Non-crossing polymers and the KPZ equation

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LETTRES SCIENCES
ÉCOLE NORMALE SUPÉRIEURE

PRL 56 889 (1986), Kardar, Parisi, Zhang

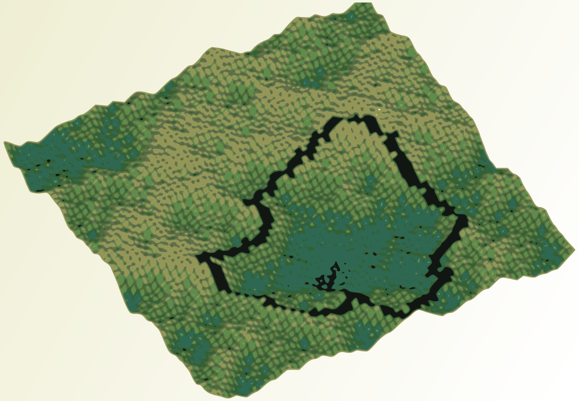
$$\partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \frac{\lambda}{2}(\partial_x h(x,t))^2 + \eta(x,t)$$

relaxation (surface tension) | lowest order non-linearity | Gaussian noise

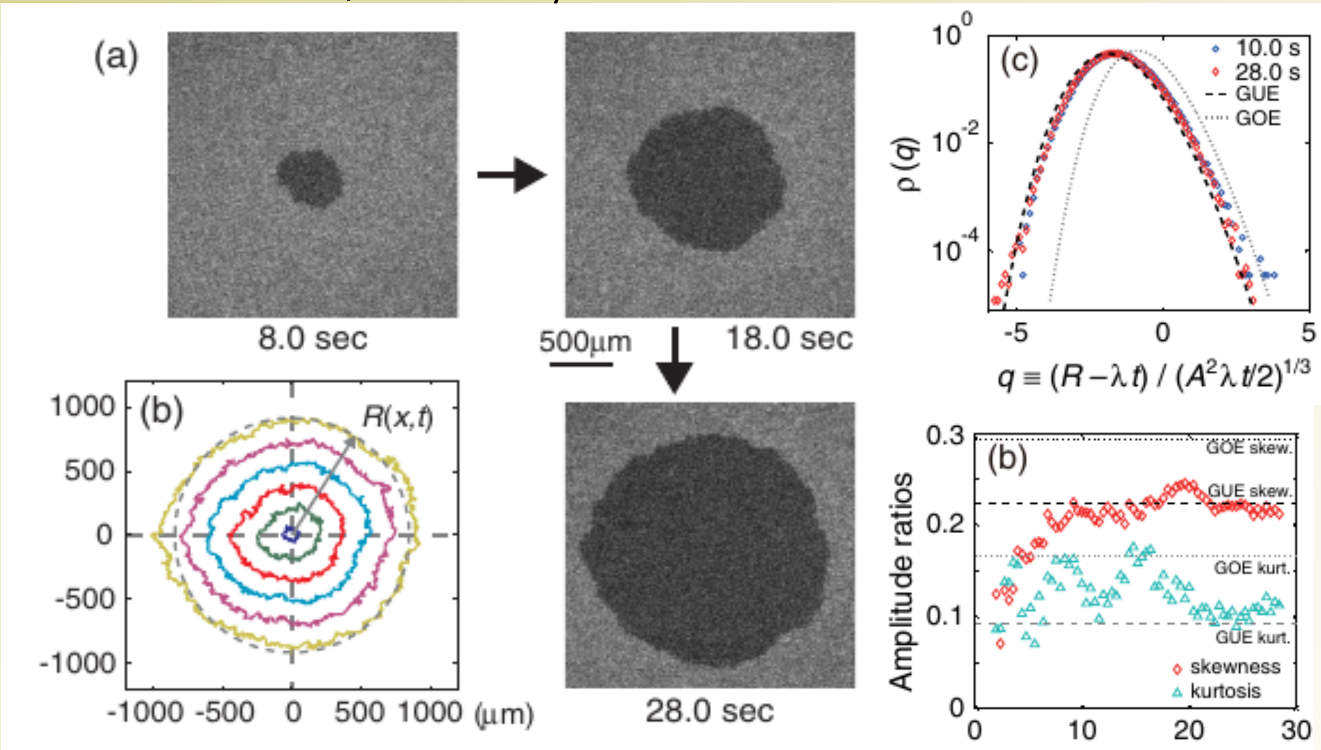
In 1D, the renormalization group provides exact exponents

$$w(L,t) = L^\xi w_0(t/L^z)$$

$$z = \frac{3}{2}, \qquad \chi = \frac{1}{2}$$



Turbulent liquid crystals - PRL 104 230601



$$h(t) = v_\infty t + \left(\frac{A^2\lambda t}{2}\right)^{1/3} \chi$$

Tracy-Widom distribution

Cole-Hopf mapping

$$\partial_t h(x,t) = \nu \partial_x^2 h(x,t) + \frac{\lambda}{2}(\partial_x h(x,t))^2 + \eta(x,t)$$

$$\lambda h(x,t) = T \ln Z(x,t)$$

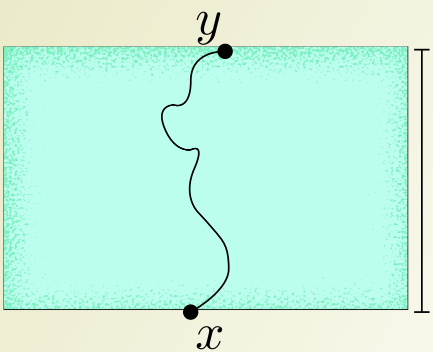
$$\partial_t Z(x,t) = \frac{T}{2k}\partial_x^2 Z(x,t) - \frac{V(x,t)}{T} Z(x,t)$$

diffusion equation in a random potential:

$$\overline{V(x,t)V(x',t')} = \delta(t-t')R(x-x')$$

directed polymer partition function

$$\partial_t Z(x,t) = \frac{T}{2k}\partial_x^2 Z(x,t) - \frac{V(x,t)}{T} Z(x,t)$$



Path integral representation (Feynman - Kac)

$$Z(x,y,t) = \int_{x(0)=x}^{x(t)=y} Dx e^{-\frac{1}{T}\int_0^t d\tau [\frac{\kappa}{2} x'(\tau)^2 + V(x(\tau),\tau)]}$$

$$\mathcal{Z}_n = \overline{Z(x_1,y_1,t)\dots Z(x_n,y_n,t)} = \langle x_1 \dots x_n | e^{-tH_n^{(rep)}} | y_1 \dots y_n \rangle$$

$$H_n^{(rep)} = -\frac{T}{2\kappa}\sum_{i=1}^{n} \partial_{x_i}^2 - \frac{1}{2T^2}\sum_{ij} R(x_i - x_j)$$

Rescaling of variables

$$x = T^3 \kappa^{-1} \tilde{x}, \quad t = 2T^5 \kappa^{-1} \tilde{t}$$

$$\boxed{H_{LL} = -\sum_{j=1}^{n} \partial_{x_j}^2 + 2c \sum_{i<j} \delta(x_i - x_j)}$$

$$\bar{c} = -c = \int du R(u), \qquad \tilde{R}(z) \to 2\bar{c}\delta(z)$$

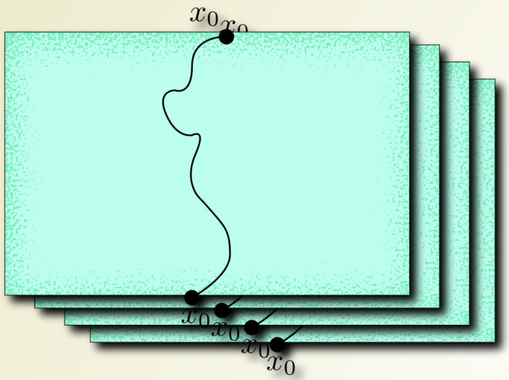
We end up with the **attractive**
Lieb-Liniger Hamiltonian
which is integrable in 1dimension!

$$\mathcal{Z}_n = \overline{Z(x_1, y_1, t) \dots Z(x_n, y_n, t)} = \langle x_1 \dots x_n | e^{-tH_n^{(rep)}} | y_1 \dots y_n \rangle$$

Hard to treat: it contains space-time correlation of the KPZ height

$$\overline{\mathcal{Z}_n} = \langle x_0 \dots x_0 | e^{-tH_{LL}} | x_0 \dots x_0 \rangle$$
$$= \sum_\mu \frac{|\langle x_0 \dots x_0 | \mu \rangle|^2}{||\mu||^2} e^{-tE_\mu}$$

decomposition in eigenstates



If we can compute the spectrum, we can find arbitrary moments...

Bethe-ansatz equations

$$\overline{\mathcal{Z}_n} = \langle x_0 \dots x_0 | e^{-tH_{LL}} | x_0 \dots x_0 \rangle$$

The initial condition is symmetric: the dynamics lies in the bosonic sector of the Hamiltonian

$$\Psi_\mu = \sum_P A_P \prod_{j=1}^{n} e^{i\mu_{P_l} x_l}$$

superposition of plane waves in each sector

The coefficient implements the scattering matrix

$$A_P = \prod_{l>k} \left(1 - \frac{\mathrm{sgn}(x_l - x_k)}{\mu_{P_l} - \mu_{P_k}} \right)$$

How to fix the values of rapidities?

The values of rapidities are fixed by boundary conditions. In the symplest case

$$\Psi_\mu(x_1 + L, \ldots, x_n) = \Psi_\mu(x_1, \ldots, x_n)$$

Bethe-Ansatz equations for the LL model

$$e^{i\mu_\alpha L} = \prod_{\beta \neq \alpha} \frac{\mu_\alpha - \mu_\beta + ic}{\mu_\alpha - \mu_\beta - ic}$$

Solutions at finite L are not easy... But in the thermodynamic limit?

String ansatz

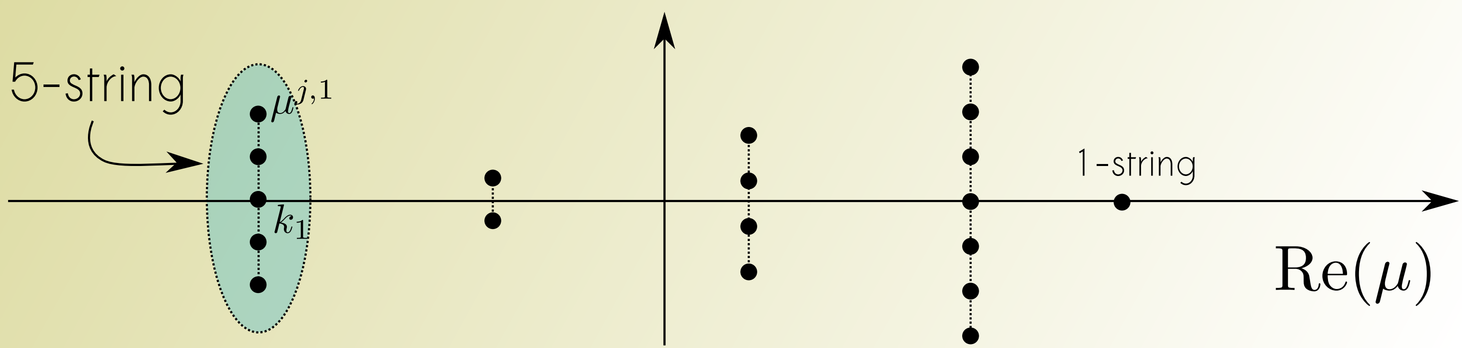
$$e^{i\mu_\alpha L} = \prod_{\beta \neq \alpha} \frac{\mu_\alpha - \mu_\beta + ic}{\mu_\alpha - \mu_\beta - ic}$$

If $\mathrm{Im}(\mu_\alpha) < 0$ for large L, we have a divergence in the LHS, which must be compensated by a pole in the RHS

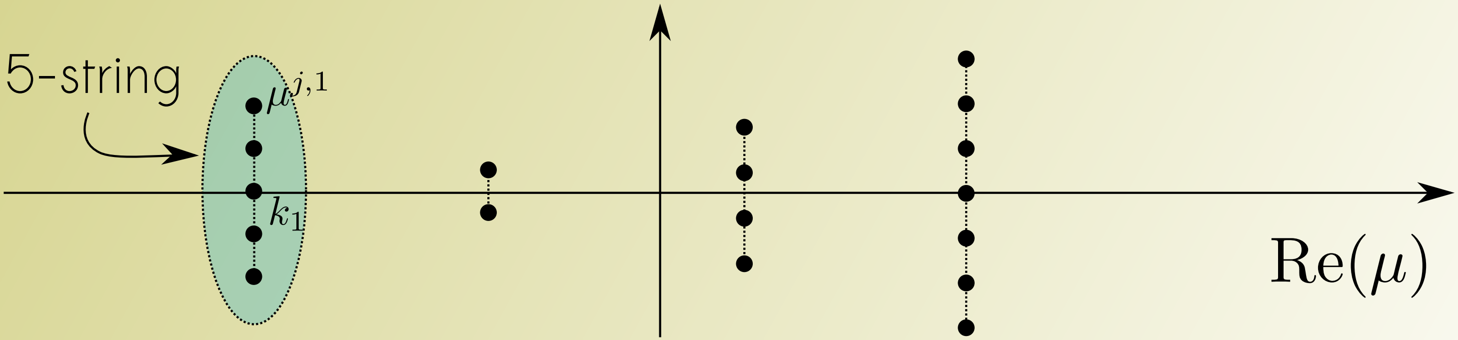
$$\mu_\alpha - \mu_\beta \simeq ic$$

$$\mu^{j,a} = k_j + \frac{i\bar{c}}{2}(m_j + 1 - 2a)$$

bound states (strings)



String features



$$E_\mu = \sum_{\alpha=1}^{n} \mu_j^2 = \sum_{j=1}^{n_s} (m_j k_j^2 - \frac{c^2}{12} m_j (m_j^2 - 1))$$ energy

$$P_\mu = \sum_{\alpha=1}^{n} \mu_j = \sum_{j=1}^{n_s} m_j k_j$$ momentum

Needed ingredients

$$\overline{\mathcal{Z}_n} = \langle x_0 \dots x_0 | e^{-tH_{LL}} | x_0 \dots x_0 \rangle = \sum_{\mu} \frac{|\Psi_\mu(x_0, \dots, x_0)|^2}{||\mu||^2} e^{-tE_\mu}$$

WF

$$\Psi_\mu(x_0, \dots, x_0) = e^{ix_0 P_\mu} \sum_P A_P = n! e^{ix_0 P_\mu}$$

Norm

$$||\mu||^2 = \frac{n!(L\overline{c})^{n_s}}{\overline{c}^n} \frac{\prod_{j=1}^{n_s} m_j^2}{\Phi(k,m)}$$

$$\Phi(k,m) = \prod_{i<j} \frac{(k_i - k_j)^2 + (m_i - m_j)^2 c^2/4}{(k_i - k_j)^2 + (m_i + m_j)^2 c^2/4}$$

General expression for moments

The sum over eigenstates becomes the sum over the possible partitioning of the n particles into strings

$$\overline{Z^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\overline{c})^{n_s}} \underbrace{\sum_{(m_1,\dots,m_{n_s})_n}}_{\text{sum over partitions}} \int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi(k,m) e^{-E_\mu t}$$

It is exact... but how to deal with it?

Partition function at fixed string number

$$\overline{Z^n} = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\overline{c})^{n_s}} \sum_{(m_1,\dots,m_{n_s})_n} \int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi(k,m) e^{-E_\mu t}$$

Use the grancanonical partition function: $\lambda = \left(\frac{\overline{c}^2 t}{4}\right)^{1/3}$

$$g(x) = 1 + \sum_{n=1}^{\infty} \frac{(e^{-\lambda x})^n}{n!} \overline{Z^n} = \overline{\exp(-e^{\lambda(x-f)})}$$

In this way we can recover the free energy distribution

$$\lim_{\lambda\to\infty} g(x) = \overline{\Theta(f-x)} = \mathrm{Prob}(f > x)$$

Exchanging the two sums, we obtain

$$g(x) = \mathrm{Det}(1 + P_0 K_x P_0)$$

$$K_x(v,v') = -\int \frac{dk}{2\pi} dy A_i(y + k^2 - x + v + v') \frac{e^{\lambda y - ix(v-v')}}{1+e^{\lambda y}}$$

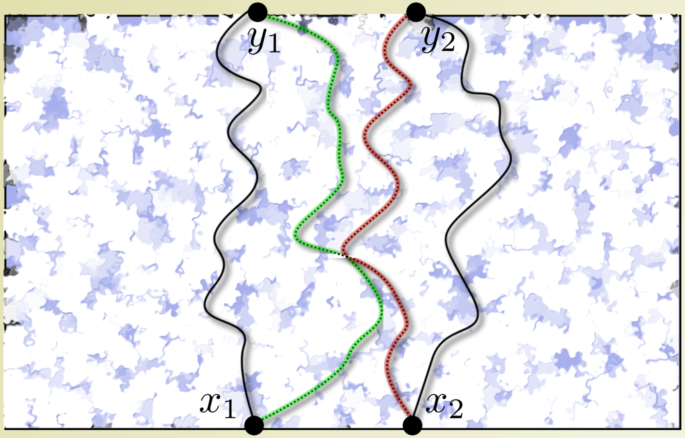
In the large time limit, one obtains

$$\boxed{\lim_{\lambda\to\infty} g(x) = \mathrm{Prob}(f > x = -2^{2/3}s) = F_2(s)}$$

EPL 90 2 (2010)
Calabrese, Le Doussal, Rosso

Tracy-Widom
GUE distribution

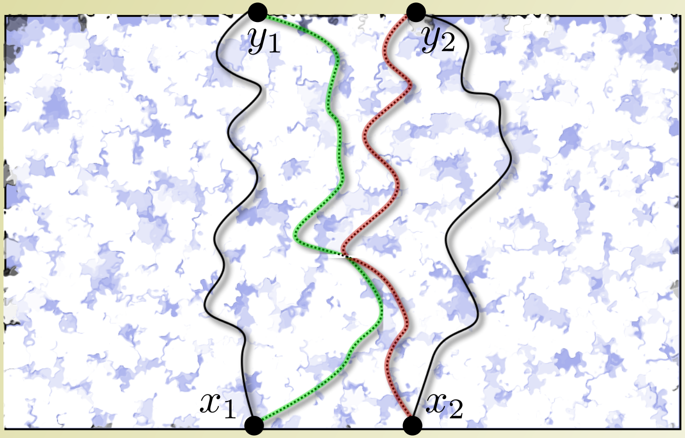
Can we use replica approach to treat non-crossing polymers?



$$p = \mathrm{Prob(polymers\ do\ not\ cross)}$$

Simplest example of interaction, together with disorder...!

Karlin-McGregor formula

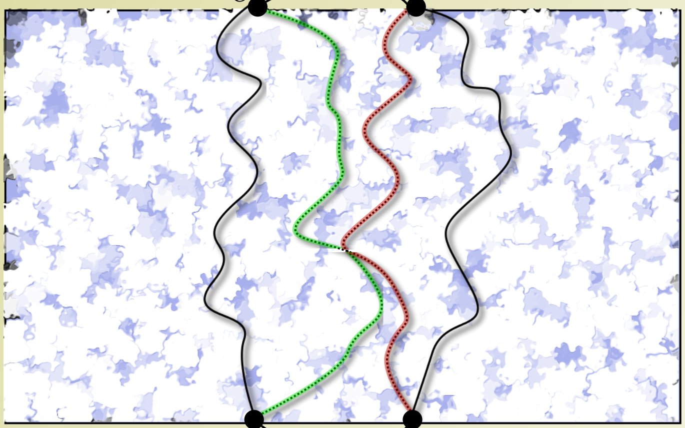


$$p = \text{Prob(polymers do not cross)} = 1 - \frac{\mathcal{Z}_{\text{crossing}}}{\mathcal{Z}} =$$

$$= 1 - \frac{Z(y_2, t|x_1)Z(y_1, t|x_2)}{Z(y_1, t|x_1)Z(y_2, t|x_2)}$$

Similar formulas for more than two polymers

Coinciding points



$$p = \frac{Z(\epsilon, t|\epsilon)Z(-\epsilon, t|-\epsilon) - Z(-\epsilon, t|\epsilon)Z(\epsilon, t|-\epsilon)}{Z(0, t|0)Z(0, t|0)} =$$

$$= \epsilon^2 \lim_{n\to 0} Z_2(\epsilon) Z(0, t|0)^{n-2}$$

Replica for non-crossing polymers

$$\bar{p} = \lim_{n \to 0} \frac{\overline{Z_2(\epsilon) Z(0,t|0)^{n-2}}}{2\epsilon^2} = \sum_{\mu} \frac{|(\partial_{x_1} - \partial_{x_2})\Psi_\mu(x)|^2}{2||\mu||^2}$$

The expression is analogous to the one for single polymer. But the bosonic sector gives a vanishing contribution!

How to build wave functions with different symmetries?

Wave function and Young tableau

$$H_{LL} = -\sum_{j=1}^{n} \partial_{x_j}^2 + 2c \sum_{i<j} \delta(x_i - x_j)$$

We look for eigen functions antisymmetric in the first two variables...

symmetric

antisymmetric

$$\Psi_\mu(x_1, x_2, \ldots, x_n) \qquad \longleftrightarrow \qquad \begin{array}{|c|c|c|c|} \hline 1 & 3 & 4 & 5 \\ \hline 2 & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

More general ansatz...

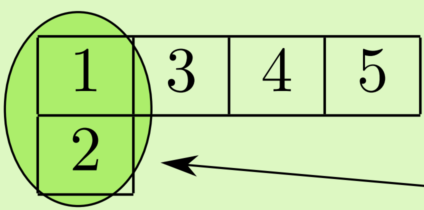
$$\Psi_\mu(x) = \sum_{P,Q} \Theta_Q(x) A_Q^P \exp(i \sum_j x_{P_j} \mu_{Q_j})$$

Nested Bethe Ansatz

$$\prod_{k\neq j}^{n} \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \frac{\mu_j - \lambda - ic/2}{\mu_j - \lambda + ic/2} = e^{i\mu_j L}$$

$$\prod_{j=1}^{n} \frac{\lambda - \mu_j - ic/2}{\lambda - \mu_j + ic/2} = 1$$

The auxliary variable λ implement the symmetry of the wave function.



In general, one auxiliary variable for every doubled column

String ansatz?

$$\prod_{k\neq j}^{n} \frac{\mu_{jk}+ic}{\mu_{jk}-ic} \times \frac{\mu_j-\lambda-ic/2}{\mu_j-\lambda+ic/2} = e^{i\mu_j L}$$

$$\prod_{j=1}^{n} \frac{\lambda-\mu_j-ic/2}{\lambda-\mu_j+ic/2} = 1$$

For large L, the first equation suggests again the presence of strings

$$\mu^{j,a} = k_j + \frac{i\bar{c}}{2}(m_j+1-2a)$$

What about the auxiliary variable?

Contour integral

$$\prod_{k\neq j}^{n} \frac{\mu_{jk} + ic}{\mu_{jk} - ic} \times \frac{\mu_j - \lambda - ic/2}{\mu_j - \lambda + ic/2} = e^{i\mu_j L}$$

$$\prod_{j=1}^{n} \frac{\lambda - \mu_j - ic/2}{\lambda - \mu_j + ic/2} = 1$$

The solution of the second equation are non trivial... But we are only interested on the sum

$$\bar{p} = \sum_{\text{strings}} \sum_{\lambda} \frac{|(\partial_{x_1} - \partial_{x_2})\Psi_{\mu}(x)|^2}{2||\mu||^2} = \sum_{\text{strings}} \oint dz P(z)$$

After summing over the auxiliary variable, we get an expression very similar to the bosonic case

$$\frac{\Psi_\mu(0)}{n!} = \mathrm{sym}\left[\prod_{i<j}\frac{\mu_{ij}+ic}{\mu_{ij}}\right] = 1$$

sum over λ

$$\frac{(\partial_{x_1}-\partial_{x_2})\Psi_\mu(0)}{n!} = \mathrm{sym}\left[\mu_{12}(\mu_{21}+ic)\prod_{i<j}\frac{\mu_{ij}+ic}{\mu_{ij}}\right] = \\ = \frac{A_1^2}{n} + O(1)$$

$$A_p = \sum_\alpha \mu_\alpha^p$$ conserved quantities of the LL

Norm is unchanged

$||\mu||^2$ $||\mu||^2$

Generalized Gibbs Ensemble

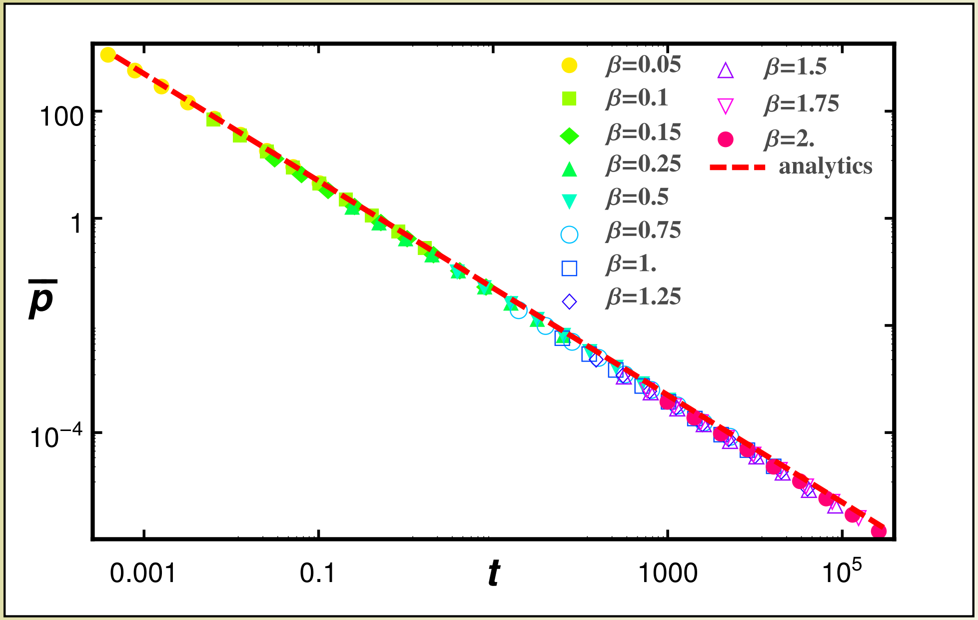
$$Z_n[\{t_1,\dots,\}] = \sum_{n_s=1}^{n} \frac{n!}{n_s!(2\pi\bar{c})^{n_s}} \sum_{(m_1,\dots,m_{n_s})_n} \int \prod_{j=1}^{n_s} \frac{dk_j}{m_j} \Phi(k,m) e^{-\sum_p A_p t_p}$$

we replace time evolution with a generalized evolution with multiples times

$$\frac{(\partial_{x_1} - \partial_{x_2})\Psi_\mu(0)}{n!} = \frac{A_1^2}{n} + O(1)$$

$$\bar{p} = \lim_{n\to 0} \partial_{t_1}^2 \frac{Z_n[\{t_1,\dots,\}]}{n} = \frac{1}{2t}$$

The average non-crossing probability is not affected by disorder!



$$\overline{p} = \frac{1}{2t}$$

Two-lines derivation

$$p = \partial_x \partial_y \ln Z(x,t|y)$$

$$\overline{\ln Z(x,t|y)} = h(t) - \frac{(x-y)^2}{4t}$$

Recipe for higher moments

In order to compute higher moments

$$\overline{p^m} =?$$

- symmetrize the polynomial in terms of conserved charges

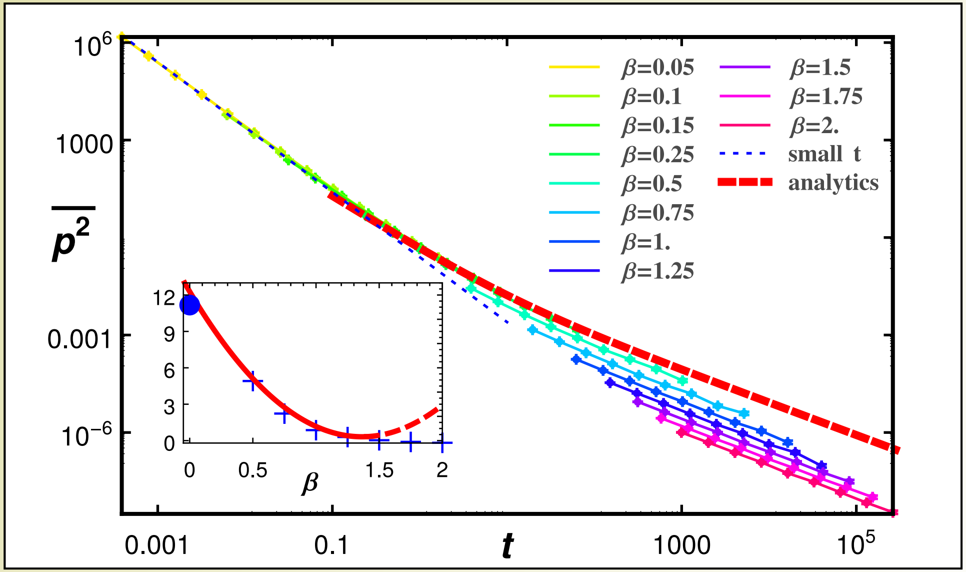
$$\frac{(\partial_{x_1} - \partial_{x_2}) \ldots (\partial_{x_{2m-1}} - \partial_{x_{2m}}) \Psi_\mu(0)}{n!} = P(\{A_p\})$$

- write the result as a set of derivatives applied to the generalized moments

$$\overline{p^m} = \lim_{n\to 0} P(\{A_p \to \partial_{t_p}\}) Z_n(t_1, \ldots)$$

Results for higher moments

$$\overline{p^2} \simeq \frac{\overline{c}^2}{12t} - \frac{2\overline{\chi_2 c}^{2/3}}{9t^{5/3}}$$



Leading order

$$\overline{p^m} \simeq \frac{4^{-m}\sqrt{\pi}\Gamma(m)^3\overline{c}^{2m-2}}{\Gamma(m+1/2)t}$$

Physical picture:
- for most of the realization:
p is exponentially small
- for a fraction 1/t
of the realization, p is O(c^2)

Conclusions

- We developed a framework based on the Nested Bethe ansatz to deal with non-crossing polymers in random media;

- We computed exactly the large times asymptotics for the moments of the non-crossing probability for two polymers;

- Agreement with numerical lattice simulations: the crossing probability is most of the time exponentially small

Open questions:

- generalization to multi-polymers
- higher order large time asymptotics: connection with random matrices?