Integrability, Topology and Discrete "Holomorphicity"

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I’m going to describe how topology provides a very useful fundamental link between integrable lattice models, conformal field theory, and discrete ``holomorphicity’’.

The moral of the story is: draw pictures – no complicated representation theory needed!

The ingredients:

- Integrability and the Yang-Baxter equation
- Knot and link invariants such as the Jones polynomial
- Discrete "holomorphicity" of lattice operators
- CFT/anyon/TQFT physics; MTC mathematics

The results:

- Discrete "holomorphicity" is best seen as current conservation.

- It is very natural when lattice models are described topologically.

- It provides a simple way to turn topological invariants into integrable Boltzmann weights, i.e. "Baxterize"

- It gives "conformal" defects in lattice models.

Integrability from the Yang-Baxter equation

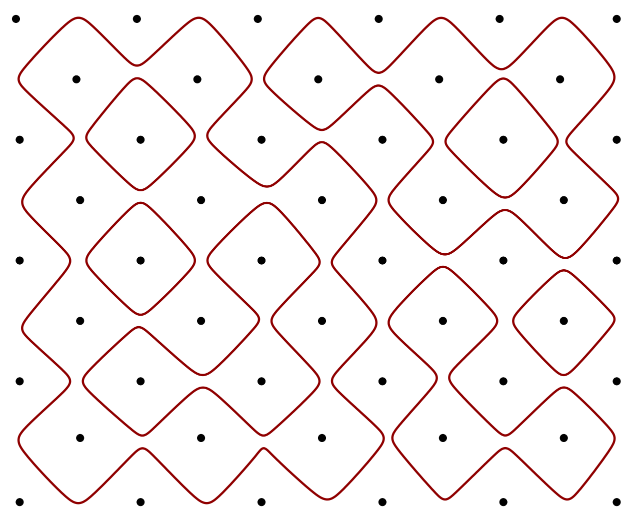
- The Boltzmann weights of a two-dimensional classical integrable model typically satisfy the Yang-Baxter equation.

- It is a functional equation; the Boltzmann weights must depend on the anisotropy/spectral/rapidity parameter u.

- Its consequence is that transfer matrices at different u commute, thus ensuring the existence of the conserved currents necessary for integrability.

The completely packed loop model/ Q-state Potts model

Every link of the square lattice is covered by non-crossing loops; the only degrees of freedom are how they avoid at each vertex.

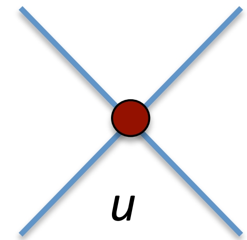


$$Z = \sum d^{n_l} v^{n_v} h^{n_h}$$

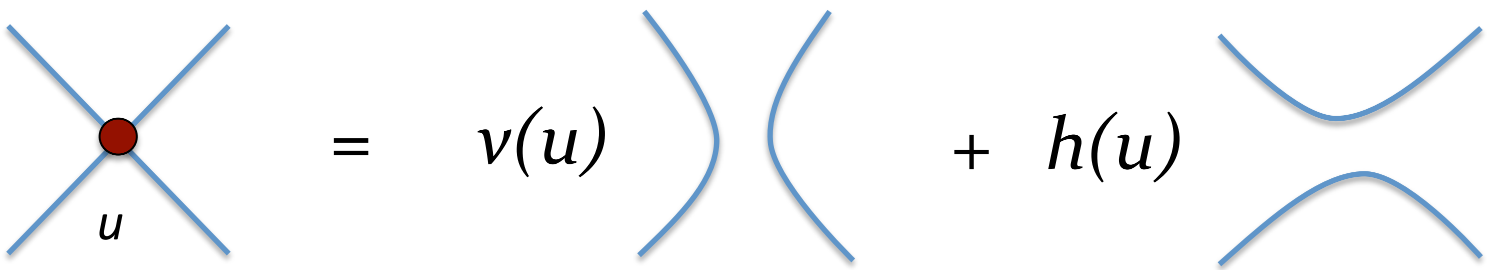
d is the weight per loop, $v(u)$ the weight per vertical avoidance, $h(u)$ the weight per horizontal avoidance.

The Boltzmann weights, pictorially

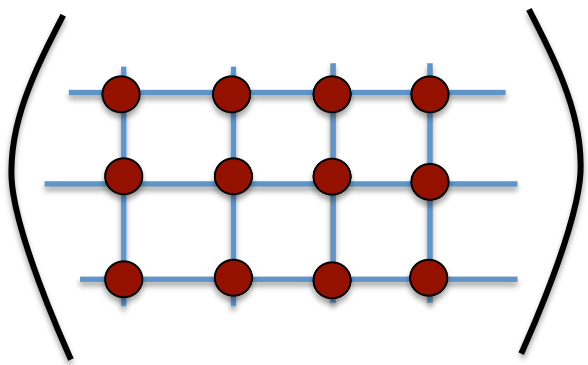
Picture the Boltzmann weights on the square lattice as



so for the completely packed loop model



$$Z = \text{eval}$$



where eval means to expand out each vertex, and sum over all loop configurations with weights $d^{n_l} v^{n_v} h^{n_h}$.

Many (all critical integrable?) lattice models can be written in a geometrical/topological form

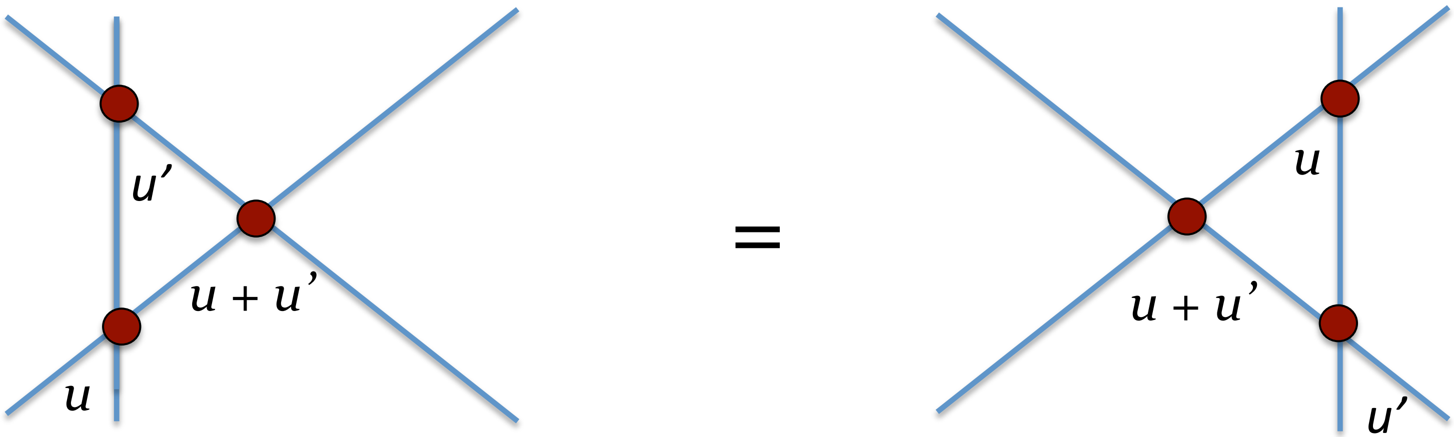
$$Z = \sum_{\text{graphs}} \text{(topological weight) x (local weights)}$$

$$\sum_{\text{completely packed loops}} d^{n_l} v^{n_v} h^{n_h}$$

- Ising/Q-state Potts models from FK expansion/TL algebra
- Ising/parafermion models in their domain wall expansion
- Height/RSOS models based on quantum-group/braid algebras

The YBE, pictorially

Sums of products of three Boltzmann weights must obey



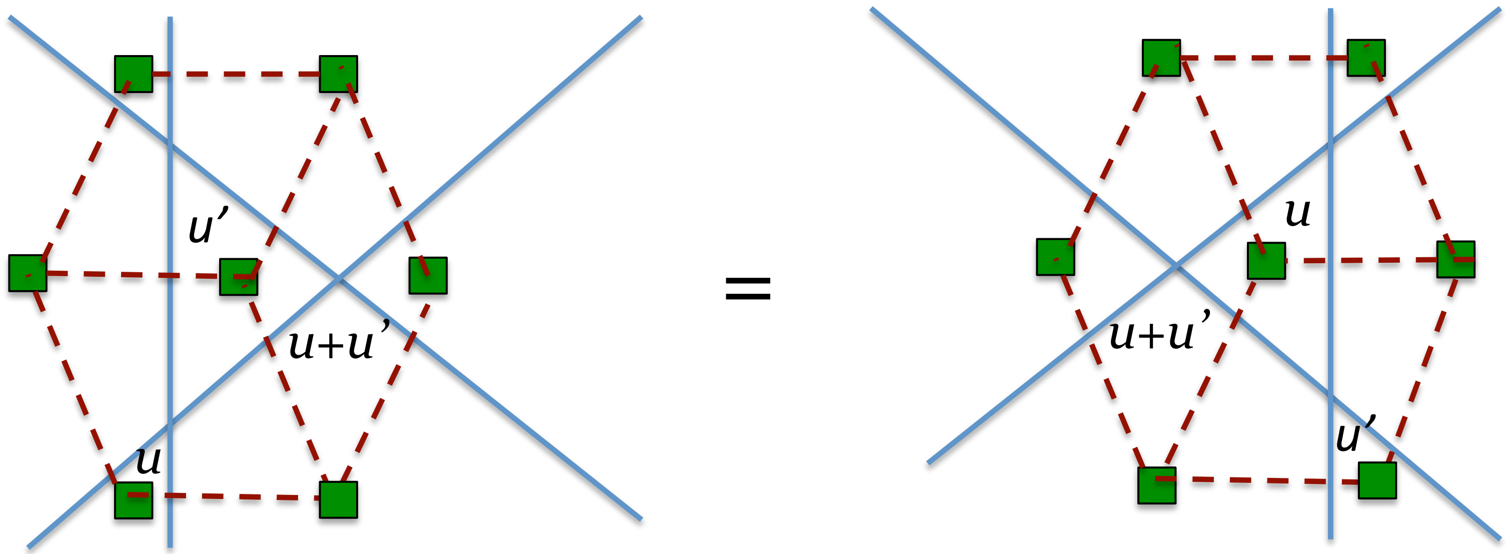
where I no longer write the eval( ).

u and u' have changed places; this leads to commuting transfer matrices.

This equation is consistent with thinking of u and u' as angles.

In terms of heights/spins:

Heights/spins live on the faces of the lattice formed by the loops:



where on both sides there is a sum over the central height.

In terms of heights/spins, the YBE remains consistent with thinking of u and u' as angles.

The YBE for completely packed loops

Plugging the Boltzmann weights into the YBE gives (wildly overconstrained) functional equations.

Setting $w(u) = \dfrac{v(u)}{h(u)}$ yields

$$w(u)w(u+u')w(u') + d\, w(u)w(u') + w(u) + w(u') - w(u+u') = 0$$

Parametrize the weight per loop by $d = q + q^{-1}$. Then

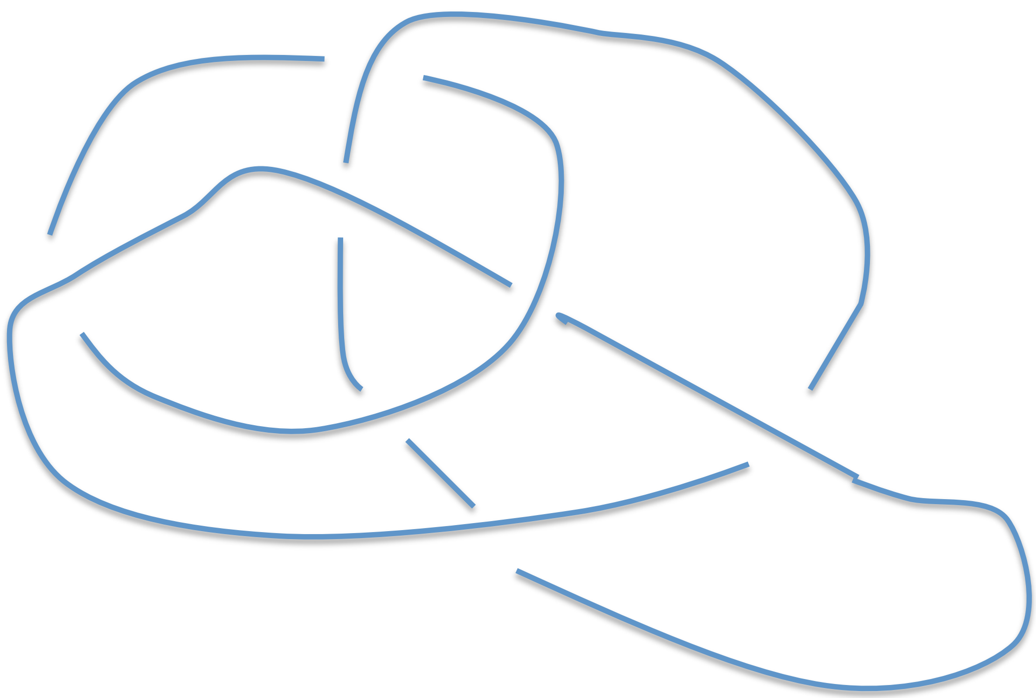
$$w(u) = \frac{qe^{-iu} - q^{-1}e^{iu}}{e^{iu} - e^{-iu}}$$

How does something so simple arise from such a complicated equation?

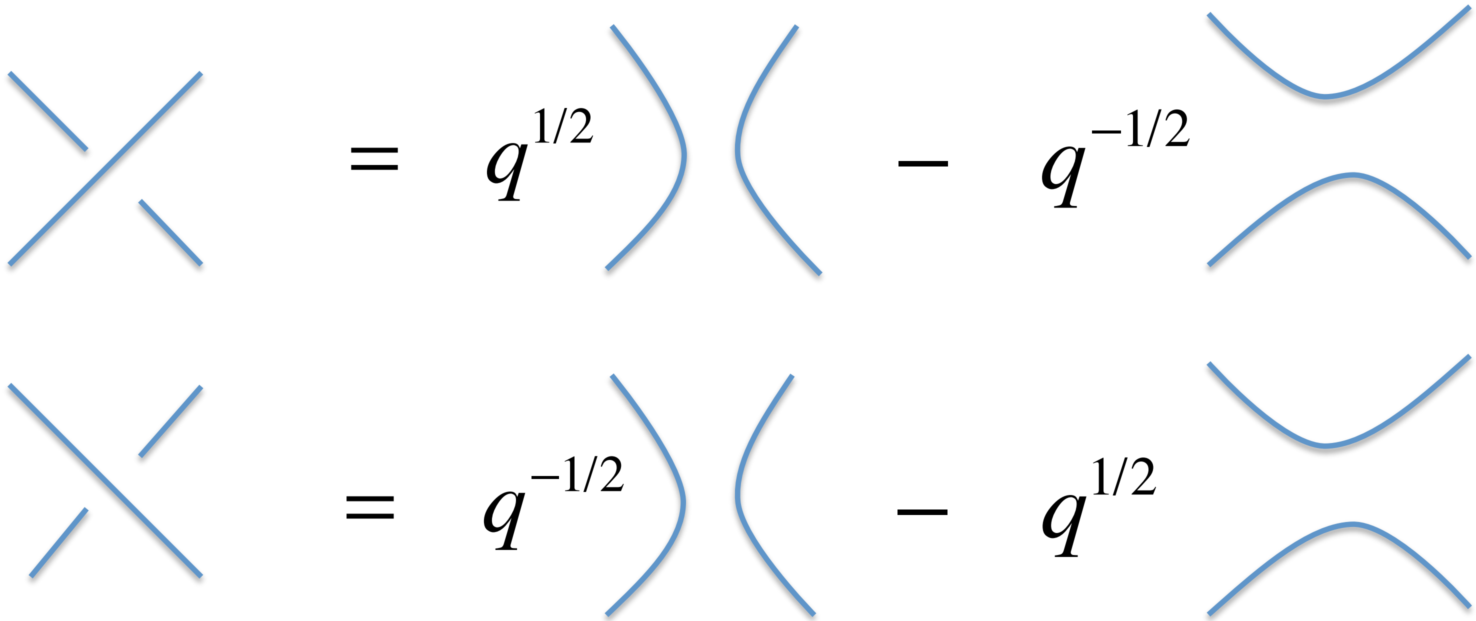
And now for something completely similar:
knot and link invariants

A knot or link invariant such as the Jones polynomial depends only on the topology of the knot.

To compute, project the knot/link onto the plane:

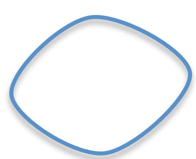


Then ``resolve'' each over/undercrossing and turn each knot/link into a sum over planar graphs. For the Jones polynomial:



This turns each link into a sum over graphs of closed loops.

To evaluate the Jones polynomial (in q), replace each loop with

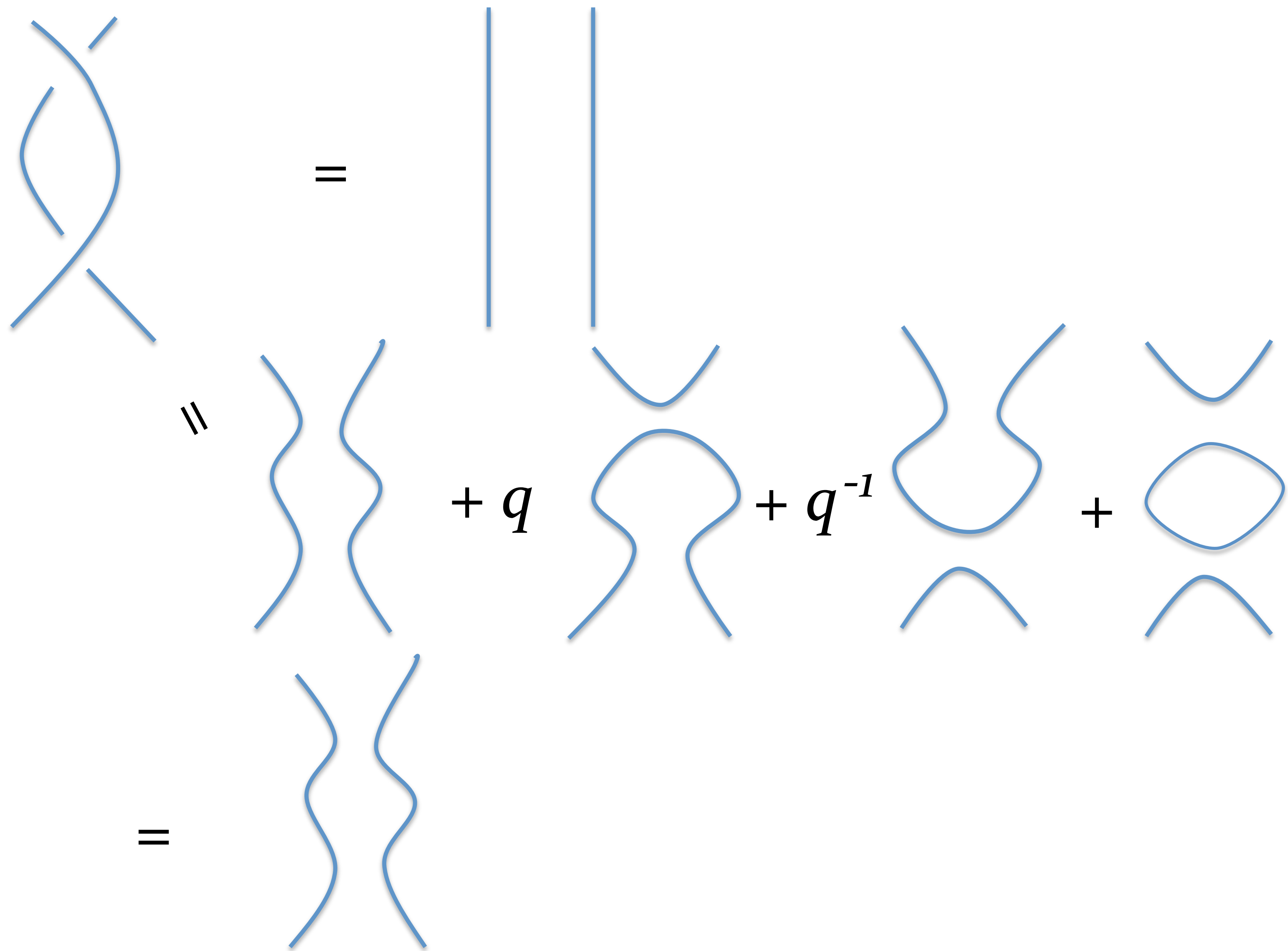


$$= q + q^{-1} = d$$

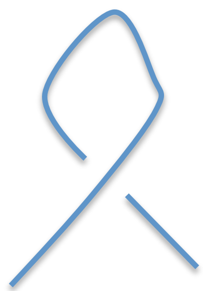
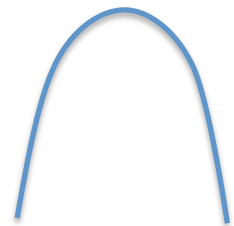
just like before!

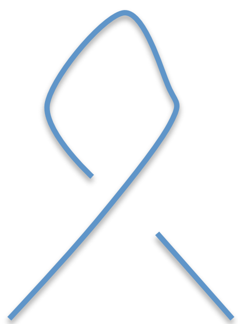
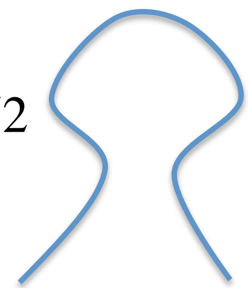
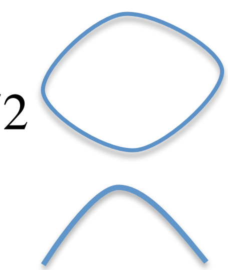
To be a topological invariant, must satisfy the Reidemeister moves:

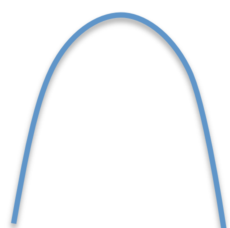
#2



One extremely important subtlety: Need

#1:  $=$ 

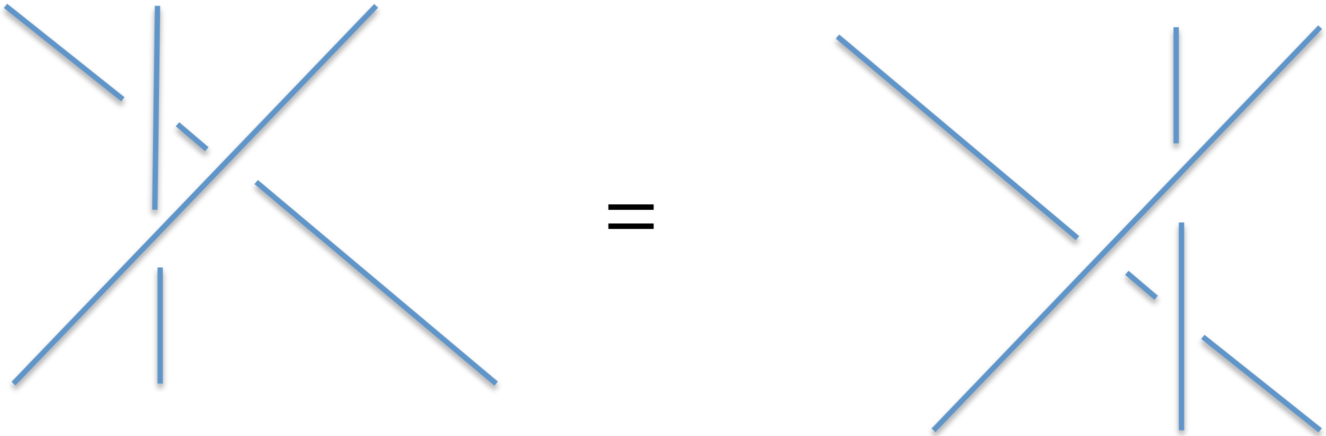
But instead:  $= q^{1/2}$  $+q^{-1/2}$ 

$= -q^{-3/2}$ 

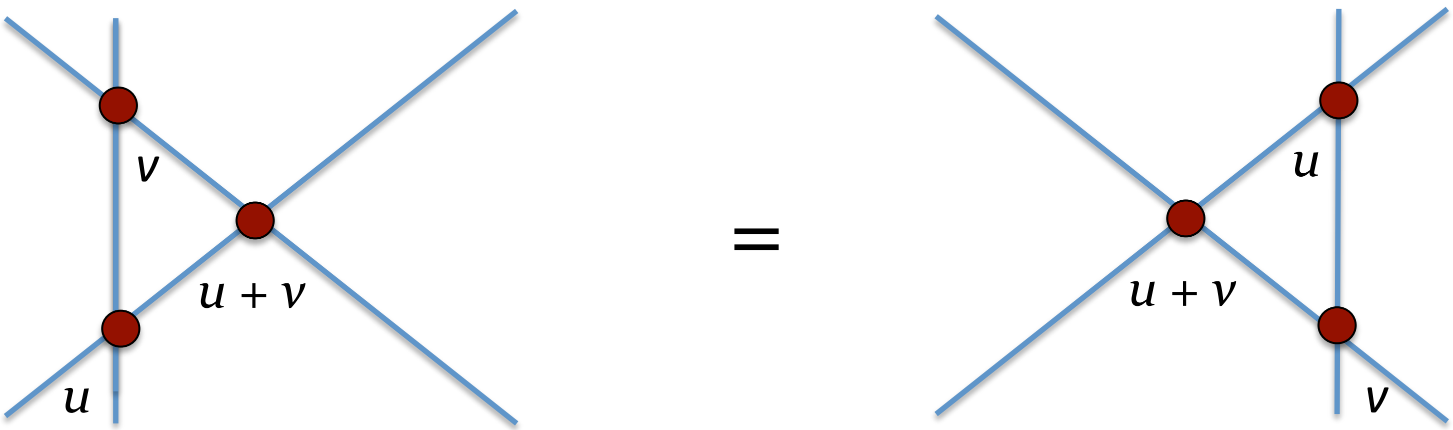
To have a topological invariant, make each link a ribbon, and keep track of twists. Then multiply by $q^{3w/2}$, where

w= #(signed twists) = writhe.

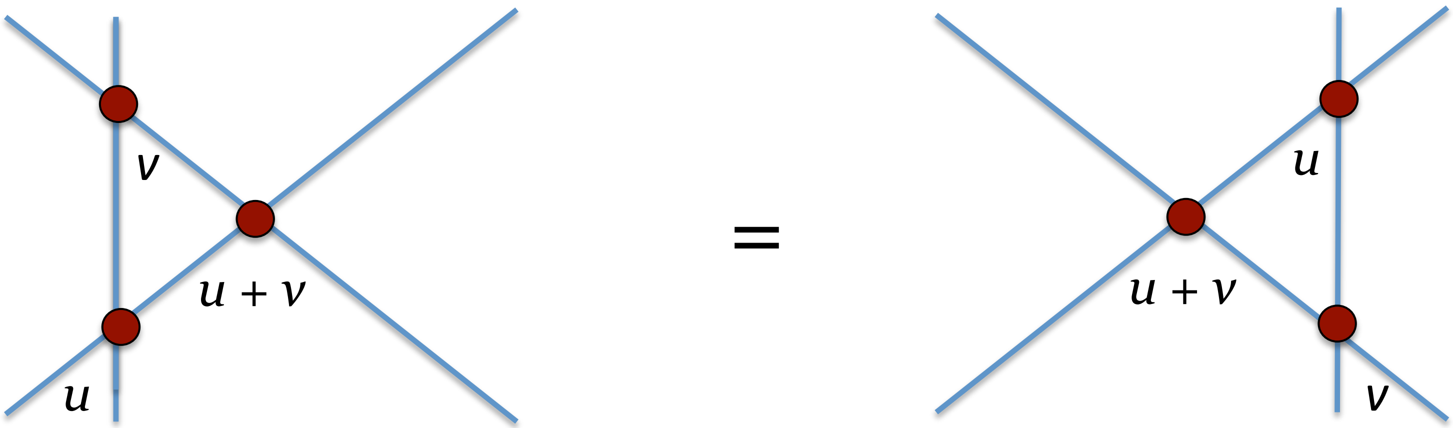
#3:

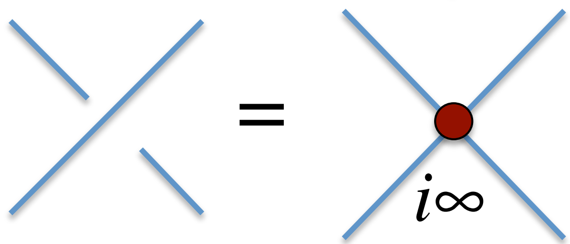


Solutions of this can be found by taking a limit of the YBE!



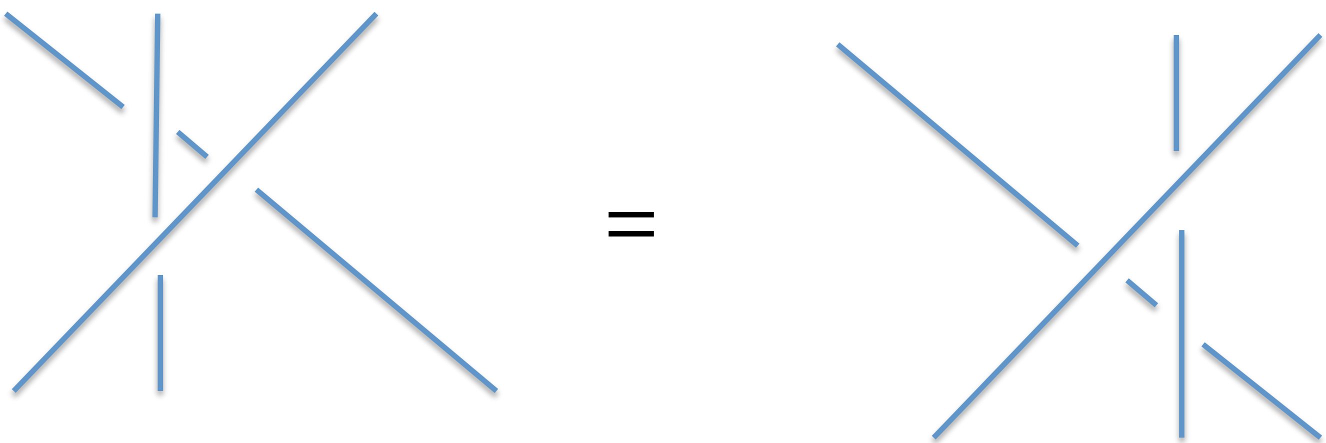
The theme of this talk is to reverse the arrow – to use topology to find (critical) solutions of the YBE!





“Baxterize”

#3:



Review by Wadati Deguchi Akutsu

And now for something newer: discrete "holmorphicity"

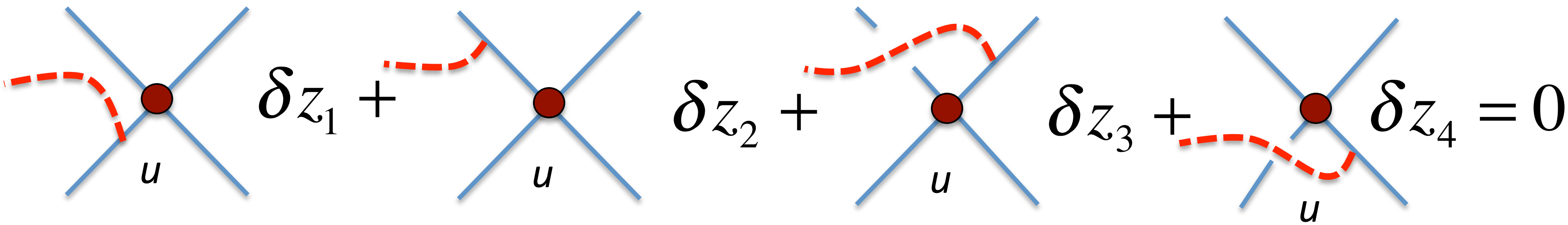
- An operator $O(z)$ in some two-dimensional lattice model is discrete "holomorphic" if its expectation values obey the lattice Cauchy-Riemann equations, i.e. around a closed path

$$\sum O(z_i)\delta z_i \quad = \quad 0$$

- An example is the fermion operator in the Ising model.

- Smirnov et al have exploited this to prove conformal invariance of the continuum limit of the Ising model.

Discrete “holomorphicity” can be described pictorially as



The trivalent vertex corresponds to the D.“H.” operator. Defining it and the crossing has been typically ad hoc (i.e. guess until you find something that works).

I will explain how CFT/anyon/TQFT physics/MTC mathematics provides a systematic and general way of defining these objects.

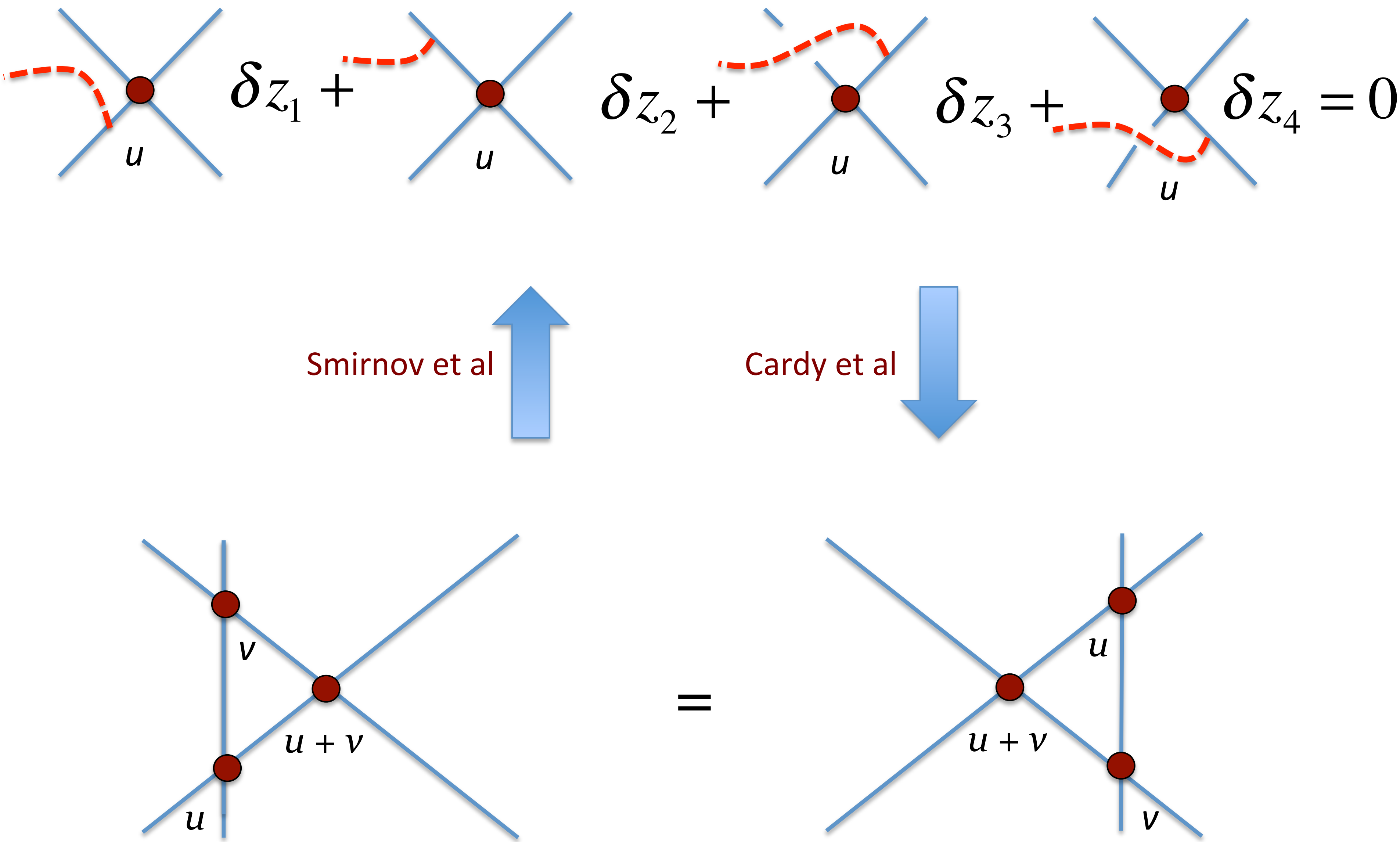
Cardy, collaborators and successors have found such discrete holomorphic operators in many integrable lattice models.

Riva and Cardy; Rajabpour and Cardy; Ikhlef and Cardy; de Gier et al; Batchelor et al; Ikhlef and Weston…

Cardy et al also reversed the order of the logic in an interesting way.

They did not require a priori that the Boltzmann weights satisfy the YBE. Requiring discrete "holomorphicity" then gives a linear condition on the Boltzmann weights.

In all the examples studied, the solution of this linear equation gives weights solving the YBE!



The reason for the quotes on “holomorphic” is that the equation is not sufficient to guarantee holomorphicity in the continuum.

The reason is obvious. There is:

one relation for each vertex, while

one operator for each link.

Twice as many variables as constraints!

Ising is a rare case where this construction does yield holomorphicity; there is another relation there.

An explicit counterexample

- By using discrete symmetries and numerics (DMRG), we found in the Hamiltonian limit explicit lattice analogs of all relevant operators of the 3-state Potts CFT, including holomorphic ones like the parafermion and the energy-momentum tensor.

  Mong, Clarke, Alicea, Lindner and Fendley

- For the $\mathbb{Z}_3$ parafermion, we showed the usual construction (order times disorder ops) does not yield a holomorphic operator – it yields a mixture of the holomorphic parafermion operator of dimensions (2/3,0) with one of dimensions (1/15,2/5). The same OPE occurs in the CFT.

- Nevertheless, we showed how to separate them. Can this be generalized to the full 2d classical 3-state Potts model?

The connection to integrability still makes this lattice ``Cauchy-Riemann'' equation fascinating and worth studying.

There's another use as well, that will allow us to rename it.

It's not like this equation has never been seen before…

QUANTUM GROUP SYMMETRIES IN TWO-DIMENSIONAL LATTICE QUANTUM FIELD THEORY

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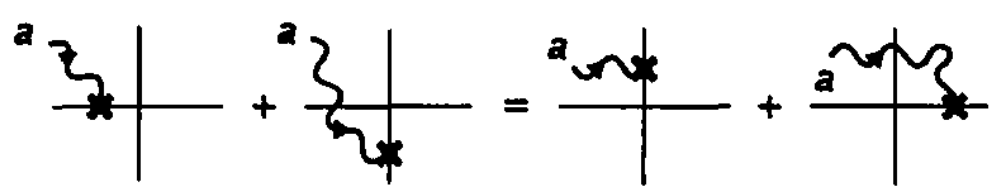
Received 2 April 1991

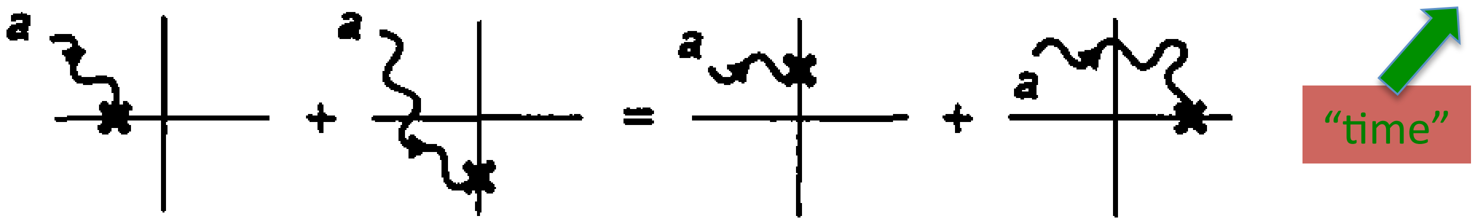
We present a general theory of non-local conserved currents in two-dimensional quantum field theory in the lattice approximation. They reflect quantum group symmetries. Various examples are studied.

The graphical representation of eqs. (2.7) and (2.8) is then

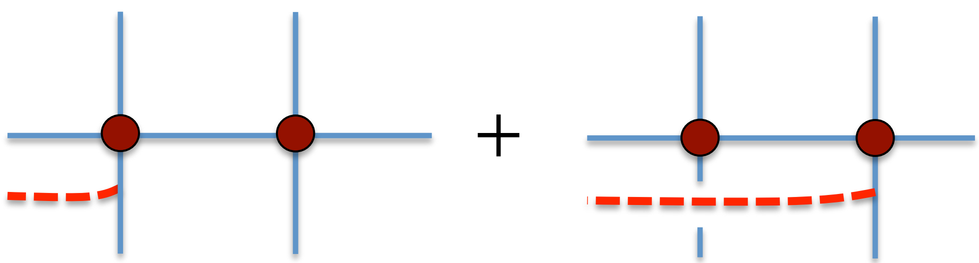


(2.10)

The discrete ``holomorphic'' operator is part of a conserved current!



Let Ψ be the sum over the operators on the vertical links:

$\Psi = \ldots +$  $+ \ldots$

Then Ψ is a zero mode! With appropriate choice of boundary conditions, it commutes with the transfer matrix/Hamiltonian:

$$[H, \Psi] \;=\; 0$$

Bernard and Felder find this conserved current/zero mode using the quantum-group symmetry present in many (all?) critical integrable models.

Subsequently this connection was illuminated, but the work is rather technical.

Ikhlef, Weston, Wheeler, and P. Zinn-Justin

The moral here today is draw pictures!

~~Discrete Holomorphicity~~

Zero mode

How are these zero modes related to integrability?

We saw how integrability is related to topology via knot/link invariants and the Yang-Baxter equation.

However, solutions of the YBE depend on a parameter, the angle u. One must ``Baxterize'' the knot invariant to obtain the Boltzmann weights.

Finding a zero mode requires solving much simpler linear equations.

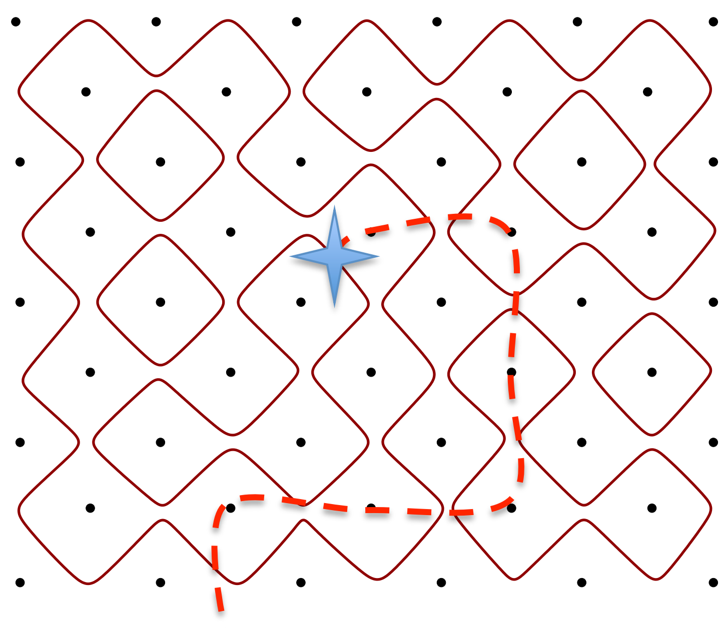
Topology allows this to be done in a very natural fashion.

This allows the results to be extended to many integrable models.

Zero mode + braiding/fusing $\stackrel{?}{=}$

integrability of a critical lattice model

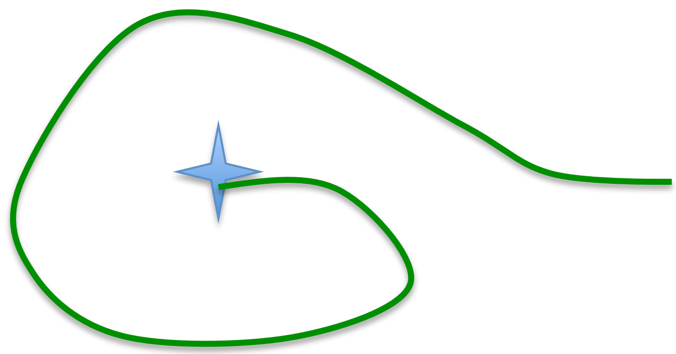
A key connection comes from looking at the way the zero modes are defined. They have a string attached:



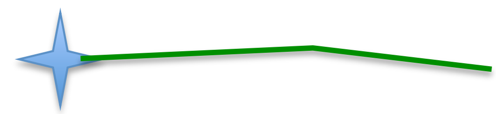

labels the operator $O(z)$ acting at point z.

In Ising, this is the familiar Jordan-Wigner string of spin flips.

The expectation value $\langle O(z) \rangle$ is independent of the string's path except for the total winding angle:

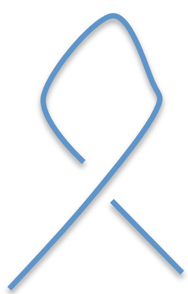


$$= e^{2\pi i h}$$

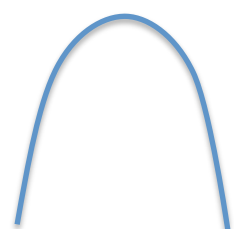


Picking up a phase under rotation of 2π is characteristic of a holomorphic object of ``dimension'' h.

It is also characteristic of the twisting of a ribbon!



$$= -q^{-3/2}$$



To make this correspondence precise, we need to understand more about the algebraic structure underlying braiding.

Luckily, this is understood extremely well.

The rules were systematized by Moore and Seiberg in order to understand chiral operators in 2D conformal field theory.

TAMING THE CONFORMAL ZOO

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Received 18 January 1989

All known rational conformal field theories may be obtained from $(2+1)$-dimensional Chern–Simons gauge theories by appropriate choice of gauge group. We conjecture that all rational field theories are classified by groups via $(2+1)$-dimensional Chern–Simons gauge theories.

1. Introduction

The problem of the classification of all conformal field theories is a useful problem to orient the research about the more interesting and more important problem of uncovering the meaning of conformal field theory, and, perhaps, string theory. An

the structure uncovered in ref. [3] is neatly summarized by 3D general covariance. In ref. [6] the connection between two- and three-dimensional theories was established only for WZW models [8] based on a simply connected compact Lie group G. In this letter we show that all known RCFT's are equivalent to some CSGT thus organizing the entire zoo of known

In the math world, the relevant structure is called a modular tensor category.

In mathematical physics, a topological quantum field theory.

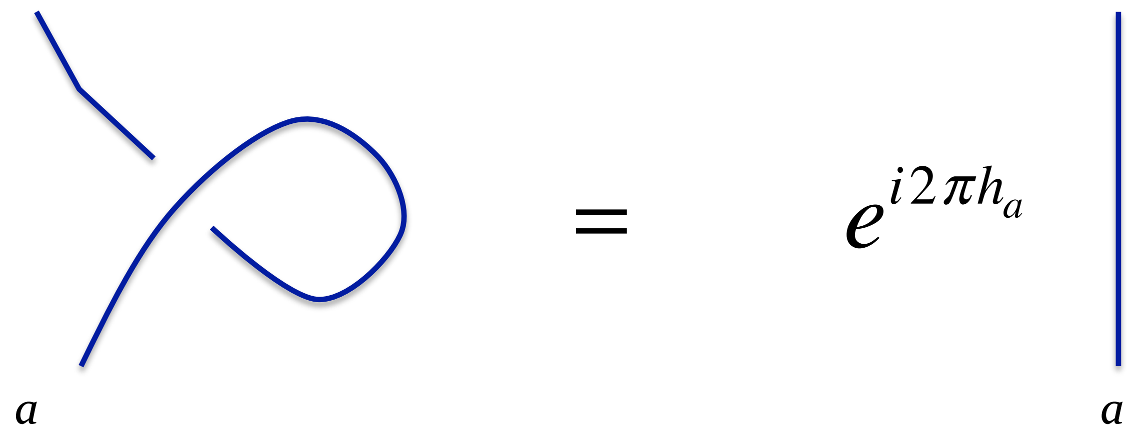
In its original context, a rational conformal field theory.

Or in the condensed-matter world nowadays, consistent braiding and fusing relations for anyons.

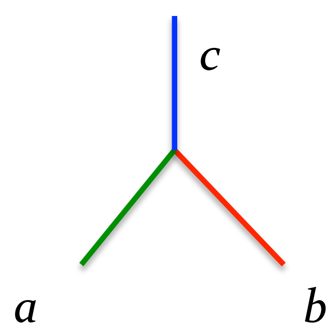
There are now many explicit examples of this structure.

The basic rules we need for RCFT/MTC/TQFT/anyons

1) spin/conformal dimension h_a of each type of anyon/operator:



2) The behavior under fusion, i.e. how to treat a combination of anyons as a single one. We use this vertex to define zero mode.



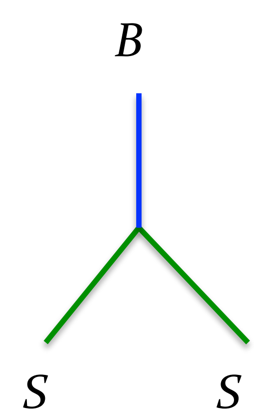
Fusion

Even in the Abelian case, fusing is non-trivial.

For example, when braiding two identical ``semions'', the wave function picks up a factor of i.

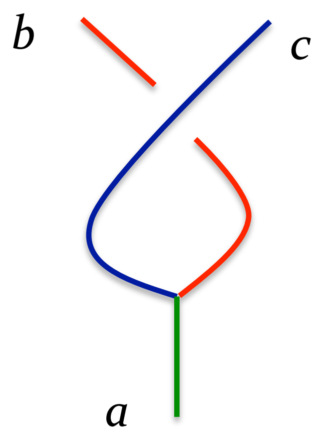
When a pair of semions is braided with another pair, the wavefunction picks up a factor of $i^4 = 1$.

Two semions make a boson!

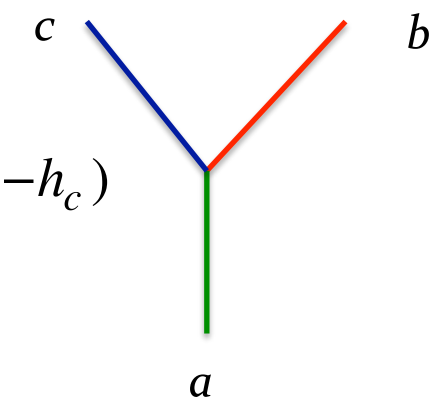


Many consistency conditions allow braiding and fusing to be found

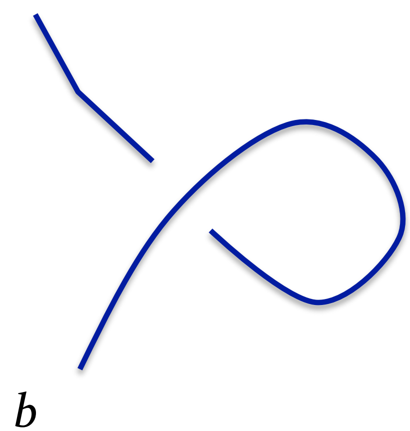
A simple one:



$$= e^{i\pi(h_a - h_b - h_c)}$$



If a is the identity this reduces to the twist:

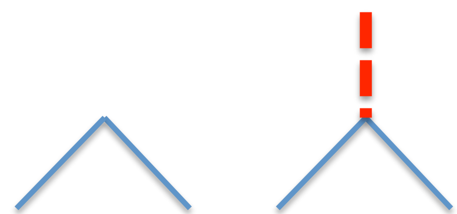


$$= \quad e^{i2\pi h_b}$$

In the non-Abelian case, fusion is a straightforward generalization of tensoring representations of Lie algebras.

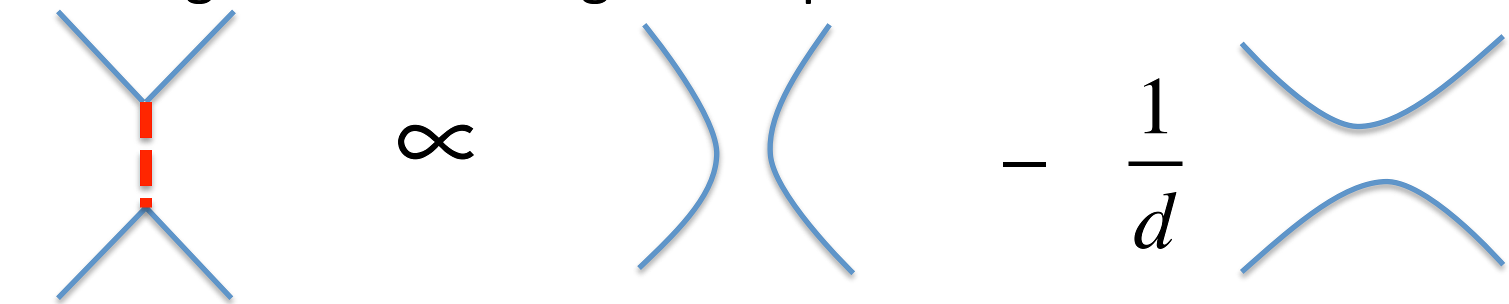
For the Jones polynomial/CPL, this is akin to spin-1/2 of sl(2):

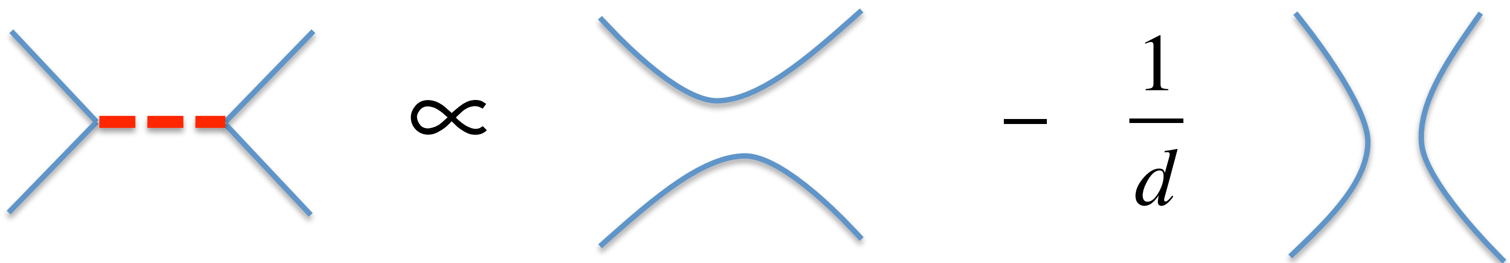
$$\frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1$$



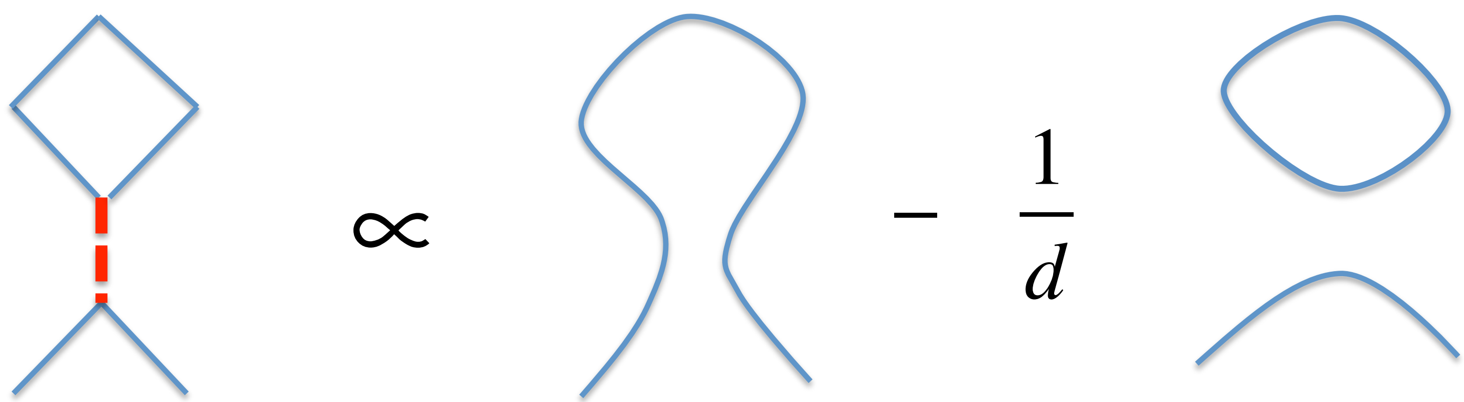
Two ways to tensor four spin-1/2 particles into an overall singlet:

Relating this to the original loop basis:



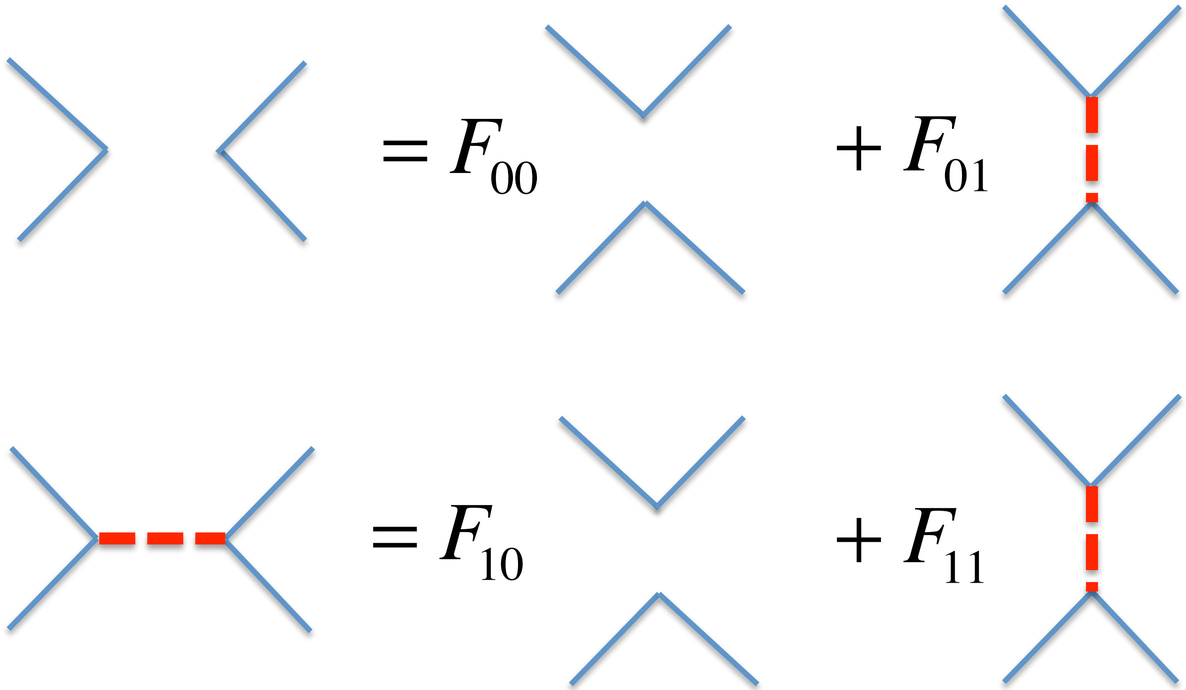


So spin 1 and spin 0 are orthogonal in the sense of:



$$= 0$$

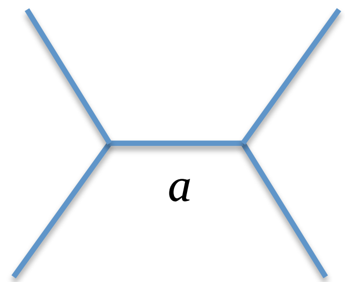
The F matrix governs this change of basis. A myriad of consistency conditions determine it. For Jones/CPL,



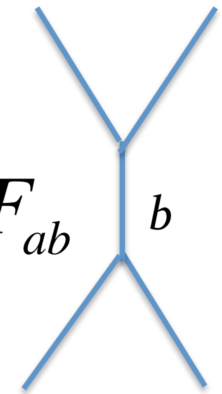
$$F = \frac{1}{d}\begin{pmatrix} 1 & \sqrt{d^2-1} \\ \sqrt{d^2-1} & -1 \end{pmatrix}$$

The braid matrix follows from F and h:

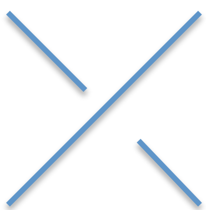
The F matrices in general:



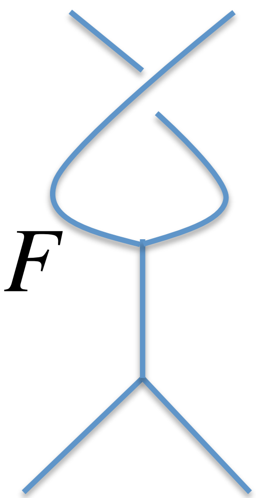
$$= \sum_b F_{ab}$$



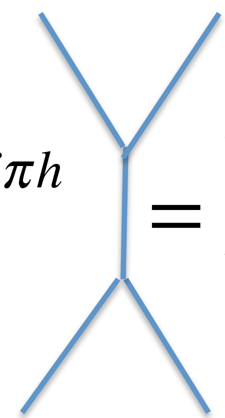
Schematically



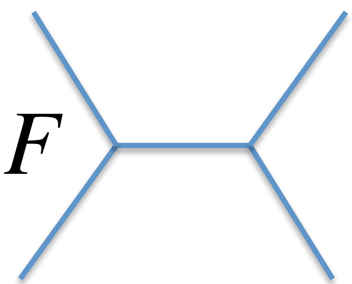
$$= \sum F$$



$$= \sum F \, e^{i\pi h}$$

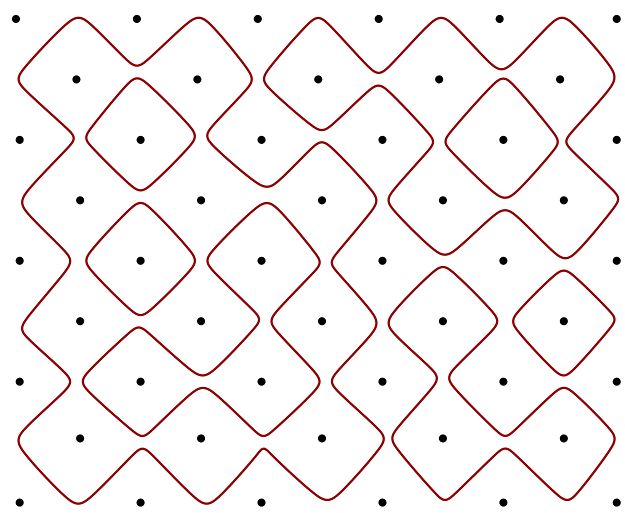


$$= \sum F e^{i\pi h} F$$



Putting all this together to get a zero mode

Lattice models in terms of geometric degrees of freedom:

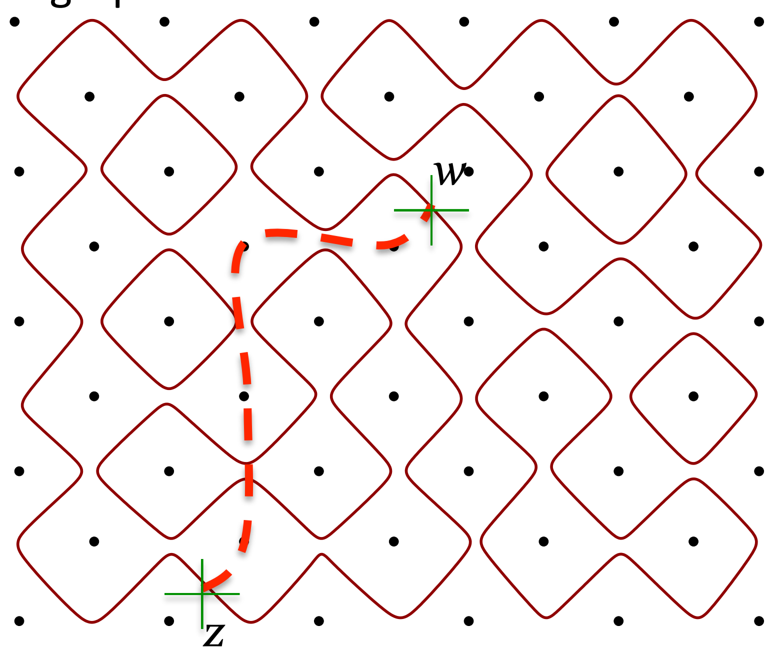


$$Z = \sum_{\text{completely packed loops}} d^{n_l} v^{n_v} h^{n_h}$$

$$= \sum_{\text{graphs}} \text{(topological weight) x (local weights)}$$

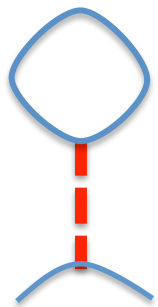
A zero mode is defined by modifying the topological part

$$\langle \psi(z)\psi(w)\rangle = \frac{1}{Z}\sum_{\text{graphs}} \text{Eval(3d graph) x (local weights)}$$



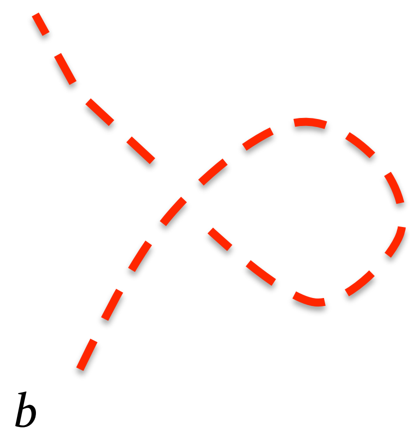
Take the string to come out of the plane, and use the braiding/fusing rules of the MTC/CFT/TQFT/anyons to evaluate the graph.

For CPL/Potts, this rule is simple: the weight is zero unless the string connects two points on the same loop.



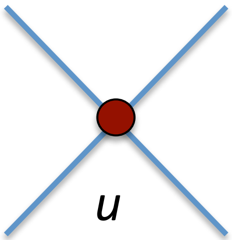
$$= 0$$

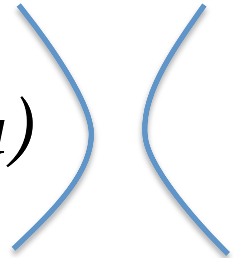
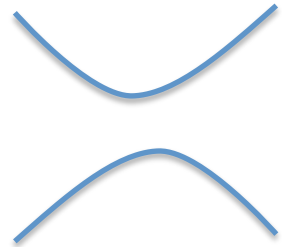
Moreover, the 3d rules also mean that the correlator is independent of the string path unless there is a twist!

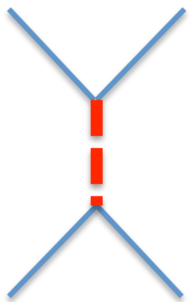
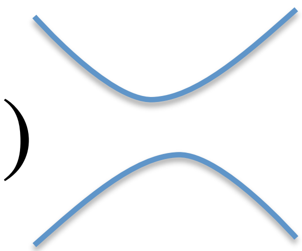


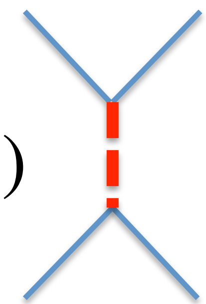
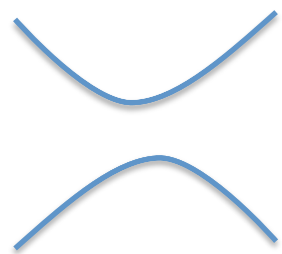
$$= \quad e^{i2\pi h_b}$$

Can use the F matrix to rewrite the Boltzmann weights, e.g.

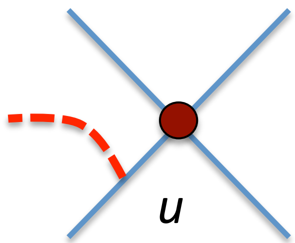


$= v(u)$  $+ \; h(u)$ 

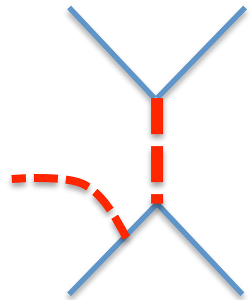
$= \dfrac{v(u)}{d}$  $+ \left(v(u)\dfrac{\sqrt{d^2-1}}{d} + h(u) \right)$ 

$\equiv \; \alpha(u)$  $+ \; \beta(u)$ 

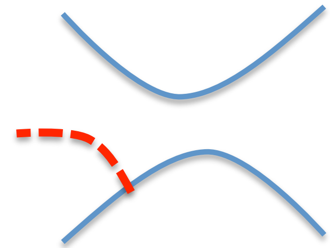
Use the F matrix to rewrite e.g.



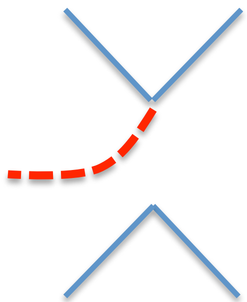
$= \alpha(u)$



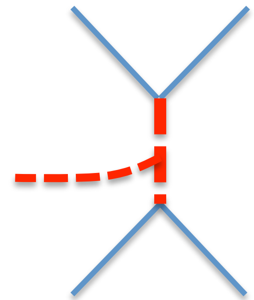
$+ \quad \beta(u)$



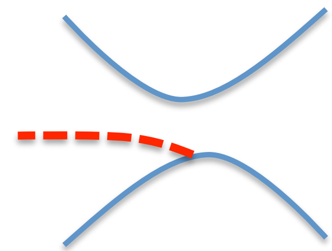
$= \alpha(u)\, F_0$



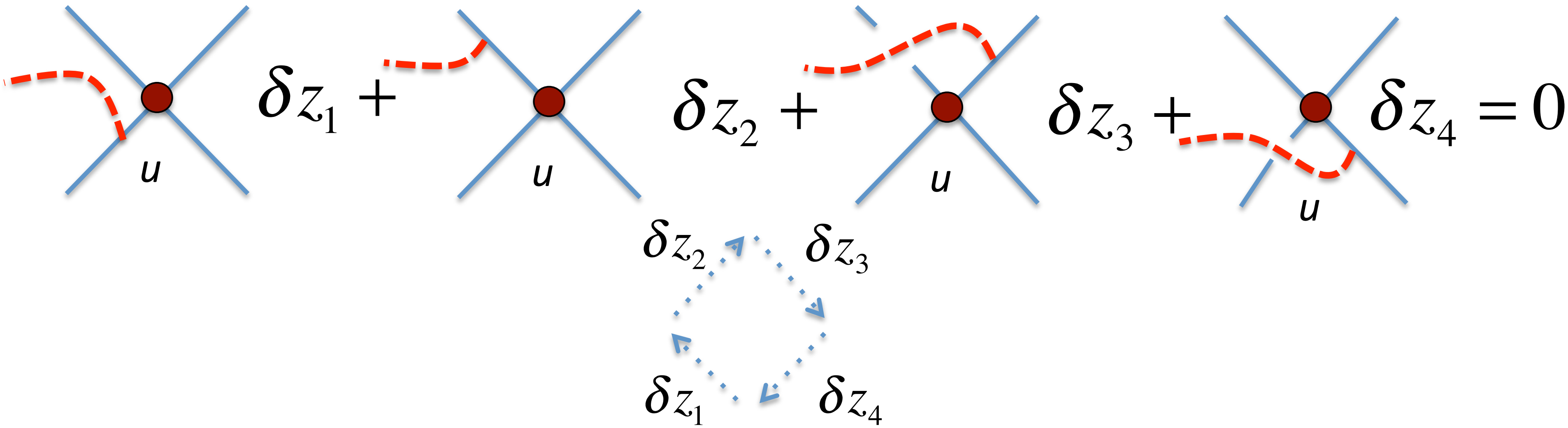
$+ \alpha(u)\, F_1$



$+ \quad \beta(u)$



All four terms in the linear zero-mode equation can be rewritten in terms of these last three graphs. Each coefficient of these graphs must vanish. So three linear equations, one unknown…



Of course there is a solution (this is a rewriting of known results):

Riva and Cardy; Ikhlef and Cardy

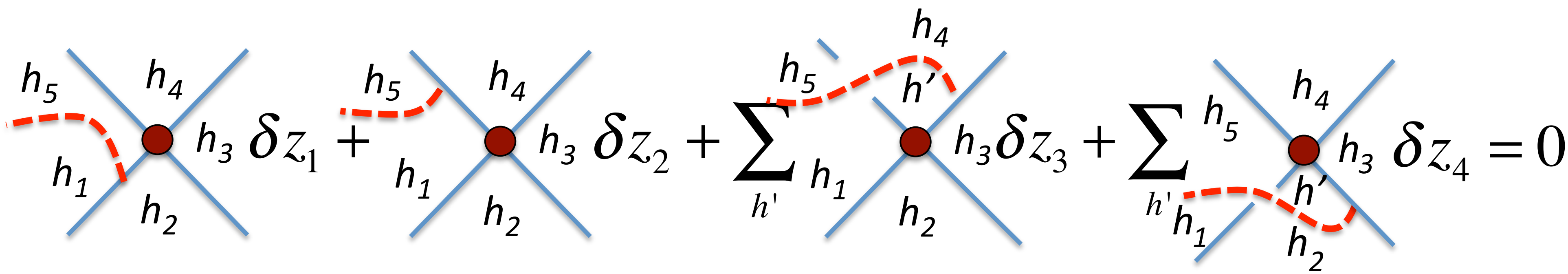
$$\frac{v(u)}{h(u)} = \frac{qe^{-iu} - q^{-1}e^{iu}}{e^{iu} - e^{-iu}}$$

Because the δz_i depend on the lattice angle, the Boltzmann weights must also. It turns out u is exactly that angle!

Rajabpour and Cardy

Height models

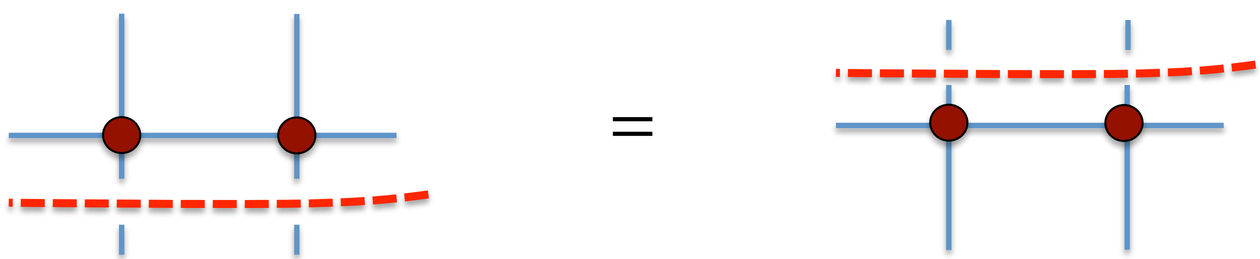
The power of this construction is that the generalization to "height" models is easy. Here the zero mode is truly a defect line.



One example is the "Fibonacci" zero mode at the A_4/hard-square/golden-chain critical point. If one had attempted this by brute force, 11 distinct equations! Using topology reduces it to one.

Topological symmetry

The “string” itself commutes with the Hamiltonian or transfer matrix:



In height models this is a generalization of Kramers-Wannier duality dubbed topological symmetry.

It’s very useful for constraining weights, but does not guarantee integrability – e.g. still commutes with staggered case.

So what good is all this?

- Provides a nice way of understanding and generalizing discretely “holomorphic” operators = zero modes.

- For example, a zero mode can be found in integrable models based on BMW algebras (completely packed models with ``nets’’ made from vector representations of A,B,C,D quantum-group algebras).

- Allows zero modes to be found in height models (RSOS/IRF etc), not just loop ones.

- Brings topology into the story in an illuminating way.

Off the critical point?

- In the Ising case, this can be generalized away from criticality – the zero mode generalizes to a ``shift'' operator Ψ satisfying

$$[H,\Psi]=(\Delta E)\Psi$$

  This is a consequence of its underlying free fermions.

- The Potts model is not free. Nevertheless, I have found such a shift operator can occur if the interactions are chiral.

- It occurs precisely for the integrable couplings! This is by far the simplest way of finding the integrable couplings in the chiral Potts Hamiltonian.

Other future directions

- Combine holomorphic (over the plane) operators with antiholomorphic (under the plane) to get the lattice analog of CFT primary fields. Can the fusion algebra be seen on the lattice? (Pasquier found the Verlinde formula before Verlinde!)

- There are many MTC/TQFT/CFT/anyon theories. Can an integrable model be found for each? Each has multiple vertices, so can a integrable model be found for other vertices ?!?

- Zero modes are important in the study of topological order. Will these ideas help find a topological quantum computer?

Getting more speculative

- Can physicists forget about quantum groups?

- Provide more candidates for some generalization of SLE?

- Starting to get at the questions: Why does SLE apply to integrable lattice models? What does geometry have to do with integrability?

- And the mother of them all: What is really the “reason” why integrable models work?