mathrm{point}}(r, s)$, a single cell at coordinate (r, s) (1-point function in the bulk); too difficult to evaluate
- $E_n^{\mathrm{rect}}(r, s)$, a $r \times s$ rectangle in a corner of the domain (Colomo–Pronko EFP); viable, but still messy
- $E_n^{\mathrm{line}}(r, s)$, a straigth segment from $(r, 0)$ to $(0, s)$; clean definition, but also quite difficult to evaluate
- $E_n^{\mathrm{RW}}(r, s)$, a directed random walk from $(r, 0)$ to $(0, s)$; **easy** to evaluate, and can be **related to $\mathbf{E_n^{\mathrm{line}}(r, s)}$**!

A simple but crucial remark

Here we have our simple but crucial remark:
Principle 3: **Probe Tangent Method**

$$A^{(1,2)}(n+1; r+1, s+1) = A(n) \binom{r+s}{r} E_n^{\mathrm{RW}}(r,s)$$

The knowledge of $A^{(1,2)}(n; r, s)$ (the "row-column" doubly-refined enumeration) is not so explicit as $A^{(1)}(n; r)$, but is well under control (see e.g. ✍ Yu. Stroganov, *A new way to deal with Izergin-Korepin determinant at root of unity*)

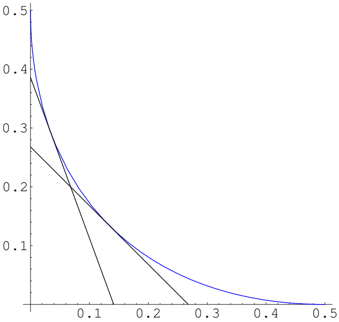
$$A^{(1,2)}(n; r, s+1) + A^{(1,2)}(n; r+1, s) - A^{(1,2)}(n; r+1, s+1) = A^{(1,3)}(n; r, s)$$

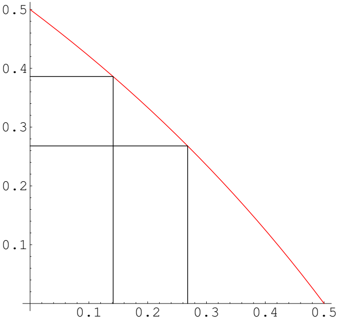
$$\begin{aligned} A^{(1,3)}(n; r, s) - A^{(1,3)}(n; r-1, s-1) &= A(n-1)^{-1} \\ &\quad \Big[A(n-1, r-1)\big(A(n,s) - A(n,s-1)\big) + (r \leftrightarrow s)\Big] \end{aligned}$$

To start: a simple transform

We want to find (the bottom-left corner of) the $\omega = 1$ arctic curve, which satisfies $x(1-x) + y(1-y) + xy = 1/4$

However, as our goal is to find it through the limit $n \to \infty$ of $E_n^{\mathrm{line}}(\rho n, \sigma n)$, we shall equivalently represent it on the (ρ, σ) plane, where it gives $(\rho, \sigma)_\theta = \left(\frac{1-\sqrt{3}\tan\theta}{2}, \frac{1-\sqrt{3}\tan(\frac{\pi}{6}-\theta)}{2}\right)$, for $\theta \in [0, \frac{\pi}{6}]$

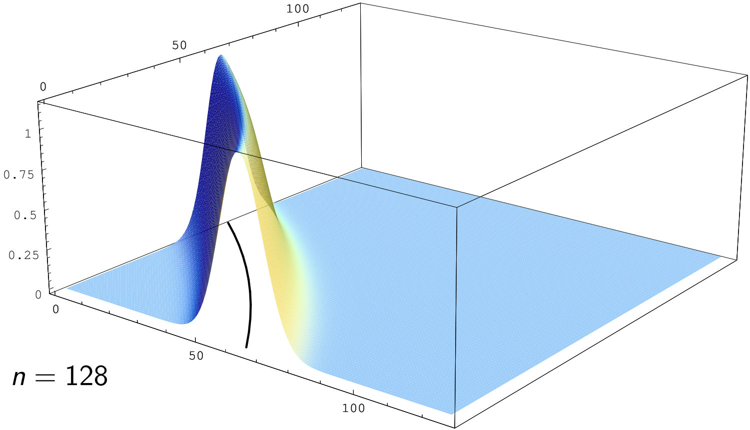




Let's have a look at $E_n^{\mathrm{RW}}(r,s)$

Let's have a look at $E_n^{\mathrm{RW}}(r,s)$, that shall converge to a step function

It is nicer to look at $-\sqrt{n}\partial_{(1,1)}E_n^{\mathrm{RW}}(r,s)$, that shall converge to a delta-function on our curve.



$n = 128$

Let's have a look at $E_n^{\mathrm{RW}}(r,s)$

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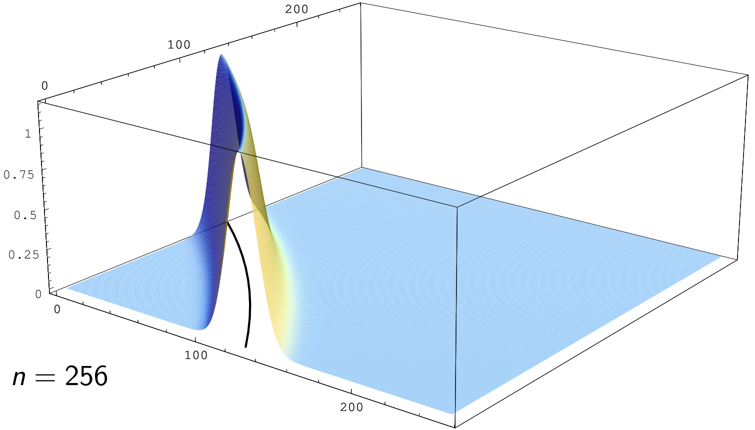
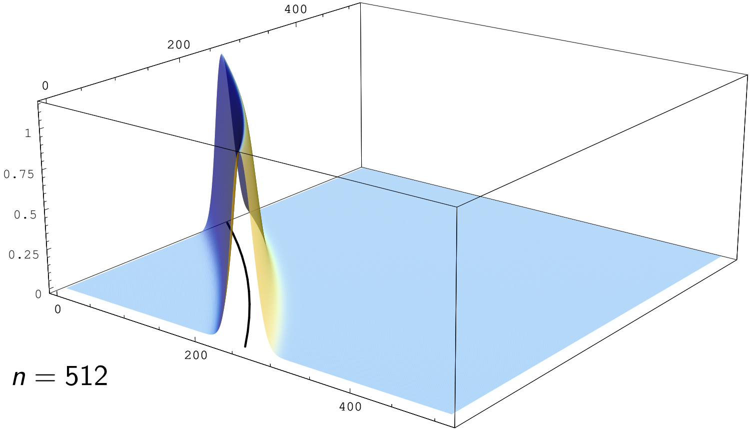


$n = 256$

Let's have a look at $E_n^{\mathrm{RW}}(r,s)$

Let's have a look at $E_n^{\mathrm{RW}}(r,s)$, that shall converge to a step function

It is nicer to look at $-\sqrt{n}\partial_{(1,1)}E_n^{\mathrm{RW}}(r,s)$, that shall converge to a delta-function on our curve.



$n = 512$

A nice accident

In fact, although $A^{(1,2)}(n; r, s)$, the row-column d-ref. enumeration is well under control, $A^{(1,3)}(n; r, s)$, the row-row d-ref. enumeration is a bit easier

By a lucky accident, we have

$$A^{(1,2)}(n; r, s+1) + A^{(1,2)}(n; r+1, s) - A^{(1,2)}(n; r+1, s+1) = A^{(1,3)}(n; r, s)$$

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By a lucky accident, we have

$$\frac{A^{(1,2)}(n;r,s+1)+A^{(1,2)}(n;r+1,s)-A^{(1,2)}(n;r+1,s+1)}{A(n-1)\binom{r+s}{r}} = \frac{A^{(1,3)}(n;r,s)}{A(n-1)\binom{r+s}{r}}$$

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$$\frac{r}{r+s}\frac{A^{(1,2)}(n;r,s+1)}{A(n-1)\binom{r+s-1}{r-1}} + \frac{s}{r+s}\frac{A^{(1,2)}(n;r+1,s)}{A(n-1)\binom{r+s-1}{r}} - \frac{A^{(1,2)}(n;r+1,s+1)}{A(n-1)\binom{r+s}{r}} = \frac{A^{(1,3)}(n;r,s)}{A(n-1)\binom{r+s}{r}}$$

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In fact, although $A^{(1,2)}(n; r, s)$, the row-column d-ref. enumeration is well under control, $A^{(1,3)}(n; r, s)$, the row-row d-ref. enumeration is a bit easier

By a lucky accident, we have

$$-\frac{r\partial_r^- + s\partial_s^-}{r+s} E_{n-1}^{\mathrm{RW}}(r,s) = \frac{A^{(1,3)}(n; r, s)}{A(n-1)\binom{r+s}{r}}$$

A nice accident

In fact, although $A^{(1,2)}(n; r, s)$, the row-column d-ref. enumeration is well under control, $A^{(1,3)}(n; r, s)$, the row-row d-ref. enumeration is a bit easier

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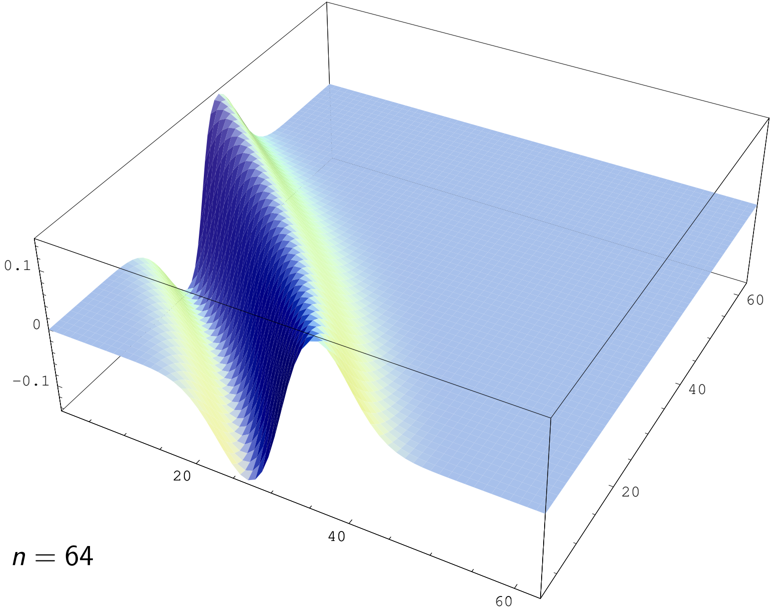
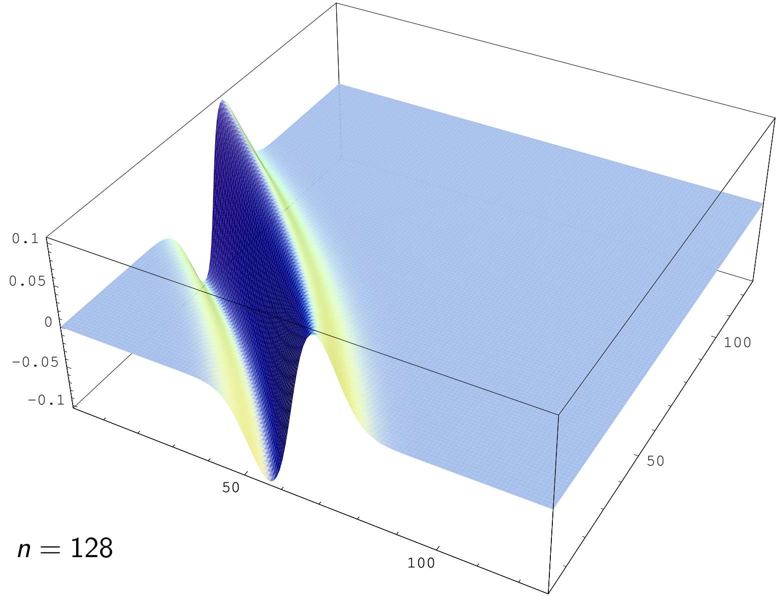
Thus $\frac{A^{(1,3)}(n;r,s)}{A(n-1)\binom{r+s}{r}}$ is sensibly larger than 0 only on the transform of the arctic curve, and its gradient along the $(1,1)$ direction shall change sign on this curve

Does this really work?



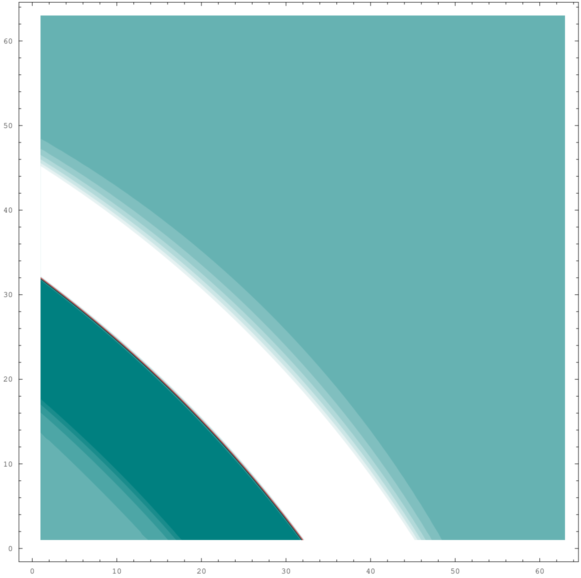
$n = 64$

Does this really work?



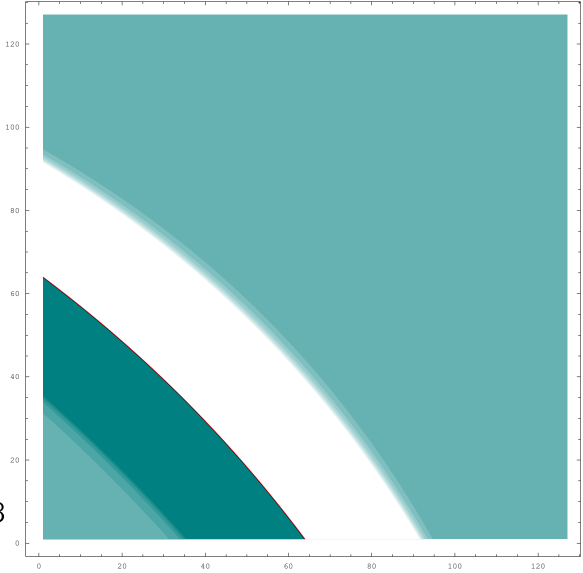
$n = 128$

Does this really work?



$n = 64$

Does this really work?



$n = 128$

From $E_n^{\mathrm{RW}}(r,s)$ to $E_n^{\mathrm{line}}(r,s)$

We now need to deduce inequalities on $E_n^{\mathrm{line}}(r,s)$ in terms of $E_n^{\mathrm{RW}}(r,s)$, in order to have a control on the arctic curve in terms of a deterministic emptiness formation probability

For any finite $\rho = r/n$, $\sigma = s/n$, the directed random walk, properly rescaled, converges to a Brownian Bridge. As such, we know the probability distribution for the maximum and minimum of the walk, in the transverse direction:

$$p(h_{\mathrm{max}} = h)\mathrm{d}h = 4h\exp(-2h^2)\mathrm{d}h$$

P. Lévy, *Sur certains processus stochastiques homogènes*, 1939
see also
J. Pitman and M. Yor, *On the distribution of ranked heights of excursions of a Brownian Bridge*, 2001
➠ Gaussian tails!

From $E_n^{\mathrm{RW}}(r,s)$ to $E_n^{\mathrm{line}}(r,s)$

A simple chain of inequalities:

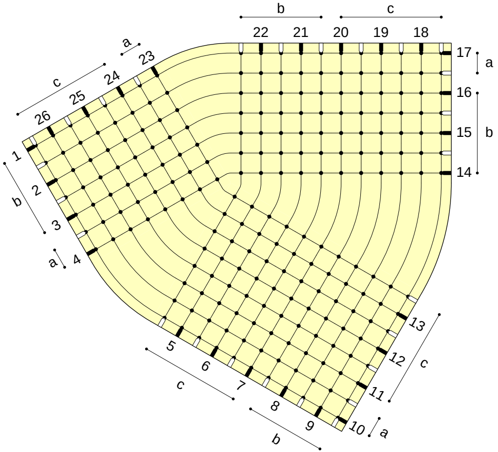
$$\begin{array}{ccccc}
\int_0^\infty p(h) E_n^{\mathrm{line}}(x+h) & \leq & E_n^{\mathrm{RW}}(x) & \leq & \int_0^\infty p(h) E_n^{\mathrm{line}}(x-h) \\
\leq & & & & \leq \\
(1-e^{-2h^2}) E_n^{\mathrm{line}}(x+h) & & & & (1-e^{-2h^2}) E_n^{\mathrm{line}}(x-h) \\
 & & \forall h > 0 & & +e^{-2h^2}
\end{array}$$

from which we get (with some other generous bounds)

$$\begin{aligned}
E_n^{\mathrm{line}}(x) & \geq \max_{h>0} \left[E_n^{\mathrm{RW}}(x+h) - \frac{e^{-2h^2}}{1-e^{-2h^2}} \right] \\
E_n^{\mathrm{line}}(x) & \leq \min_{h>0} \left[E_n^{\mathrm{RW}}(x-h) + \frac{e^{-2h^2}}{1-e^{-2h^2}} \right]
\end{aligned}$$

Well ok... what about some new result?

The severe bottleneck for obtaining arctic curves in new geometries is the absence of exact formulas for the refined enumerations...
...but we have a nice candidate, our favourite triangoloid domain!



Well ok... what about some new result?

The severe bottleneck for obtaining arctic curves in new geometries is the absence of exact formulas for the refined enumerations...
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This domain arises from the work of L. Cantini and myself on the classification of domains for which the Razumov–Stroganov correspondence holds.

As a corollary, the enumeration of all configurations factorises into $\sum_\pi \Psi_\pi = A_n \cdot \Psi_{\pi_{\min}}$. And $\Psi_{\pi_{\min}}$ is equal the number of lozenge tilings of a hexagon, $M_{a,b,c}$.

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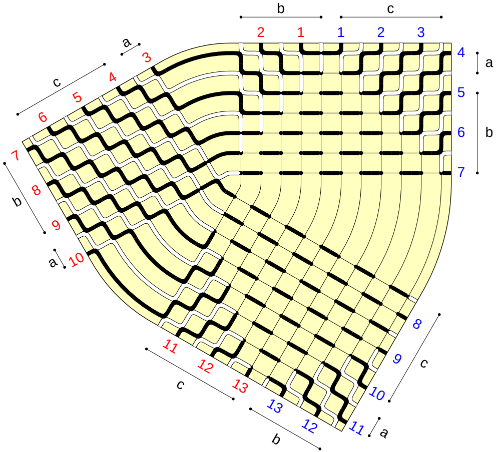

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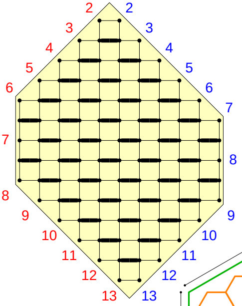
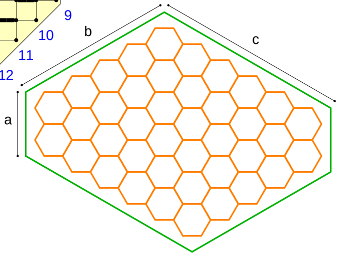


Thus $A_{a,b,c} = A_{a+b+c} M_{a,b,c}$
But in fact more is true: call $n = a + b + c$,
$A_{a,b,c}(r) = \sum_{r'} A(n, r - r') M_{a,b,c}(r')$

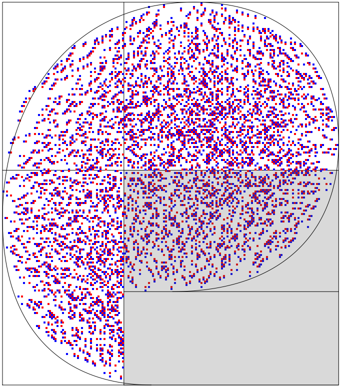


The arctic curve for the triangoloid

Very easy to find the position of tangence points κ_i.
Then, finding the arc between two of these points is harder but feasible (through the entropic method)... finally you get a parametric expression (here $a = 1 - b - c$, $p \in [0, 1]$, $q = 1 - p$)

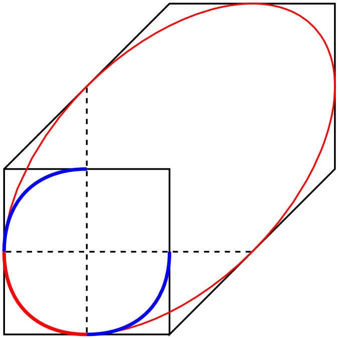
$$x(b, c, p) = \frac{3 - c}{2} - \frac{2 - p}{2\sqrt{1 - pq}} - \frac{(1 - c)(1 - (pb + qc)) - 2pbc}{2\sqrt{(pb - qc)^2 - 2(pb + qc) + 1}} ;$$

$$y(b, c, p) = x(c, b, 1 - p) .$$



Analytic continuation

The surprises are not over...
Just like the arc of the Colomo–Pronko Arctic Curve can be completed to a certain ellipse...



$$x(1-x)+y(1-y)+xy=1/4$$

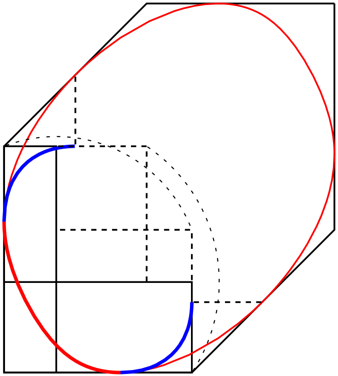
Analytic continuation

The surprises are not over...

Just like the arc of the Colomo–Pronko Arctic Curve can be completed to a certain ellipse...

...we can try to continue analytically our curve. We get a closed curve composed of 6 arcs, for the intervals $p \in (-\infty, 0], [0, 1], [1, +\infty)$, and a $\pm$-choice for square roots.

This curve is framed into a hexagonal box, with side-slopes $0, 1, \infty$ and nice rational tangence points.



The shear phenomenon

Fact:

Consider a given arc of the triangoloid arctic curve $\mathcal{C}$ (the one "near vertex A")

The two other arcs of $\mathcal{C}$ (the ones "near vertices B and C") do coincide with the 45-degree shear of the neighbouring arcs in the boxed analytic continuation of the first arc.

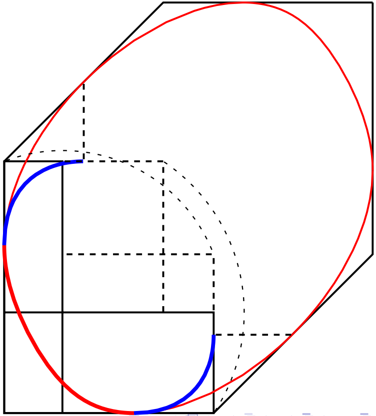
This fact is of course true also in Colomo–Pronko ellipse, but here it sounds much more striking: we have two free parameters (b/a and c/a), and the single arcs do not have a polynomial Cartesian representation

It is believable that this points towards the universality of the shear phenomenon, for any tangent point of the arctic curve $\mathcal{C}$ on its boxing domain Λ, for $\omega = 1$ ASM.

The shear phenomenon

$$x(b,c,p) = \frac{3-c}{2} - \frac{2-p}{2\sqrt{1-pq}} - \frac{(1-c)(1-(pb+qc))-2pbc}{2\sqrt{(pb-qc)^2-2(pb+qc)+1}}\,;$$

$$y(b,c,p) = x(c,b,1-p)\,.$$



What else?

The strategy of the Tangent Method in principle applies also to models beyond the 6-Vertex Model. What you need is a formulation of your configurations in terms of (interacting) non-intersecting paths, that form a sort of "rainbow".

Given this, the 1-point boundary correlation function ('refined enumeration') corresponds to evaluating the large deviation for the most external of these paths to reach a given point on the boundary.

In doing this, it produces with large probability a straight segment tangent to the arctic curve.

Let us illustrate this with another example: the 'Cauchy formula' that Petrov has shown us on monday...

Cauchy identity for generalised Hall–Littlewood polynomials

✍ A. Borodin, *On a Family of Symmetric Rational Functions*;
I. Corwin and L. Petrov, *Stochastic Higher Spin Vertex Models on the Line*;
A. Borodin and L. Petrov, *(in preparation)*

Let q, s be 'global' parameters, and u_i, v_j be spectral parameters associated to horizontal spin-$1/2$ spectral lines. Let we have also a bundle of vertical q-boson lines, with spectral parameter set to 1. Let us adopt the integrable stochastic weights, discussed in Corwin and Petrov talks, and let us assume $\left|\frac{u_i-s}{1-su_i}\right|\left|\frac{v_j-s}{1-sv_j}\right| \leq 1$.

$\frac{1-suq^g}{1-su}$ $\quad$ $\frac{(1-s^2q^{g-1})u}{1-su}$ $\quad$ $\frac{u-sq^g}{1-su}$ $\quad$ $\frac{1-q^{g+1}}{1-su}$

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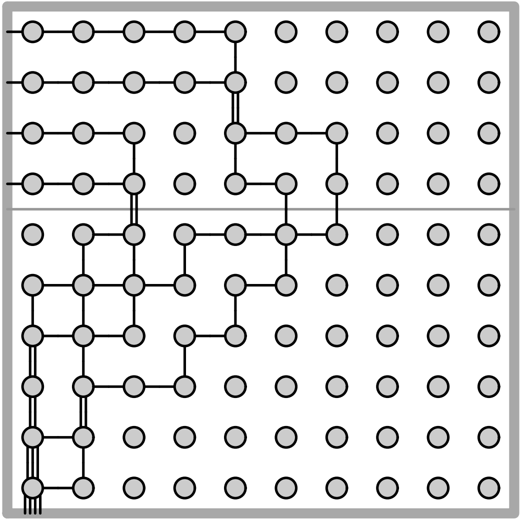
Then we have, among many other things, [Coroll. 4.7 in Borodin]

$$S_{k,n}(\vec{u}, \vec{v}) := \sum_{\lambda} c(\lambda) F_{\lambda}(\vec{u}) G_{\lambda}(\vec{v}) = \frac{(q;q)_k}{\prod_i (1 - s u_i)} \prod_{i,j} \frac{1 - q u_i v_j}{1 - u_i v_j} \,,$$

for F and G describing a suitable geometry, as depicted in the figure.

Cauchy identity for generalised Hall–Littlewood polynomials



Let us consider
$u_i = u$, $v_j = v$ for all i, j.

For real-positive weights, which are a 'large set' in $(u, v, q, s) \in \mathbb{R}^4$, the Cauchy identity can be seen as a partition-funct. calculation for configs (F, G) with a probabilistic measure.

What are the arctic curves associated to these configs?

Cauchy identity for generalised Hall–Littlewood polynomials

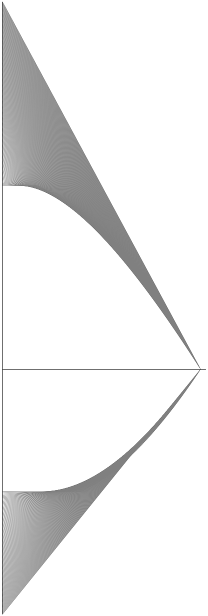
By the magic of the Cauchy identity, it is easy to extract the refined generating functions, for the only turn in the first or in the last row. The equivalent of an isolated walk is here given by a hypergeometric series (instead of $\binom{r+l}{r}$, we have here $\sum_m x^m \binom{r}{m}\binom{l}{m}$, the same happens for ASM's at $\omega \neq 1$).

We can apply, e.g., the Geometric Tangent Method, and obtain the limit arctic curve.

The expression in terms of $(u, v, s, q, k/n)$ is too long for being written down here. The top part of the curve is the portion of height k of an infinite curve that does not depend on k (similar facts hold for the bottom part). Scaling n to 1, it reads...

Cauchy identity for generalised Hall–Littlewood polynomials

$$\ell^*(r) = ab\frac{(2+2a-2b+bc-bcr)-R}{2(1+a-b)(1+a-b+bc)}$$
$$\cdot\frac{2b(c-1)+(1+a)(2-c-cr-R/b)}{(2(1-b)(1-b+a+bc)+abc(1+r)+aR)}$$
$$a = \frac{(1-q)(1-s^2)v}{(qv-s)(1-sv)}$$
$$b = \frac{(v-s)(u-s)}{(1-vs)(1-us)}$$
$$c = \frac{(1-q)(1-s^2)u}{(u-s)(1-qsu)}$$
$$R^2 = \frac{bc}{a}\left(abc(1+r)^2+4r(1-b)(1+a-b+bc)\right)$$



We can have a look at some specific values, e.g. $(q, s, u, v, k/n) = (1/3, 1/3, 1/2, 3/4, 2/3)$:

$$\ell_+(r) = \frac{1}{322}(6 + 29r + 10\sqrt{16 + 262r + 16r^2})$$
$$\ell_-(r) = \frac{1}{644}(87 + 12r + 20\sqrt{36 + 393r + 16r^2})$$