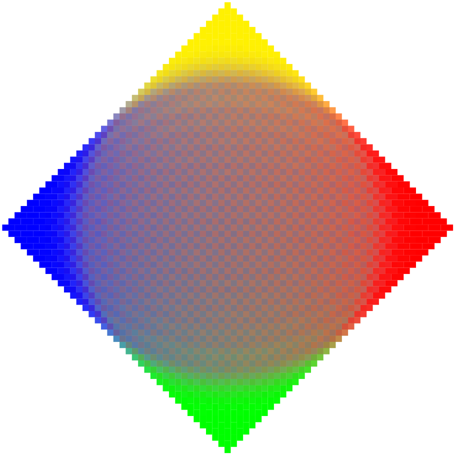


Dimer models: monomers, arctic curve and CFT

Nicolas Allegra (Groupe de physique statistique, IJL Nancy)

July 2, 2015

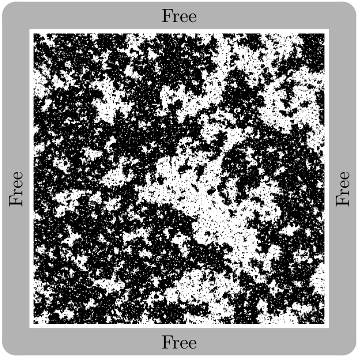
1. Critical phenomena on rectangle geometry
   - Generalities about boundary conditions
   - Critical free energy and CFT
   - Correlation functions

2. Dimer models
   - Free boson theory
   - Free fermion theory
   - Corner free energy and exponents

3. Arctic circle phenomena and curved Dirac field
   - Arctic Circle
   - Exact Calculations
   - Asymptotic and field theory correspondence: toy model

4. Conclusions

Critical systems: Example of the Ising model



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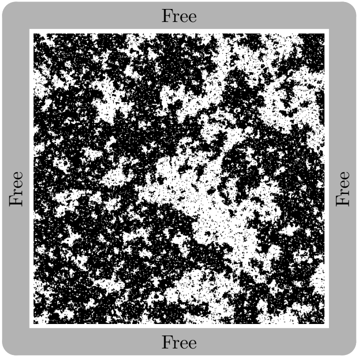

Some interesting questions

- Change of boundary conditions $\rightarrow$ change of the critical behavior
- Expression of the free energy at the critical point
- Magnetization profile at the critical point
- Spin/energy correlation exponents close to a surface or corner

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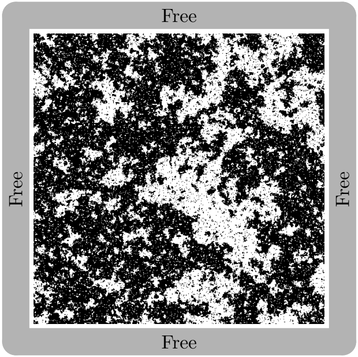

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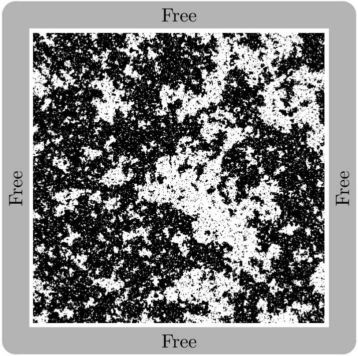
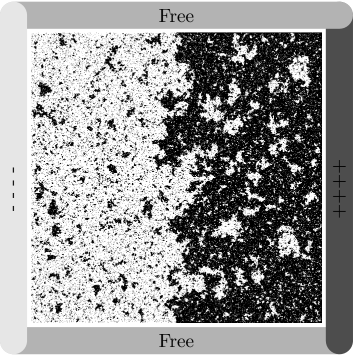
Ising model $\to c = 1/2$ CFT





Boundary condition changing operators (bcc)

- $(- - -)$ to $(+ + +)$ or (Free) to $(+ + +)$ or $(- - -)$ to (Free)
- Ψ *bcc* primary operators of the $c = 1/2$ CFT
- Kac table $c = 1/2 \to h_{bcc} = \{0, 1/2, 1/16\}$
- $\Psi_{+free} = \sigma$ and $\Psi_{+-} = \epsilon$ with $h_{+free} = 1/2$ and $h_{+-} = 1/16$

Boundary conformal field theory (Cardy '84)

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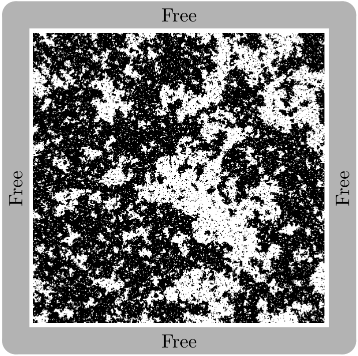

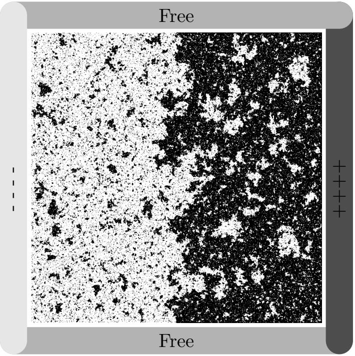

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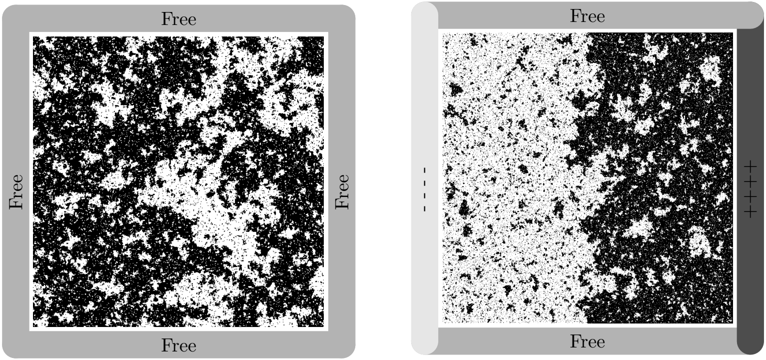

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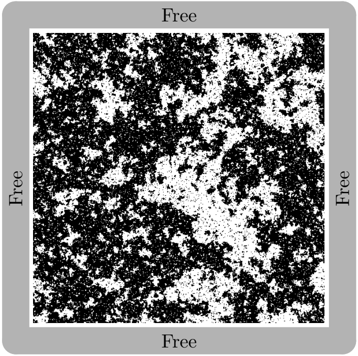

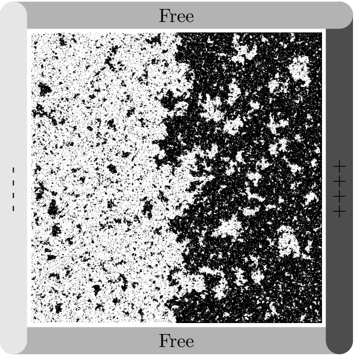

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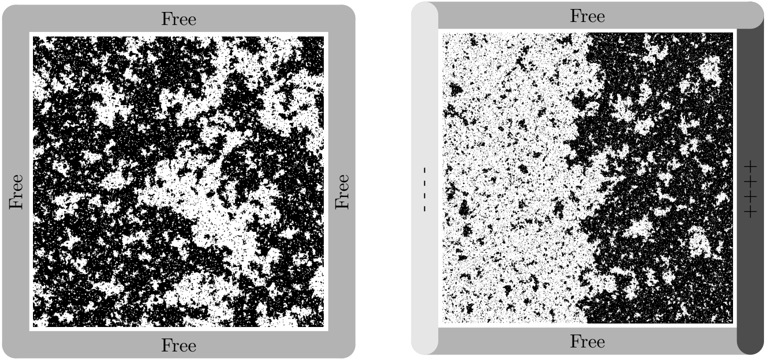

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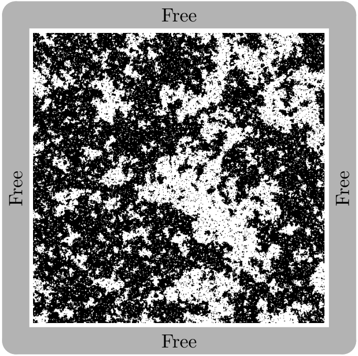

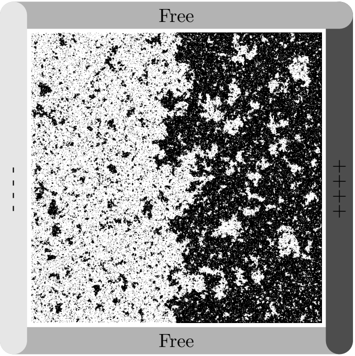

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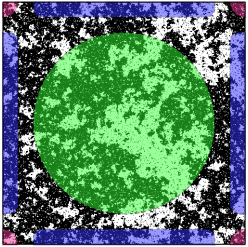
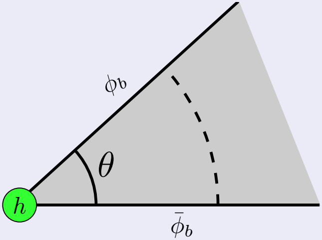


Free energy decomposition

$$\mathcal{F} = L^2 f_{\mathrm{bulk}} + L f_{\mathrm{surface}} + f_{\mathrm{corner}}$$

f_{bulk} and f_{surface} can be obtained by BA/TM but not f_{corner}

Corner free energy with a bcc operator



- CFT predicts $f_{\mathrm{corner}} = \left[\frac{\pi}{\theta} h_{bcc} + \frac{c}{24}\left(\frac{\theta}{\pi} - \frac{\pi}{\theta}\right)\right] \log L$ universal (Cardy Peschel)
- Valid close to criticality $L \to \xi$
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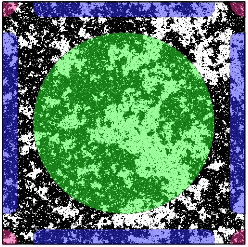

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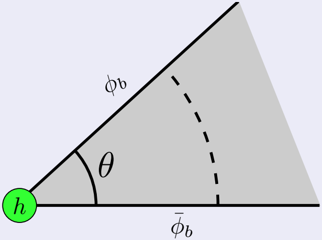

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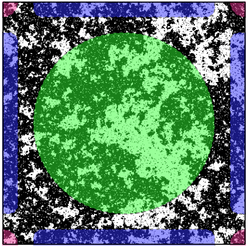

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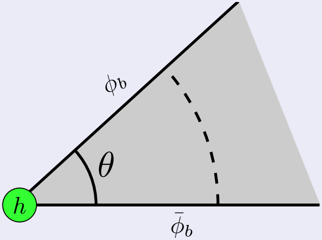

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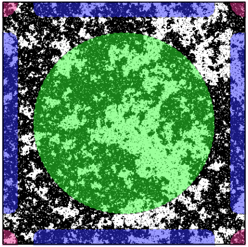

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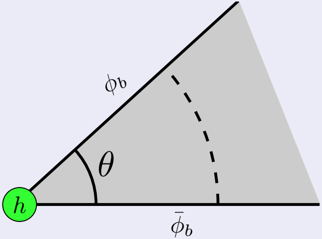

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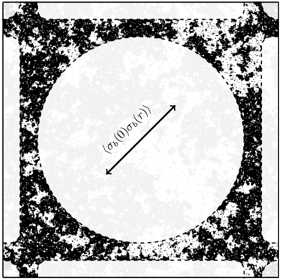
Bulk surface and corner correlations



Correlations and scaling dimensions (spin σ and energy ϵ)

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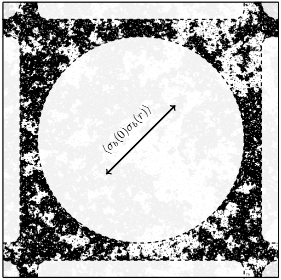

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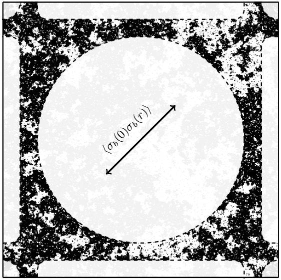

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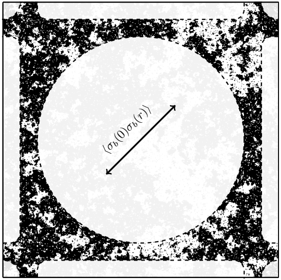

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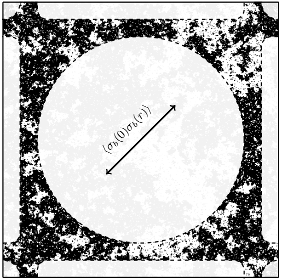

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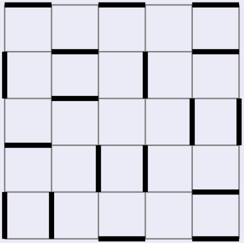
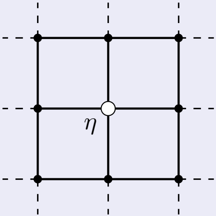
Classical dimer model on the square lattice

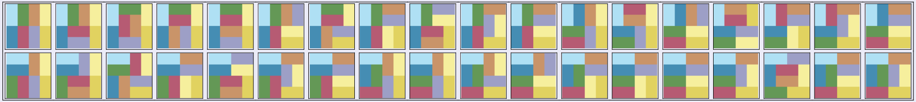




$$\boxed{\mathcal{H} = -\frac{t}{2}\sum_{ij}\eta_i A_{ij}\eta_j,}$$

- Ising model with nilpotent variables ($\eta^2 = 0$) instead of spins ($\sigma^2 = 1$)
- A_{ij} Connectivity (Adjacent) matrix
- Partition function $\int \mathcal{D}\eta \exp -\mathcal{H} = \sqrt{\mathrm{perm} A}$

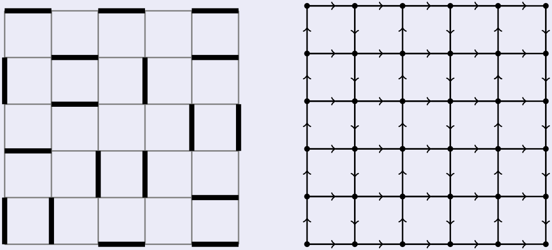
Selected chronology for dimer model on the square lattice (1937-2015)



- Model of absorption of dimer molecules on a $2d$ substract (Fowler and Rushbrooke '37)
- Partition function (Kasteleyn, Fisher and Temperley 1961)
- Solution by transfer matrix (Lieb '67)
- Correlation functions dimer-dimer and monomer-monomer (Fisher Stephenson, Hartwig '68)
- General correlation functions in terms of Ising correlations (Perk and Capel '77)
- Solution by Grassmann variables (Hayn Plechko '93)
- One monomer at the boundary by spanning tree mapping (Tzeng and Wu '02)
- Arbitrary number of monomers at the boundary (Priezzhev Ruelle '08)
- Arbitrary number of monomers anywhere (N.A and Fortin '14)

Other development: General monomer-dimer model (Heilmann and Lieb '70, Baxter '68) Quantum dimer model (Roshkar and Kivelson '88). Interacting dimer model (Alet et Al '05, Fradkin et Al '06)...

Kasteleyn orientation

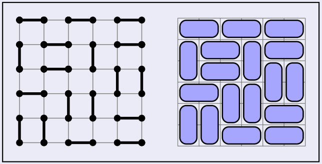


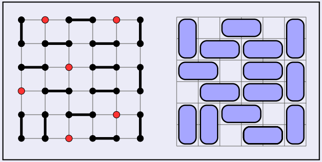
$$\boxed{\mathcal{H} = -\frac{t}{2} \sum_{ij} a_i K_{ij} a_j}$$

- Ising model with Grassmann variables ($a^2 = 0$ and $\{a_i, a_j\} = 0$)
- K_{ij} Kasteleyn (weighted Adjacent) matrix
- Partition function $\int \mathcal{D}a \exp -\mathcal{H} = \sqrt{\det K}$

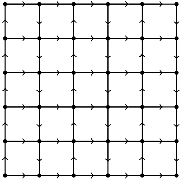
Kasteleyn modification for the monomer-dimer model

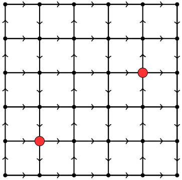
Modification of the orientation matrix K' induced by monomers

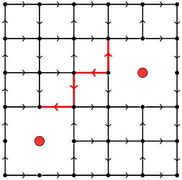




- Monomers on boundary $\rightarrow$ K' is still a Kasteleyn Matrix
- Monomers creates changing-sign lines $\rightarrow$ K' no more a Kasteleyn Matrix







Pfaffian perturbation theory (Fisher Stephenson 1963)

- $\mathrm{pf}^2(K') = \mathrm{pf}^2(K).\det(1 + K^{-1}E)$ where $K' = K + E$
- dimer-dimer correlation $\langle d(r)d(0)\rangle \sim r^{-2}$
- monomer-monomer correlation $\langle m(r)m(0)\rangle \sim r^{-1/2}$

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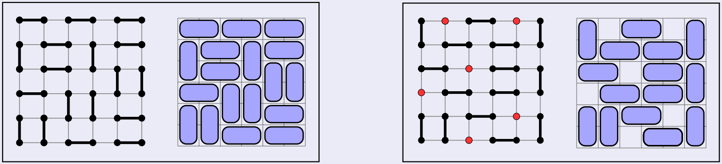

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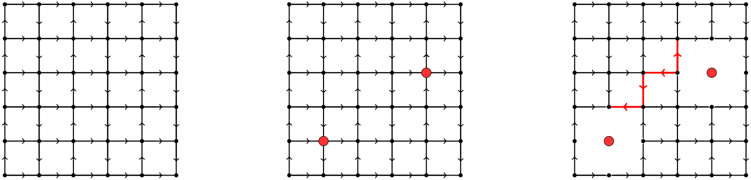

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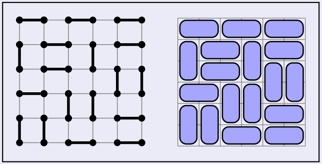
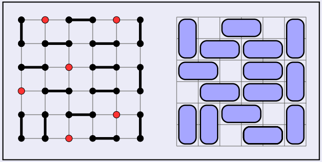
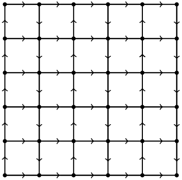
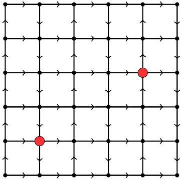
Modification of the orientation matrix K' induced by monomers

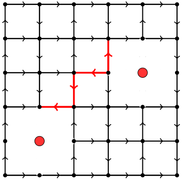




- Monomers on boundary → K' is still a Kasteleyn Matrix
- Monomers creates changing-sign lines → K' no more a Kasteleyn Matrix







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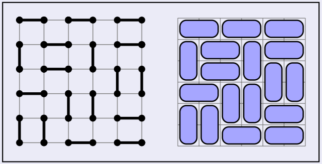

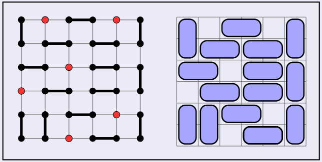

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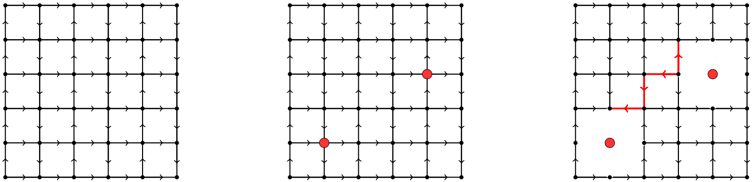

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Grassmann variables and Berezin integration

Nilpotent and variables $\{\eta\}$ and Grassmann variables $\{\theta\}$

- $[\eta_i, \eta_j] = 0, \quad \eta_i^2 = 0$
- $\int \mathrm{d}\eta = 0$
- $\int \mathrm{d}\eta.\eta = 1$

- $\{\theta_i, \theta_j\} = 0, \quad \theta_i^2 = 0$
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Berezin Integration over Grassmann variables $\{\theta_i\}$

- Berezin Integration: if $f = f_1 + \theta_i f_2$ then $f_2 = \frac{\partial f}{\partial \theta_i}$ and

$$\int \mathrm{d}\theta_i f(\theta_i) = \frac{\partial f}{\partial \theta_i}$$

- Gaussian Integration

$$\det(A) = \int \prod_\alpha \mathrm{d}\theta_\alpha \mathrm{d}\bar{\theta}_\alpha \exp\Big(\sum_{\alpha\beta} \theta_\alpha A_{\alpha\beta} \bar{\theta}_\beta\Big)$$

$$\mathrm{pf}(A) = \int \prod_\alpha \mathrm{d}\theta_\alpha \exp\Big(\frac{1}{2} \sum_{\alpha\beta} \theta_\alpha A_{\alpha\beta} \theta_\beta\Big)$$

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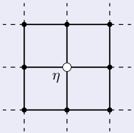
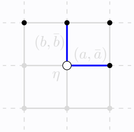
Plechko partition function (using nilpotent variables)



$$\rightarrow \quad \mathcal{Q}_0(L) = \int \prod_{m,n}^{L} \mathrm{d}\eta_{mn}(1 + t_x \eta_{mn}\eta_{m+1n})(1 + t_y \eta_{mn}\eta_{mn+1})$$

Fermionization using Grassmann variables



$$1 + t_x \eta_{mn}\eta_{m+1n} = \int \mathrm{d}\bar{a}_{mn}\mathrm{d}a_{mn} e^{a_{mn}\bar{a}_{mn}}(1 + a_{mn}\eta_{mn})(1 + t_x \bar{a}_{mn}\eta_{m+1n})$$

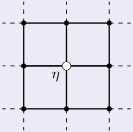
$$= \mathrm{Tr}_{\{a,\bar{a}\}} A_{mn}\bar{A}_{m+1n}$$

$$1 + t_y \eta_{mn}\eta_{mn+1} = \int \mathrm{d}\bar{b}_{mn}\mathrm{d}b_{mn} e^{b_{mn}\bar{b}_{mn}}(1 + b_{mn}\eta_{mn})(1 + t_y \bar{b}_{mn}\eta_{mn+1})$$

$$= \mathrm{Tr}_{\{b,\bar{b}\}} B_{mn}\bar{B}_{mn+1}$$

Grassmann formulation of the dimer model

Plechko partition function (using nilpotent variables)



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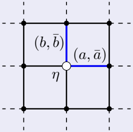

$$\begin{aligned} 1 + t_x \eta_{mn}\eta_{m+1n} &= \int \mathrm{d}\bar{a}_{mn}\mathrm{d}a_{mn} e^{a_{mn}\bar{a}_{mn}} (1 + a_{mn}\eta_{mn})(1 + t_x \bar{a}_{mn}\eta_{m+1n}) \\ &= \mathrm{Tr}_{\{a,\bar{a}\}} A_{mn}\bar{A}_{m+1n} \\ 1 + t_y \eta_{mn}\eta_{mn+1} &= \int \mathrm{d}\bar{b}_{mn}\mathrm{d}b_{mn} e^{b_{mn}\bar{b}_{mn}} (1 + b_{mn}\eta_{mn})(1 + t_y \bar{b}_{mn}\eta_{mn+1}) \\ &= \mathrm{Tr}_{\{b,\bar{b}\}} B_{mn}\bar{B}_{mn+1} \end{aligned}$$

Grassmann variables factorization

- Associativity $(\mathcal{O}_1\bar{\mathcal{O}}_2)(\mathcal{O}_2\bar{\mathcal{O}}_3)(\mathcal{O}_3\bar{\mathcal{O}}_4) = \mathcal{O}_1(\bar{\mathcal{O}}_2\mathcal{O}_2)(\bar{\mathcal{O}}_3\mathcal{O}_3)\bar{\mathcal{O}}_4$
- Mirror ordering $(\mathcal{O}_1\bar{\mathcal{O}}_1)(\mathcal{O}_2\bar{\mathcal{O}}_2)(\mathcal{O}_3\bar{\mathcal{O}}_3) = \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\bar{\mathcal{O}}_3\bar{\mathcal{O}}_2\bar{\mathcal{O}}_1$

$$
\begin{aligned}
\prod_{m,n}^{L}(A_{mn}\bar{A}_{m+1n})(B_{mn}\bar{B}_{mn+1}) &= \overrightarrow{\prod_{n=1}}(A_{1n}\bar{A}_{2n})(B_{1n}\bar{B}_{1n+1})(A_{2n}\bar{A}_{3n})(B_{2n}\bar{B}_{2n+1})\cdots \\
&= \overrightarrow{\prod_{n=1}}(A_{1n}\bar{A}_{2n})(A_{2n}\bar{A}_{3n})\cdots(B_{1n}B_{2n}\cdots\bar{B}_{2n+1}\bar{B}_{1n+1}) \\
&= \overrightarrow{\prod_{n=1}}(B_{1n}(A_{1n}\bar{A}_{2n})B_{2n}(A_{2n}\bar{A}_{3n})\cdots\bar{B}_{2n+1}\bar{B}_{1n+1}) \\
&= \overrightarrow{\prod_{n=1}}(\bar{B}_{Ln}\cdots\bar{B}_{2n}\bar{B}_{1n})(B_{1n}A_{1n}\bar{A}_{2n}B_{2n}A_{2n}\bar{A}_{3n}\cdots\bar{A}_{Ln}B_{Ln}A_{Ln})
\end{aligned}
$$

Mirror symmetry

$$
\mathcal{Q}_0 = \mathrm{Tr}_{\{a,\bar{a},b,\bar{b},\eta\}}\overrightarrow{\prod_{n}}\Big(\overleftarrow{\prod_{m}}\bar{B}_{mn}\overrightarrow{\prod_{m}}\bar{A}_{mn}B_{mn}A_{mn}\Big).
$$

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$$

Solution and fermion field theory

Grassmann partition function

$$\mathcal{Q}_0 = \int \mathcal{D}[a]\,\mathcal{D}[\bar{a}]\,\mathcal{D}[b]\,\mathcal{D}[\bar{b}] \overrightarrow{\prod_{m,n}} L_{mn}[a,\bar{a},b,\bar{b}] = \int \mathcal{D}[a]\,\mathcal{D}[\bar{a}]\,\mathcal{D}[b]\,\mathcal{D}[\bar{b}]\,\mathcal{D}[c] \exp \sum_{mn} c_{mn} L_{mn}$$

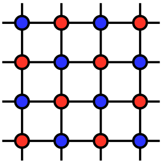
$$= \int \prod_{mn} \mathrm{d}c_{mn} \exp \sum_{mn} (t_x c_{mn} c_{m+1n} + i t_y c_{mn} c_{mn+1})$$

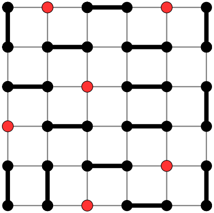
$$= \int \mathcal{D}[c] \exp \sum_{mn} \mathcal{S}_0[c_{mn}] \quad \rightarrow \text{ Kasteleyn solution}$$

Field theory $\rightarrow$ free fermions

$$\mathcal{S}_0[\psi_\alpha, \psi_\beta] = \frac{1}{2} \int \mathrm{d}x\mathrm{d}y\ \psi_\alpha M_{\alpha\beta} \psi_\beta$$

- $\psi_{\alpha,\beta}$ Complex fermions $\in$ even/odd sub-lattice
- such that $\langle \psi_\alpha \psi_\alpha \rangle = \langle \psi_\beta \psi_\beta \rangle = 0$
- $\langle \psi_\alpha(0) \psi_\beta(r) \rangle = M^{-1}_{\alpha\beta}(r)$





Modification of the partition function induced by monomers insertion

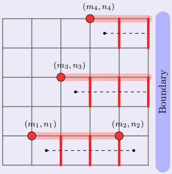
$$\mathcal{Q}_{2n}(L) = \int \prod_{m,n}^{L} \mathrm{d}\eta_{mn}(1 + t_x \eta_{mn}\eta_{m+1n})(1 + t_y \eta_{mn}\eta_{mn+1}) \prod_{\{r_i\}} \eta_{m_i,n_i}$$

- $L_{mn} \to L_{mn} + h_i$
- Change of sign from r_i to the boundary $m_i = L$

Partition function of the dimer model with $2n$ monomers

$$\mathcal{Q}_{2n} = \int \mathcal{D}[c]\,\mathcal{D}[h]\exp\left(\mathcal{S}_0 + \sum_{\{r_i\}} c_{m_i n_i} h_i + 2t_y \sum_{\{r_i\}} \sum_{m=m_i+1}^{L} (-1)^{m+1} c_{mn_i-1} c_{mn_i}\right).$$

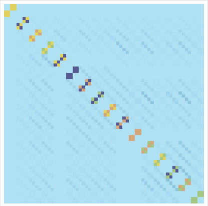
- "Free fermion" action $\mathcal{S}_0$
- "Grassmann Magnetic field"
- "Topological defect line"



Pfaffian formulation

$$\boxed{\mathcal{Q}_{2n} = \mathrm{pf}(W)\mathrm{pf}(C)}$$

- $W^{\mu\nu}_{\alpha\beta} = \delta_{\alpha\beta} M^{\mu\nu}_{\alpha} + V^{\mu\nu}_{\alpha\beta} \qquad \rightarrow \qquad W =$
- $\dim(W) = L^2 \times L^2$, $\dim(C) = 2n \times 2n$



Partition function with monomers

Partition function of the dimer model with $2n$ monomers

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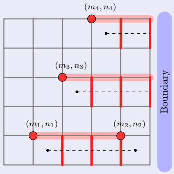

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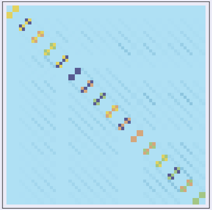
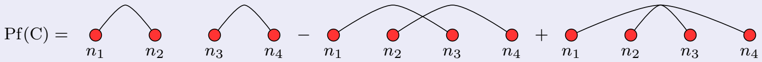
- $W^{\mu\nu}_{\alpha\beta} = \delta_{\alpha\beta} M^{\mu\nu}_{\alpha} + V^{\mu\nu}_{\alpha\beta} \quad \rightarrow \quad W =$ 
- $\dim(W) = L^2 \times L^2$, $\dim(C) = 2n \times 2n$

Partition function of $2n$ boundary monomers and exact $2n$-point correlations

- If $\{r_i\} \in \partial\mathcal{B} \rightarrow W = M$

$$\mathcal{Q}_{2n} = \mathcal{Q}_0.\mathrm{pf}(C) \qquad \frac{\mathcal{Q}_{2n}}{\mathcal{Q}_0} = \langle c_1 c_2 \ldots c_{2n}\rangle_0 = \mathrm{pf}(C) \quad \text{where} \quad |\mathrm{pf}(C)| < 1$$

- Exemple: 4-point correlations and Wick decomposition



2-point function and Majorana Fermions

$$C_{ij} = \frac{4\left[(-1)^{n_i} - (-1)^{n_j}\right]}{(L+1)^2} \sum_{p,q=1}^{L/2} \frac{i^{1+n_i+n_j}\, t_y \cos\frac{\pi q}{L+1} \sin^2\frac{\pi p}{L+1}}{t_x^2 \cos^2\frac{\pi p}{L+1} + t_y^2 \cos^2\frac{\pi q}{L+1}} \sin\frac{\pi q n_i}{L+1} \sin\frac{\pi q n_j}{L+1}$$

- Asymptotically $\langle c_i c_j\rangle_0 \sim \frac{-2}{\pi d_{ij}}$ if $n_i, n_j \notin$ same sublattice
- 2 Complex chiral free fermions $\langle \psi(x)\psi^\dagger(y)\rangle = -\frac{2}{\pi(x-y)}$ and $\langle \psi(x)\psi(y)\rangle = 0$

Partition function of $2n$ boundary monomers and exact $2n$-point correlations

- If $\{r_i\} \in \partial\mathcal{B} \rightarrow W = M$

$$\mathcal{Q}_{2n} = \mathcal{Q}_0.\mathrm{pf}(C) \qquad \frac{\mathcal{Q}_{2n}}{\mathcal{Q}_0} = \langle c_1 c_2 \dots c_{2n} \rangle_0 = \mathrm{pf}(C) \quad \text{where} \quad |\mathrm{pf}(C)| < 1$$

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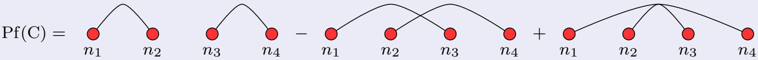
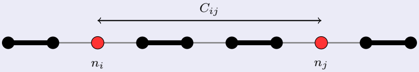


Cauchy determinant and superposition principle

- Asymptotically $C_{ij} = \frac{2}{\pi |z_i - w_j|}$ ($z_i/w_i \in$ odd/even sublattice) then

$$\left\langle c_1 c_2 \dots c_n \right\rangle_0 = \left(\frac{-2}{\pi}\right)^n \det\left(\frac{1}{z_i - w_j}\right) = \frac{\prod_{i<j}(z_i - z_j)\prod_{k<l}(w_k - w_l)}{\prod_{p<q}(z_p - w_q)}$$

- Coulomb Gase: same/opposite sublattice = same/opposite charge

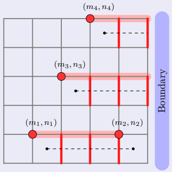


Partition function of $2n$ monomers and $2n$-point correlations

- If $\{r_i\} \notin \partial\mathcal{B} \rightarrow \mathcal{Q}_{2n}\mathcal{Q}_0^{-1} = \mathrm{pf}(WM^{-1}).\mathrm{pf}(C) = \langle c_1 c_2 \ldots c_{2n}\rangle_I$

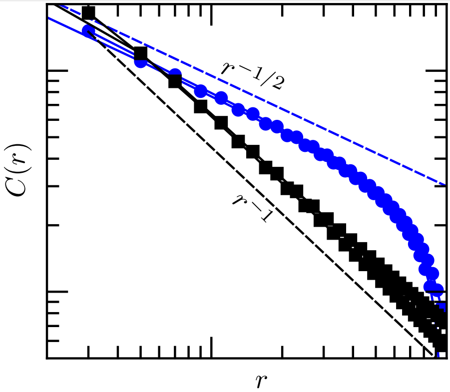
$$\langle c_1 c_2 \ldots c_{2n}\rangle_I = \left\langle \prod_{\{r_i\}} c_i \exp\left(2t_y \sum_{m=m_i+1}^{L} (-1)^{m+1} c_{mn_i-1} c_{mn_i}\right)\right\rangle_0$$

- Asymptotically $\langle c_i c_j\rangle_I = \mathcal{Q}(r_i, r_j)\mathcal{Q}_0^{-1} \sim d_{ij}^{-1/2}$



Bulk vs surface criticality

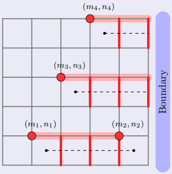
- bulk correlation $\sim r^{-1/2}$
- surface correlation $\sim r^{-1}$
- corner correlation $\sim r^{-1}$, r^{-2} or r^{-3}



Partition function of the dimer model with $2n$ monomers

$$\mathcal{Q}_{2n} = \int \mathcal{D}[c,h] \exp\left(\mathcal{S}_0 + \sum_{\{r_i\}} c_{m_i n_i} h_i + 2t_y \sum_{\{r_i\}} \sum_{m=m_i+1}^{L} (-1)^{m+1} c_{mn_i - 1} c_{mn_i} \right).$$

- $c = 1$ Free fermion action $\mathcal{S}_0$
- Grassmann Magnetic field
- Topological defect line



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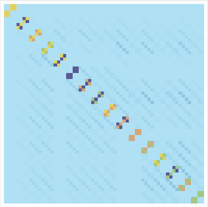
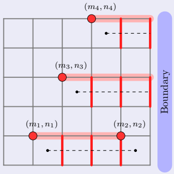


Partition function with monomers

Partition function of the dimer model with $2n$ monomers

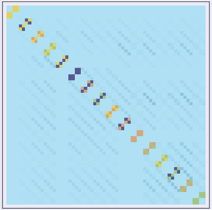
$$\mathcal{Q}_{2n} = \int \mathcal{D}[c,h] \exp\left(\mathcal{S}_0 + \sum_{\{r_i\}} c_{m_i n_i} h_i + 2t_y \sum_{\{r_i\}} \sum_{m=m_i+1}^{L} (-1)^{m+1} c_{mn_i-1} c_{mn_i}\right).$$

- $c = 1$ Free fermion action $\mathcal{S}_0$
- Grassmann Magnetic field
- Topological defect line

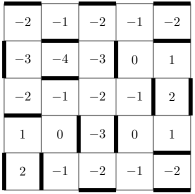


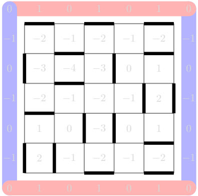
Pfaffian formulation

$$\boxed{\mathcal{Q}_{2n} = \mathrm{pf}(W)\mathrm{pf}(C)}$$

- $W^{\mu\nu}_{\alpha\beta} = \delta_{\alpha\beta} M^{\mu\nu}_{\alpha} + V^{\mu\nu}_{\alpha\beta}$ $\quad\rightarrow\quad$ $W =$ 
- $\dim(W) = L^2 \times L^2$, $\dim(C) = 2n \times 2n$

Bosonic formulation of the dimer model





$c = 1$ Free boson theory

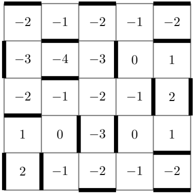
- Action: $\mathcal{S}[\phi] = \frac{g}{2} \int \mathrm{d}x\mathrm{d}y \; \left(\nabla\phi\right)^2$ where g stiffness
- Vertex operators: $V_{e,m}(z) =: \mathrm{e}^{ie\phi + im\psi}:$ where $\partial_i \psi = \epsilon_{ij}\partial_j\phi$
- Scaling dimensions: $x_g(e,m) = \frac{e^2}{4\pi g} + \pi g m^2$
- Comparison with exact KFT results fixes $g = 1/4\pi$

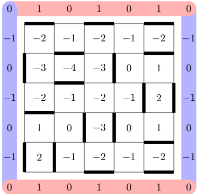
bcc operators

- Change of boundary conditions in each corner $h_{bcc} = \frac{g}{2\pi}\Delta\phi_b^2 = 1/32$
- Crucial for the extrapolation of the central charge

Now we can look at the corner free energy !

Bosonic formulation of the dimer model





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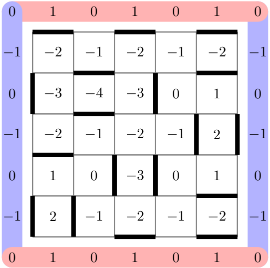
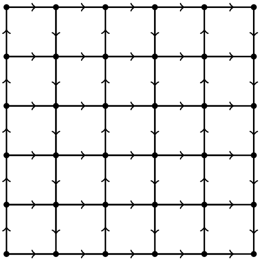
Corner free energy and CFT



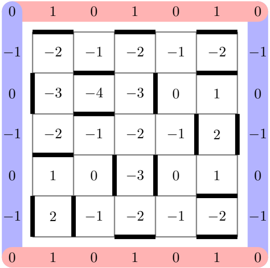


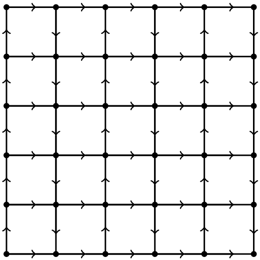
Even size dimer model $\to$ Kasteleyn theory

$$Q_0 = \sqrt{\det K} = \prod_{p,q=1}^{L} \left[4\cos^2 \frac{\pi p}{L+1} + 4\cos^2 \frac{\pi q}{L+1} \right]$$

- Asymptotic of Q_0 gives $f_{\mathrm{corner}} = 0$
- $4\left[\frac{\pi}{\theta} h_{bcc} + \frac{c}{24}\left(\frac{\theta}{\pi} - \frac{\pi}{\theta}\right)\right] = 0$ with 4 bcc operators with $h_{bcc} = 1/32 \to c = 1$

Corner free energy and CFT





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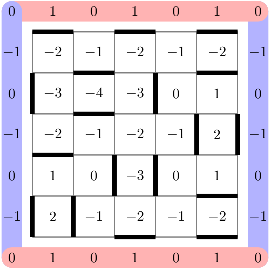

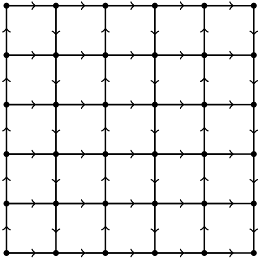

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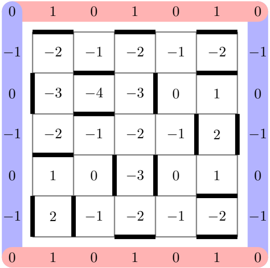

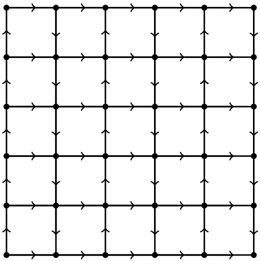

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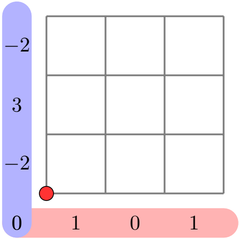

Odd size lattice with one monomer at the boundary (Tzeng-Wu)

$$Q_1 = \prod_{p,q=1}^{L-1} \left[4\cos^2 \frac{\pi p}{L+1} + 4\cos^2 \frac{\pi q}{L+1} \right]$$

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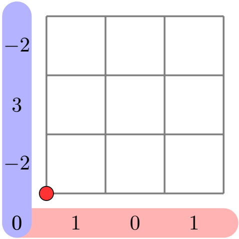
A similar analysis can be done in a $c = -2$ formalism !

Corner free energy and CFT



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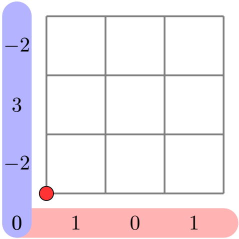

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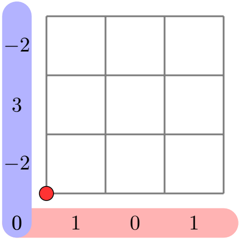

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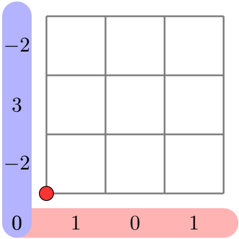

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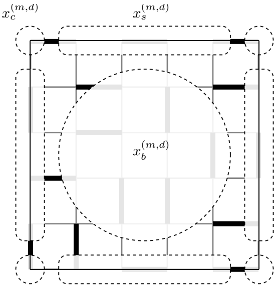
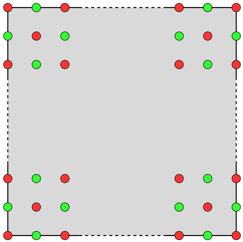
Critical exponents (bulk surface and corner)

Monomer and dimer scaling dimensions

scaling dimension ($g_{\mathrm{free}} = 1/4\pi$)	bulk	surface	corner
$x^{(d)}$	1	1	2
$x^{(m)}$	1/4	1/2	1/2 or 3/2

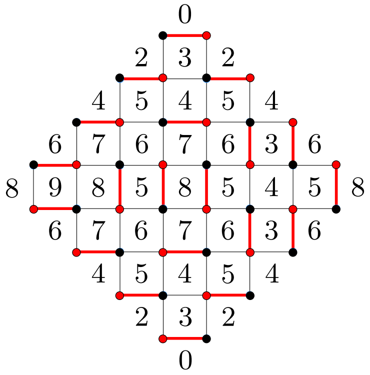
- The monomer corner scaling dimension is not unique (Why ? IDK)
- In perfect agreement with the height mapping formulation
- Relation between corner and surface dimensions $x_c = \frac{\pi}{\theta} x_s$ satisfied



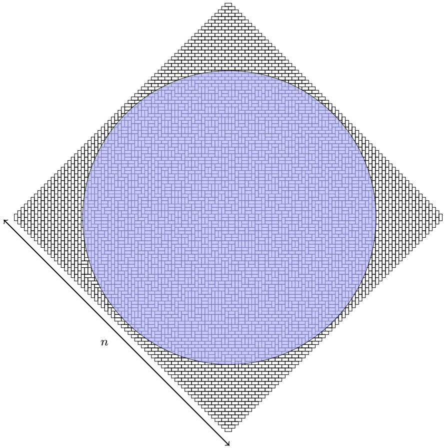


Aztec diamond dimer

- Highly constrained configurations
- Highly excited boundaries → Non conformal boundaries
- Bipartite planar lattice → Kasteleyn still holds → free fermion
- $\mathcal{S}[\phi] = \frac{g(x,y)}{2} \int \mathrm{d}x\mathrm{d}y \, \left(\nabla\phi\right)^2$



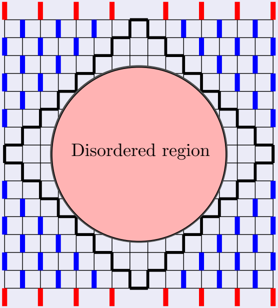
Arctic circle !!!

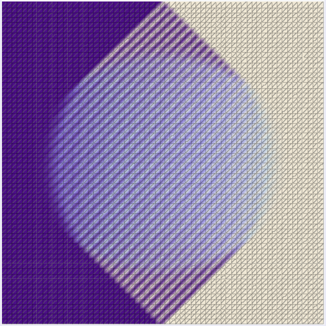


Main math results

- Mapping to non-intersecting paths $\rightarrow Z = 2^{n(n+1)/2}$ (Why so simple ?)
- Gaussian fluctuations (bulk $\sim$ square lattice)
- Boundary fluctuations $\rightarrow$ corner growth process $\rightarrow$ GUE ensemble

Arctic circle phenomenon in the dimer model





- $2d$ statistical problem $\rightarrow$ $1d$ quantum chain in imaginary time
- Transfer matrix $\mathcal{T}$ $\rightarrow$ Quantum hamiltonian $\mathcal{H} = -\log \mathcal{T}$
- Particular initial and final state $|\psi_0\rangle$ $\rightarrow$ Domain wall initial state

Strategy

- Step I $\rightarrow$ Compute fermion correlators exactly on the lattice
- Step II $\rightarrow$ Manage to study the scaling behavior (x/R and y/R fixed, $R \rightarrow \infty$)
- Step III $\rightarrow$ Make a correspondance to correlators in a Dirac field theory

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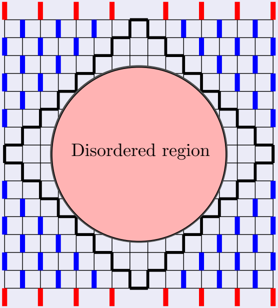

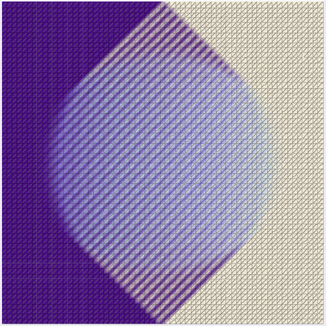

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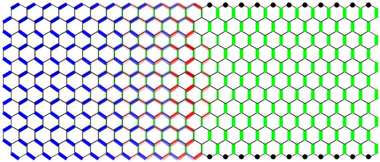
What kind of model can we tackle ?

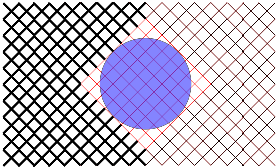
Single band models (XX chain and hexagonal dimers)

$$H = \int \frac{\mathrm{d}k}{2\pi} \varepsilon(k) c^\dagger(k) c(k) \tag{1}$$



Two bands models (6-vertex and square dimers)

$$H = \int \frac{\mathrm{d}k}{2\pi} \varepsilon_+(k) a^\dagger(k) a(k) + \varepsilon_-(k) b^\dagger(k) b(k) \tag{2}$$



Single Band expression

- we are dealing with a **free fermion** problem, so every correlator can be reduced to a combination of **two-point functions** thanks to Wick's theorem. Therefore, the quantity of interest is the propagator

$$\left\langle c^\dagger(x,y)c(x',y')\right\rangle \equiv \begin{cases} \dfrac{\langle\psi|e^{-(R-y)H}\, c_x^\dagger\, e^{-(y-y')H}\, c_{x'}\, e^{-(R+y')H}|\psi\rangle}{\langle\psi|e^{-2R\,H}|\psi\rangle} & (y>y') \\[2ex] -\dfrac{\langle\psi|e^{-(R-y')H}\, c_{x'}\, e^{-(y'-y)H}\, c_x^\dagger\, e^{-(R+y)H}|\psi\rangle}{\langle\psi|e^{-2R\,H}|\psi\rangle} & (y<y') \end{cases}$$

- going to momentum space, and using methods that are familiar from bosonization, one gets the **key technical result**

$$\left\langle c^\dagger(k,y)c(k',y')\right\rangle \equiv \frac{e^{iR(\tilde\varepsilon(k)-\tilde\varepsilon(k'))}\, e^{-(y\varepsilon(k)-y'\varepsilon(k'))}}{2i\,\sin\left(\frac{k-k'}{2}-i0^+\right)} \tag{3}$$

where $\varepsilon(k)$ is the **dispersion relation** and $\tilde\varepsilon(k)$ is its **Hilbert transform**,

$$\tilde\varepsilon(k) \equiv \text{p.v.}\int_{-\pi}^{\pi}\frac{dk'}{2\pi}\varepsilon(k')\cot\left(\frac{k-k'}{2}\right).$$

Hamiltonian in k space

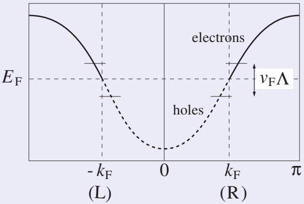
$$H = \int \frac{\mathrm{d}k}{2\pi} \varepsilon(k) c^\dagger(k) c(k)$$

The low-energy theory is defined in terms of creation and annihilation operators in the vicinity of the Fermi points

Slow fields ψ_R and ψ_L

$$c(x) = \sqrt{a}(\psi_R(x,t)e^{ik_F x} + \psi_L(x,t)e^{-ik_F x})$$
$$c^\dagger(x) = \sqrt{a}(\psi_R^\dagger(x,t)e^{ik_F x} + \psi_L^\dagger(x,t)e^{-ik_F x})$$

such that $\{\psi_R(x,t), \psi_R^\dagger(x',t)\} = \delta(x-x')$... etc



ψ_L and $\psi_R \to (1+1d)$ Dirac field theory

$$\mathcal{L} = i\bar{\Psi}(\gamma^0\partial_t - v\gamma^1\partial_x\Psi)$$

with Ψ Dirac spinor with ψ_L and ψ_R component.

$1d$ electron gas in one slide

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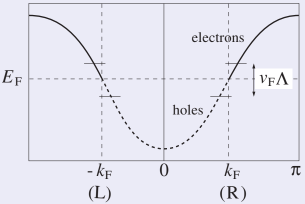

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Asymptotic analysis: General framework

Scaling regime (x/R and y/R fixed, $R \to \infty$)

$$\left\langle c^\dagger(x,y)c(x',y')\right\rangle = \frac{e^{-\frac{1}{2}[\sigma(x,y)+\sigma(x',y')]}}{2\pi i}\left[\frac{e^{-i[\varphi(x,y)-\varphi(x',y')]}}{2\sin\left(\frac{z(x,y)-z(x',y')}{2}\right)} - \frac{e^{i[\varphi^*(x,y)-\varphi^*(x',y')]}}{2\sin\left(\frac{z^*(x,y)-z^*(x',y')}{2}\right)}\right]$$

Propagators of Dirac field $\Psi^\dagger = \left(\psi^\dagger \ \overline{\psi}^\dagger\right)$

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+ Gauge transformation $\Psi(x,y) \to e^{i\,\mathrm{Re}\varphi(x,y)\gamma^5}\, e^{-\mathrm{Im}\varphi(x,y)}\,\Psi(x,y)$ et $\Psi^\dagger(x,y) \to \Psi^\dagger(x,y)e^{-i\,\mathrm{Re}\varphi(x,y)\gamma^5}\, e^{\mathrm{Im}\varphi(x,y)}$

This is familiar to boundary CFT expert $\to$ correlators on a strip+non flat metric

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Dirac action on a $2d$ curved metric

Curved Dirac field

$$S = \frac{1}{2\pi} \int d^2x \sqrt{\det g}\, e_a^\mu \left[\frac{i}{2} \overline{\Psi} \gamma^a \overleftrightarrow{\partial}_\mu \Psi \right] . \tag{4}$$

Here e_a^μ is the tetrad, and $(d^2x\sqrt{\det g})$ is the volume element. The spin connection drops out of the two-dimensional Dirac action. We are free to chose the coordinate system, and it is natural to take the coordinates x^1, x^2 such that

$$\begin{cases} x^z &= x^1 + i\, x^2 = z(x,y) \\ x^{\bar{z}} &= x^1 - i\, x^2 = z^*(x,y) . \end{cases}$$

In this coordinate system, we take the following tetrad:

$$e_a^\mu = e^{-\sigma} \delta_{a\mu} ,$$

where σ is the function $\sigma(x,y)$ that appeared previously; note that the metric is simply

$$ds^2 = e^{2\sigma} \left[(dx^1)^2 + (dx^2)^2 \right] .$$

Exemple: Metric for the XX chain (dimer, 6vertex..much more complicated)

$$e^{\sigma(x,y)} = \sqrt{R^2 - x^2 - y^2}$$

Perspectives: In progress or not

Interesting questions

- Connection with the bosonic theory ?
- Study of boundary correlations ?
- Explore the field theory more carefully, partition function ?
- Can we tell something interesting about the real time quench ?
- What remains true in the interacting case and what is wrong ?