Worshop « Theoretical Cosmology in the Era of Large Survey »
Focus Week on Dark Energy and Modified Gravity

The Galileo Galilei Institute for Theoretical Physics, Firenze
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The Equations of State for Dark Sector Perturbations

f(R) as a dark energy fluid

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The Equation of State Approach to cosmological perturbation for Dark Energy

Modified Gravity Models

Dark Energy Fluid

Equations of State for Perturbations

$$\Gamma_{\mathrm{de}} = \Gamma_{\mathrm{de}}(\Delta_{\mathrm{de}}, \Theta_{\mathrm{de}}, \Delta_{\mathrm{m}}, \Theta_{\mathrm{m}})$$

$$\Pi^{\mathrm{S}}_{\mathrm{de}} = \Pi^{\mathrm{S}}_{\mathrm{de}}(\Delta_{\mathrm{de}}, \Theta_{\mathrm{de}}, \Delta_{\mathrm{m}}, \Theta_{\mathrm{m}})$$

The Λ Cold Dark Matter model

Background dynamics

$$\Omega_{\rm b} + \Omega_{\rm rad} + \Omega_{\rm cdm} + \Omega_{\rm de} = 1 + \frac{\rm K}{a^2 H^2}$$

$$\Omega_{\rm i} \equiv \frac{\rho_{\rm i}}{3H^2}$$

$$w_{\rm i} \equiv \frac{P_{\rm i}}{\rho_{\rm i}}$$

$$\rho_{\rm i}' + 3(1 + w_{\rm i})\rho_{\rm i} = 0$$

$$\frac{\ddot{a}}{a} = -\frac{\rho + 3P}{6M_{\rm Pl}^2}$$

Equation of state parameter at the background level

w<-1/3 → ACCELERATION

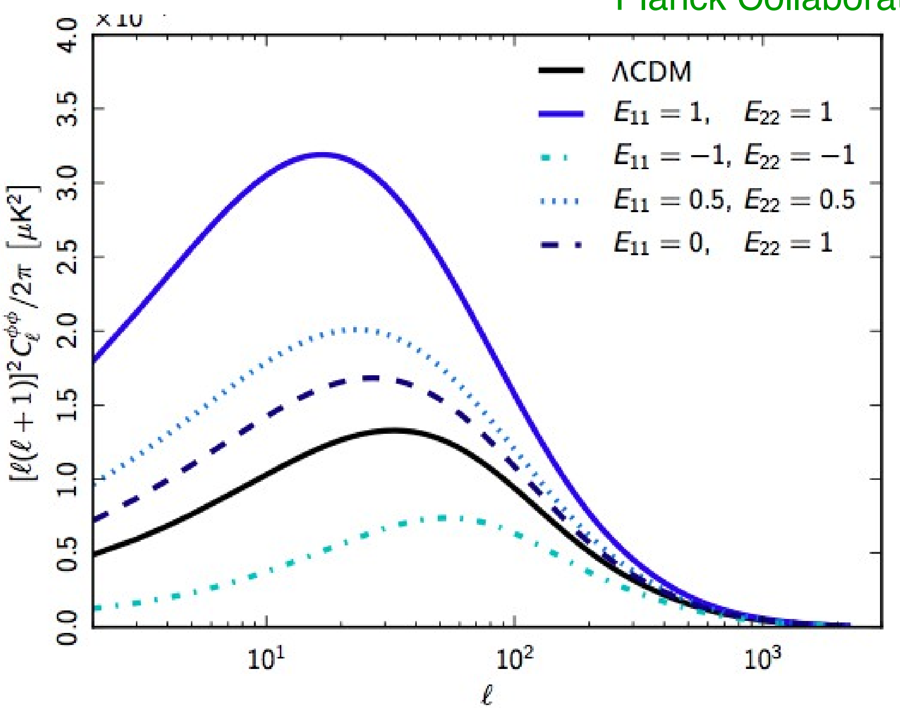
w = -1 : COSMOLOGICAL CONSTANT

w = w(t) : QUINTESSENCE

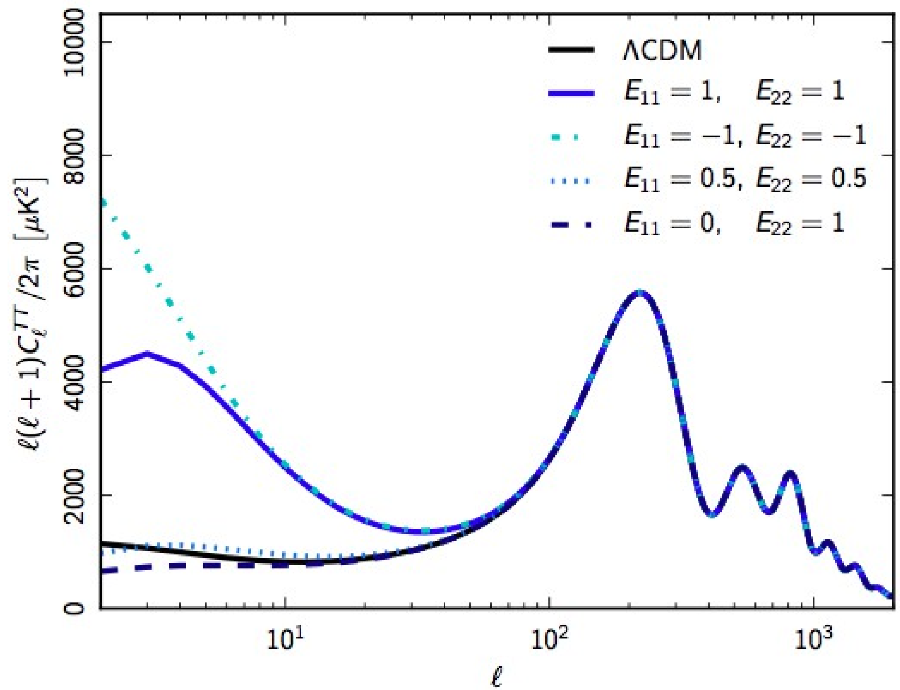
Perturbation parametrization

The evolution of cosmological perturbations encodes extra information about the nature of dark energy.

Planck Collaboration [astro-ph/1502.01590]



Lensing potential



Integrated Sachs Wolf Effect

+ Modifications of the **CMB B-mode amplitude** and scale dependence

Amendola-Ballesteros-Pettorino [astro-ph/1405.7004]

Perturbation parametrization

The evolution of cosmological perturbations encodes extra information about the nature of dark energy.

$$ds^2 = a^2[-(1+2\Psi)dt^2 + (1-2\Phi)dx^2]$$

Gravitational potentials in the conformal Newtonian gauge:

$$-k^2\Psi = 4\pi G a^2 \mu(a,\mathbf{k})\rho\Delta$$

$$\eta(a,\mathbf{k}) = \Phi/\Psi$$

ΛCDM:

$$\eta(a,\mathbf{k}) = \mu(a,\mathbf{k}) = 1$$

Parametrization:

$$\mu(a,k) = 1 + f_1(a)\frac{1+c_1(\lambda H/k)^2}{1+(\lambda H/k)^2}$$

$$\eta(a,k) = 1 + f_2(a)\frac{1+c_2(\lambda H/k)^2}{1+(\lambda H/k)^2}$$

$$f_i(a) = E_{ii}\Omega_{de}(a)$$

OR

$$f_i(a) = E_{i1} + E_{i2}(1-a)$$

The equations of state for dark sector perturbations

The equation of state at the perturbative level (formalism)

$$G_{\mu\nu} = \kappa(T_{\mu\nu} + D_{\mu\nu})$$

Effective stress-energy tensor of the dark sector

The equations of state for dark sector perturbations

The equation of state at the perturbative level (formalism)

$$G_{\mu\nu} = \kappa(T_{\mu\nu} + D_{\mu\nu})$$

Effective stress-energy tensor of the dark sector

First order linear perturbation of the stress energy tensor:

$$\delta D^{\mu}{}_{\nu} = (\rho\delta + \delta P)\, u^{\mu} u_{\nu} + (\rho + P)\,(u_{\nu}\delta u^{\mu} + u^{\mu}\delta u_{\nu}) + \delta P \delta^{\mu}{}_{\nu} + P\Pi^{\mu}{}_{\nu}$$

Differential equations for the evolution of cosmological perturbations:

$$\Delta'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}}$$

$$\hat{\Theta}'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - \epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}} - 3\Gamma_{\mathrm{de}}$$

$$\hat{\Theta}'_{\mathrm{m}} = -\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3Y$$

$$\Delta'_{\mathrm{m}} = -g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3X$$

The equations of state for dark sector perturbations

The equation of state at the perturbative level (formalism)

$$G_{\mu\nu} = \kappa(T_{\mu\nu} + D_{\mu\nu})$$

Effective stress-energy tensor of the dark sector

First order linear perturbation of the stress energy tensor:

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Differential equations for the evolution of cosmological perturbations:

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$$\hat{\Theta}'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - \epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}} - 3\Gamma_{\mathrm{de}}$$

$$\hat{\Theta}'_{\mathrm{m}} = -\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3Y$$

$$\Delta'_{\mathrm{m}} = -g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3X$$

where the **anisotropic stress** and the **entropy perturbation** are specified as:

$$\boxed{\begin{aligned} \Gamma_{\mathrm{de}} &= \Gamma_{\mathrm{de}}(\Delta_{\mathrm{de}}, \Theta_{\mathrm{de}}, \Delta_{\mathrm{m}}, \Theta_{\mathrm{m}}) \\ \Pi^{\mathrm{S}}_{\mathrm{de}} &= \Pi^{\mathrm{S}}_{\mathrm{de}}(\Delta_{\mathrm{de}}, \Theta_{\mathrm{de}}, \Delta_{\mathrm{m}}, \Theta_{\mathrm{m}}) \end{aligned}}$$

Equation of state for perturbations

The stress-energy tensor of f(R) gravity

Action of f(R) gravity in the Jordan frame

Action of matter fields

$$S = \frac{1}{2\kappa}\int \mathrm{d}^4x\,\sqrt{-g}\,\{\mathcal{R} + f(\mathcal{R})\} + S_{\mathrm{m}}$$

The stress-energy tensor of f(R) gravity

$$S = \frac{1}{2\kappa} \int \mathrm{d}^4 x \, \sqrt{-g} \, \{\mathcal{R} + f\,(\mathcal{R})\} + S_{\mathrm{m}}$$

<u>Field equations</u>

$$G_{\mu\nu} = \kappa \left(T_{\mu\nu} + U_{\mu\nu} \right)$$

The stress-energy tensor of f(R) gravity

$$S = \frac{1}{2\kappa} \int \mathrm{d}^4 x \sqrt{-g} \left\{ \mathcal{R} + f\left(\mathcal{R}\right) \right\} + S_{\mathrm{m}}$$

$$G_{\mu\nu} = \kappa \left(T_{\mu\nu} + U_{\mu\nu} \right)$$

<u>Stress-energy tensor of f(R) gravity</u>

$$\boxed{\kappa U_{\mu\nu} \equiv \tfrac{1}{2} f g_{\mu\nu} - \left(R_{\mu\nu} + g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_{\mathcal{R}}}$$

Notations:

$$f' = \frac{df}{d \ln a} \qquad f_{\mathcal{R}} = \frac{df}{d\mathcal{R}}$$

The stress-energy tensor of f(R) gravity

$$S = \frac{1}{2\kappa} \int \mathrm{d}^4 x \sqrt{-g} \, \{\mathcal{R} + f(\mathcal{R})\} + S_{\mathrm{m}}$$

$$G_{\mu\nu} = \kappa \left(T_{\mu\nu} + U_{\mu\nu} \right)$$

$$\kappa U_{\mu\nu} \equiv \tfrac{1}{2} f g_{\mu\nu} - \left(R_{\mu\nu} + g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_{\mathcal{R}}$$

$$\Omega_{\mathrm{de}} = -\frac{f}{6H^2} + (1 - \epsilon_{\mathrm{H}}) f_{\mathcal{R}} - f'_{\mathcal{R}}$$

$$w_{\mathrm{de}} + 1 = -\frac{1}{3\Omega_{\mathrm{de}}} \left(2\epsilon_{\mathrm{H}} f_{\mathcal{R}} + (1 + \epsilon_{\mathrm{H}}) f'_{\mathcal{R}} - f''_{\mathcal{R}} \right)$$

where $\Omega = \dfrac{\rho}{3H^2}$ and $\epsilon_{\mathrm{H}} \equiv -\dfrac{H'}{H}$

FRW universe – Friedmann equation

$$\Omega_{\rm m} + \Omega_{\rm de} = 1$$
$$w_{\rm m}\Omega_{\rm m} + w_{\rm de}\Omega_{\rm de} = \tfrac{2}{3}\epsilon_{\rm H} - 1$$

where $\boxed{\Omega = \dfrac{\rho}{3H^2} \quad \text{and} \quad \epsilon_{\rm H} \equiv -\dfrac{H'}{H}}$

First order linear perturbations - Geometry

Calculations are done in Fourier space,
in both synchronous and conformal Newtonian gauges.

<u>Synchronous gauge</u>

$$\boxed{\delta g_{ij} = a^2 h_{ij}}$$

First order linear perturbations - Geometry

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Synchronous gauge

$$\boxed{\delta g_{ij} = a^2 h_{ij}}$$

Basis matrices

$$\sigma_{ij} = \hat{k}_i \hat{k}_j - \frac{1}{3}\delta_{ij}$$

$$v_{ij}^{(1)} = 2\hat{k}_{(i}\hat{l}_{j)}$$

$$v_{ij}^{(2)} = 2\hat{k}_{(i}\hat{m}_{j)}$$

$$e_{ij}^{\times} = 2\hat{l}_{[i}\hat{m}_{j]}$$

$$e_{ij}^{+} = \hat{l}_i\hat{l}_j - \hat{m}_i\hat{m}_j$$

First order linear perturbations - Geometry

Calculations are done in Fourier space,
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Synchronous gauge

$$\boxed{\delta g_{ij} = a^2 h_{ij}}$$

Two scalar modes

Two vector modes

Two tensor modes

$$h_{ij} = \frac{1}{3} h \delta_{ij} + h_{||} \sigma_{ij} + h^{\mathrm{V}} \cdot v_{ij} + h^{\mathrm{T}} \cdot e_{ij}$$

$$6\eta \equiv h_{||} - h$$

First order linear perturbations - Geometry

Calculations are done in Fourier space,
in both synchronous and conformal Newtonian gauges.

Conformal Newtonian gauge

$$\delta g_{00} = -2a^2\psi$$

$$\delta g_{ij} = -2a^2\phi\delta_{ij} + h^{\mathrm{V}} \cdot v_{ij} + h^{\mathrm{T}} \cdot e_{ij}$$

First order linear perturbations - Geometry

Calculations are done in Fourier space,
in both synchronous and conformal Newtonian gauges.

<u>Appearance of an additional perturbed d.o.f . due to f(R)</u>

Bean-Bernat-Pogosian-Silvestri-Trodden [astro-ph/0611321]

$$\chi \equiv -\frac{f'_{\mathcal{R}}}{\bar{\epsilon}_{\mathrm{H}}} \frac{\delta\mathcal{R}}{6H^2}$$

A key point in our analysis is that we eliminate this geometrical d.o.f in the benefit of the perturbed fluid d.o.f.

Notation: $\bar{\epsilon}_{\mathrm{H}} = -\frac{\mathcal{R}'}{6H^2}$ hence, $\chi = f_{\mathcal{R}\mathcal{R}}\delta\mathcal{R}$

First order linear perturbations - Fluid

First order perturbation of a generic stress-energy tensor

$$\delta D^{\mu}{}_{\nu} = (\rho\delta + \delta P)\, u^{\mu} u_{\nu} + (\rho + P)\,(u_{\nu}\delta u^{\mu} + u^{\mu}\delta u_{\nu}) + \delta P \delta^{\mu}{}_{\nu} + P\Pi^{\mu}{}_{\nu}$$

First order linear perturbations - Fluid

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Density contrast
$\delta \equiv \delta\rho/\rho$

Hubble flow
$u_{\nu} = (-1, \vec{0})$

Perturbed velocity field
$\delta u_{\nu} = (0, \delta u_i)$

Scalar mode of the perturbed velocity field: $\theta \equiv \dfrac{\mathrm{i}k^j \delta u_j}{k^2}$

Perturbed pressure, recast into the **gauge invariant entropy perturbation**

Gauge invariant dimensionless linear combination

$$\Delta \equiv \delta + 3\,(1 + w)\, H\theta$$

Dimensionless perturbed velocity field

$$\Theta \equiv 3\,(1 + w)\, H\theta$$

$$w\Gamma = \frac{\delta\hat{P}}{\rho} - \frac{dP}{d\rho}\left(\Delta - \hat{\Theta}\right)$$

Anisotropic stress

$$\Pi_{ij} = \Pi^{\mathrm{S}}\sigma_{ij} + \Pi^{\mathrm{V}} \cdot v_{ij} + \Pi^{\mathrm{T}} \cdot e_{ij}$$

One scalar mode   Two vector/tensor modes

Gauge invariant notations

	Synchronous gauge	Conformal Newtonian gauge
T	$\frac{h'_{\parallel}}{2\mathrm{K}^2}$	0
Y	$T' + \epsilon_{\mathrm{H}} T$	ψ
Z	$\eta - T$	ϕ
X	$Z' + Y$	$Z' + Y$
W	$X' - \epsilon_{\mathrm{H}}(X+Y)$	$X' - \epsilon_{\mathrm{H}}(X+Y)$
$\hat{\chi}$	$\chi_s + f'_{\mathcal{R}} T$	χ_c
$\hat{\chi}'$	$\chi'_s + (f''_{\mathcal{R}} - \epsilon_{\mathrm{H}} f'_{\mathcal{R}})T$	$\chi'_c - f'_{\mathcal{R}}\psi$

f(R) sector

$$\delta g_{00} = -2a^2\psi$$

$$\delta g_{ij} = -2a^2\phi\delta_{ij} + h^{\mathrm{V}} \cdot v_{ij} + h^{\mathrm{T}} \cdot e_{ij}$$

Gauge invariant notations

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T	$\dfrac{h'_{\parallel}}{2\mathrm{K}^2}$	0
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Z	$\eta - T$	ϕ
X	$Z' + Y$	$Z' + Y$
W	$X' - \epsilon_{\mathrm{H}}(X+Y)$	$X' - \epsilon_{\mathrm{H}}(X+Y)$
$\hat{\chi}$	$\chi_s + f'_{\mathcal{R}} T$	χ_c
$\hat{\chi}'$	$\chi'_s + (f''_{\mathcal{R}} - \epsilon_{\mathrm{H}} f'_{\mathcal{R}})T$	$\chi'_c - f'_{\mathcal{R}}\psi$

f(R) sector

Fluid variables

	S.G.	C.N.G.
$\hat{\Theta}$	$\Theta_s + 3(1+w)T$	Θ_c
$\delta\hat{P}$	$\delta P_s + P'_s T$	δP_c

Example 1: Expression of χ and the perturbed Ricci scalar

$$\chi \equiv -\frac{f'_{\mathcal{R}}}{\bar{\epsilon}_{\rm H}} \frac{\delta\mathcal{R}}{6H^2} = f_{\mathcal{R}\mathcal{R}}\delta\mathcal{R}$$

$$a^2\delta\mathcal{R} = \ddot{h} + 3\mathcal{H}\dot{h} - 4k^2\eta$$

$$a^2\delta\mathcal{R} = -6\ddot{\phi} - 6\mathcal{H}\left(\dot{\psi} + 3\dot{\phi}\right) - 12\left(\dot{\mathcal{H}} + \mathcal{H}^2\right)\psi - 4k^2\phi + 2k^2\psi$$

$$\delta\mathcal{R} = -6H^2(W + 4X - \tfrac{1}{3}K^2(Y - 2Z) - \bar{\epsilon}_{\rm H}T)$$

Gauge invariant notation	Synchronous gauge	Conformal Newtonian gauge
$\hat{\chi}$	$\chi + f'_{\mathcal{R}}T$	χ

$$\hat{\chi} = \frac{f'_{\mathcal{R}}}{\bar{\epsilon}_{\rm H}}\left\{W + 4X - \tfrac{1}{3}\mathrm{K}^2(Y - 2Z)\right\}$$

$$\mathrm{K} \equiv \frac{k}{aH}$$

Example 2: Space-Time projection of the perturbed field equations

$$2X = \Omega_{\mathrm{m}}\hat{\Theta}_{\mathrm{m}} + \Omega_{\mathrm{de}}\hat{\Theta}_{\mathrm{de}}$$

In the conformal Newtonian gauge:

$$\delta g_{00} = -2a^2\psi$$

$$\delta g_{ij} = -2a^2\phi\delta_{ij} + h^{\mathrm{V}} \cdot v_{ij} + h^{\mathrm{T}} \cdot e_{ij}$$

$$X = Z' + Y = \phi' + \psi$$

$$\hat{\Theta} = \Theta_c = 3H(1+w)\theta$$

$$\phi' + \psi = \tfrac{3}{2}H\left\{(1+w_{\mathrm{de}})\theta_{\mathrm{de}} + (1+w_{\mathrm{m}})\theta_{\mathrm{m}}\right\}$$

Example 2: Space-Time projection of the perturbed field equations

$$2X = \Omega_{\rm m}\hat{\Theta}_{\rm m} + \Omega_{\rm de}\hat{\Theta}_{\rm de}$$

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$$X = Z' + Y = \phi' + \psi$$

$$\hat{\Theta} = \Theta_c = 3H(1+w)\theta$$

$$\phi' + \psi = \frac{3}{2}H\left\{(1+w_{\rm de})\theta_{\rm de} + (1+w_{\rm m})\theta_{\rm m}\right\}$$

Example 3: Gauge invariant entropy perturbation

$$w\Gamma = \frac{\delta\hat{P}}{\rho} - \frac{dP}{d\rho}\left(\Delta - \hat{\Theta}\right)$$

Conformal Newtonian gauge:

$$w\Gamma = \left(\frac{\delta P}{\delta\rho} - \frac{dP}{d\rho}\right)\delta$$

Synchronous gauge:

$$w\Gamma = \frac{\delta P}{\rho} + \frac{P'}{\rho}T - \frac{dP}{d\rho}\left(\delta + 3H(1+w)\theta - 3H(1+w)\theta - 3(1+w)T\right)$$

$$w\Gamma = \left(\frac{\delta P}{\delta\rho} - \frac{dP}{d\rho}\right)\delta + \frac{P'}{\rho}T + 3(1+w)\frac{dP}{d\rho}T = \left(\frac{\delta P}{\delta\rho} - \frac{dP}{d\rho}\right)\delta$$

Perturbed field equations

$$\delta G_{\mu\nu} = \kappa\left(\delta T_{\mu\nu} + \delta U_{\mu\nu}\right)$$

$$-\tfrac{2}{3}\mathrm{K}^2 Z = \Omega_{\mathrm{m}}\Delta_{\mathrm{m}} + \Omega_{\mathrm{de}}\Delta_{\mathrm{de}}$$

$$2X = \Omega_{\mathrm{m}}\hat{\Theta}_{\mathrm{m}} + \Omega_{\mathrm{de}}\hat{\Theta}_{\mathrm{de}}$$

$$\tfrac{2}{3}W + 2X - \tfrac{2}{9}\mathrm{K}^2\left(Y - Z\right) = \Omega_{\mathrm{m}}(\delta\hat{P}_{\mathrm{m}}/\rho_{\mathrm{m}}) + \Omega_{\mathrm{de}}(\delta\hat{P}_{\mathrm{de}}/\rho_{\mathrm{de}})$$

$$\tfrac{1}{3}\mathrm{K}^2\left(Y - Z\right) = \Omega_{\mathrm{m}} w_{\mathrm{m}} \Pi^{\mathrm{S}}_{\mathrm{m}} + \Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{S}}_{\mathrm{de}}$$

Scalar

$$\tfrac{1}{6}h^{\mathrm{V}\prime\prime} + (\tfrac{1}{2} - \tfrac{1}{6}\epsilon_{\mathrm{H}})h^{\mathrm{V}\prime} = \Omega_{\mathrm{m}} w_{\mathrm{m}} \Pi^{\mathrm{V}}_{\mathrm{m}} + \Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{V}}_{\mathrm{de}}$$

Vector

$$\tfrac{1}{6}h^{\mathrm{T}\prime\prime} + (\tfrac{1}{2} - \tfrac{1}{6}\epsilon_{\mathrm{H}})h^{\mathrm{T}\prime} + \tfrac{1}{3}\mathrm{K}^2 h^{\mathrm{T}} = \Omega_{\mathrm{m}} w_{\mathrm{m}} \Pi^{\mathrm{T}}_{\mathrm{m}} + \Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{T}}_{\mathrm{de}}$$

Tensor

Perturbed stress-energy tensor of f(R) gravity

$$\kappa U_{\mu\nu} \equiv \tfrac{1}{2} f g_{\mu\nu} - \left(R_{\mu\nu} + g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_{\mathcal{R}}$$

$$\begin{aligned}\kappa \delta U_{\mu\nu} = & -f_{\mathcal{R}} \delta R_{\mu\nu} + \tfrac{1}{2} f \delta g_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} f_{\mathcal{R}} \delta \mathcal{R} - f_{\mathcal{R}\mathcal{R}} R_{\mu\nu} \delta \mathcal{R} \\ & + \delta \left(\nabla_\mu \nabla_\nu f_{\mathcal{R}} \right) - \left(\Box f_{\mathcal{R}} \right) \delta g_{\mu\nu} - g_{\mu\nu} \delta \left(\Box f_{\mathcal{R}} \right)\end{aligned}$$

$$\kappa\delta U_{\mu\nu} = -f_{\mathcal{R}}\delta R_{\mu\nu} + \tfrac{1}{2} f\,\delta g_{\mu\nu} + \tfrac{1}{2} g_{\mu\nu} f_{\mathcal{R}}\delta\mathcal{R} - f_{\mathcal{R}\mathcal{R}} R_{\mu\nu}\delta\mathcal{R}$$
$$+\delta\left(\nabla_\mu\nabla_\nu f_{\mathcal{R}}\right) - \left(\Box f_{\mathcal{R}}\right)\delta g_{\mu\nu} - g_{\mu\nu}\delta\left(\Box f_{\mathcal{R}}\right)$$

Perturbed fluid variables of the f(R) fluid
(VECTOR and TENSOR)

$$\Omega_{\rm de} w_{\rm de} \Pi^{\rm V}_{\rm de} = -\tfrac{1}{6} f_{\mathcal{R}} h^{\rm V}{}'' - \tfrac{1}{6}\left\{(3-\epsilon_{\rm H}) f_{\mathcal{R}} + f'_{\mathcal{R}}\right\} h^{\rm V}{}'$$
$$\Omega_{\rm de} w_{\rm de} \Pi^{\rm T}_{\rm de} = -\tfrac{1}{6} f_{\mathcal{R}} h^{\rm T}{}'' - \tfrac{1}{6}\left\{(3-\epsilon_{\rm H}) f_{\mathcal{R}} + f'_{\mathcal{R}}\right\} h^{\rm T}{}' - \tfrac{1}{6} f_{\mathcal{R}} {\rm K}^2 h^{\rm T}$$

Dimensionless wavenumber: $\mathrm{K} \equiv \dfrac{k}{aH}$

$$\kappa\delta U_{\mu\nu} = -f_{\mathcal{R}}\delta R_{\mu\nu} + \frac{1}{2}f\delta g_{\mu\nu} + \frac{1}{2}g_{\mu\nu}f_{\mathcal{R}}\delta\mathcal{R} - f_{\mathcal{R}\mathcal{R}}R_{\mu\nu}\delta\mathcal{R} \\ +\delta\left(\nabla_\mu\nabla_\nu f_{\mathcal{R}}\right) - \left(\Box f_{\mathcal{R}}\right)\delta g_{\mu\nu} - g_{\mu\nu}\delta\left(\Box f_{\mathcal{R}}\right)$$

Perturbed fluid variables of the f(R) fluid
(VECTOR and TENSOR)

$$\Omega_{\mathrm{de}}w_{\mathrm{de}}\Pi^{\mathrm{V}}_{\mathrm{de}} = -\frac{1}{6}f_{\mathcal{R}}h^{\mathrm{V}\prime\prime} - \frac{1}{6}\left\{(3-\epsilon_{\mathrm{H}})f_{\mathcal{R}} + f'_{\mathcal{R}}\right\}h^{\mathrm{V}\prime}$$

$$\Omega_{\mathrm{de}}w_{\mathrm{de}}\Pi^{\mathrm{T}}_{\mathrm{de}} = -\frac{1}{6}f_{\mathcal{R}}h^{\mathrm{T}\prime\prime} - \frac{1}{6}\left\{(3-\epsilon_{\mathrm{H}})f_{\mathcal{R}} + f'_{\mathcal{R}}\right\}h^{\mathrm{T}\prime} - \frac{1}{6}f_{\mathcal{R}}\mathrm{K}^2h^{\mathrm{T}}$$

To get the EoS in the tensor and vector sectors, one replaces h" thanks to the fields equations:

$$\frac{1}{6}h^{\mathrm{V}\prime\prime} + (\frac{1}{2} - \frac{1}{6}\epsilon_{\mathrm{H}})h^{\mathrm{V}\prime} = \Omega_{\mathrm{m}}w_{\mathrm{m}}\Pi^{\mathrm{V}}_{\mathrm{m}} + \Omega_{\mathrm{de}}w_{\mathrm{de}}\Pi^{\mathrm{V}}_{\mathrm{de}}$$

$$\frac{1}{6}h^{\mathrm{T}\prime\prime} + (\frac{1}{2} - \frac{1}{6}\epsilon_{\mathrm{H}})h^{\mathrm{T}\prime} + \frac{1}{3}\mathrm{K}^2h^{\mathrm{T}} = \Omega_{\mathrm{m}}w_{\mathrm{m}}\Pi^{\mathrm{T}}_{\mathrm{m}} + \Omega_{\mathrm{de}}w_{\mathrm{de}}\Pi^{\mathrm{T}}_{\mathrm{de}}$$

Perturbed field equations

$$\tfrac{1}{6}h^{\mathrm{V}\prime\prime} + (\tfrac{1}{2} - \tfrac{1}{6}\epsilon_{\mathrm{H}})h^{\mathrm{V}\prime} = \Omega_{\mathrm{m}} w_{\mathrm{m}} \Pi^{\mathrm{V}}_{\mathrm{m}} + \Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{V}}_{\mathrm{de}}$$

$$\tfrac{1}{6}h^{\mathrm{T}\prime\prime} + (\tfrac{1}{2} - \tfrac{1}{6}\epsilon_{\mathrm{H}})h^{\mathrm{T}\prime} + \tfrac{1}{3}\mathrm{K}^2 h^{\mathrm{T}} = \Omega_{\mathrm{m}} w_{\mathrm{m}} \Pi^{\mathrm{T}}_{\mathrm{m}} + \Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{T}}_{\mathrm{de}}$$

Perturbed fluid variables of the f(R) fluid
(VECTOR and TENSOR)

$$\Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{V}}_{\mathrm{de}} = -\tfrac{1}{6} f_{\mathcal{R}} h^{\mathrm{V}\prime\prime} - \tfrac{1}{6}\left\{(3-\epsilon_{\mathrm{H}})f_{\mathcal{R}} + f'_{\mathcal{R}}\right\} h^{\mathrm{V}\prime}$$

$$\Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{T}}_{\mathrm{de}} = -\tfrac{1}{6} f_{\mathcal{R}} h^{\mathrm{T}\prime\prime} - \tfrac{1}{6}\left\{(3-\epsilon_{\mathrm{H}})f_{\mathcal{R}} + f'_{\mathcal{R}}\right\} h^{\mathrm{T}\prime} - \tfrac{1}{6} f_{\mathcal{R}} \mathrm{K}^2 h^{\mathrm{T}}$$

Equations of state for perturbation in the vector and tensor sectors:

$$\Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{V}}_{\mathrm{de}} = -\frac{1}{6}\frac{f'_{\mathcal{R}}}{1+f_{\mathcal{R}}} h^{\mathrm{V}\prime}$$

$$\Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{T}}_{\mathrm{de}} = -\frac{1}{6}\frac{f'_{\mathcal{R}}}{1+f_{\mathcal{R}}} h^{\mathrm{T}\prime} + \frac{1}{3}\frac{f_{\mathcal{R}}}{1+f_{\mathcal{R}}} \mathrm{K}^2 h^{\mathrm{T}}$$

$$\kappa\delta U_{\mu\nu} = -f_{\mathcal{R}}\delta R_{\mu\nu} + \tfrac{1}{2}f\delta g_{\mu\nu} + \tfrac{1}{2}g_{\mu\nu}f_{\mathcal{R}}\delta\mathcal{R} - f_{\mathcal{R}\mathcal{R}}R_{\mu\nu}\delta\mathcal{R}$$
$$+\delta\left(\nabla_\mu\nabla_\nu f_{\mathcal{R}}\right) - \left(\Box f_{\mathcal{R}}\right)\delta g_{\mu\nu} - g_{\mu\nu}\delta\left(\Box f_{\mathcal{R}}\right)$$

Perturbed fluid variables of the f(R) sector
(SCALAR)

$$\Omega_{\rm de}\Delta_{\rm de} = -g_{\rm K}\epsilon_{\rm H}\hat{\chi} + f'_{\mathcal{R}}X + \tfrac{2}{3}f_{\mathcal{R}}{\rm K}^2 Z$$

$$\Omega_{\rm de}\hat{\Theta}_{\rm de} = \hat{\chi}' - \hat{\chi} - 2f_{\mathcal{R}}X$$

$$\Omega_{\rm de}(\delta\hat{P}_{\rm de}/\rho_{\rm de}) = \tfrac{1}{3}\hat{\chi}'' + (\tfrac{2}{3} - \tfrac{1}{3}\epsilon_{\rm H})\hat{\chi}' - \left(1 - \tfrac{1}{3}\epsilon_{\rm H} - \tfrac{2}{9}{\rm K}^2\right)\hat{\chi}$$
$$-\tfrac{2}{3}f_{\mathcal{R}}W - 2(f_{\mathcal{R}} + \tfrac{1}{3}f'_{\mathcal{R}})X + \tfrac{2}{9}f_{\mathcal{R}}{\rm K}^2\left(Y - Z\right)$$

$$\Omega_{\rm de}w_{\rm de}\Pi^{\rm S}_{\rm de} = -\tfrac{1}{3}{\rm K}^2\hat{\chi} - \tfrac{1}{3}f_{\mathcal{R}}{\rm K}^2\left(Y - Z\right)$$

Anisotropic stress of the dark sector

Start with the field equation:

$$\tfrac{1}{3}\mathrm{K}^2\left(Y - Z\right) = \Omega_{\mathrm{m}} w_{\mathrm{m}} \Pi^{\mathrm{S}}_{\mathrm{m}} + \Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{S}}_{\mathrm{de}}$$

Assume no matter anisotropic stress.

In the C.N.G.:

$$X = \phi' + \psi$$
$$Y = \psi$$
$$Z = \phi$$

From the projection of the stress energy tensor of the dark sector:

$$\Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{S}}_{\mathrm{de}} = -\tfrac{1}{3}\mathrm{K}^2\hat{\chi} - \tfrac{1}{3} f_{\mathcal{R}} \mathrm{K}^2 \left(Y - Z\right)$$

Hence we deduce the expression of Y in terms of Z and χ: $Y = Z - \dfrac{1}{1 + f_{\mathcal{R}}}\hat{\chi}$

Combined to the field equation, this yields the expression of the anisotropic stress in terms of χ:

$$\boxed{\Omega_{\mathrm{de}} w_{\mathrm{de}} \Pi^{\mathrm{S}}_{\mathrm{de}} = \frac{1}{3}\frac{\mathrm{K}^2}{1 + f_{\mathcal{R}}}\hat{\chi}}$$

$$\Omega_{\rm de} w_{\rm de} \Pi^{\rm S}_{\rm de} = \frac{1}{3}\frac{{\rm K}^2}{1+f_{\mathcal{R}}}\hat{\chi}$$

From the projection of the stress energy tensor of the dark sector we also get:

$$\Omega_{\rm de}\Delta_{\rm de} = -g_{\rm K}\epsilon_{\rm H}\hat{\chi} + f'_{\mathcal{R}}X + \tfrac{2}{3}f_{\mathcal{R}}{\rm K}^2 Z$$

where $g_{\rm K} \equiv 1 + \dfrac{{\rm K}^2}{3\epsilon_{\rm H}}$

In the C.N.G.:

$$X = \phi' + \psi$$
$$Y = \psi$$
$$Z = \phi$$

allowing to eliminate χ. Then X and Z are written in terms of the perturbed fluid variable thanks to the field equations:

$$-\tfrac{2}{3}{\rm K}^2 Z = \Omega_{\rm m}\Delta_{\rm m} + \Omega_{\rm de}\Delta_{\rm de}$$

$$2X = \Omega_{\rm m}\hat{\Theta}_{\rm m} + \Omega_{\rm de}\hat{\Theta}_{\rm de}$$

This yields the equation of state for the dark sector anisotropic stress:

$$w_{\rm de}\Pi^{\rm S}_{\rm de} = \frac{1}{3g_{\rm K}\epsilon_{\rm H}}{\rm K}^2\left\{\Delta_{\rm de} - \frac{f'_{\mathcal{R}}}{2(1+f_{\mathcal{R}})}\hat{\Theta}_{\rm de} + \frac{\Omega_{\rm m}}{\Omega_{\rm de}}\frac{f_{\mathcal{R}}}{1+f_{\mathcal{R}}}\Delta_{\rm m} - \frac{\Omega_{\rm m}}{\Omega_{\rm de}}\frac{f'_{\mathcal{R}}}{2(1+f_{\mathcal{R}})}\hat{\Theta}_{\rm m}\right\}$$

Entropy perturbation in the dark sector

The field equation for the pressure perturbation is

$$\tfrac{2}{3}W + 2X - \tfrac{2}{9}\mathrm{K}^2\left(Y - Z\right) = \Omega_{\mathrm{m}}(\delta\hat{P}_{\mathrm{m}}/\rho_{\mathrm{m}}) + \Omega_{\mathrm{de}}(\delta\hat{P}_{\mathrm{de}}/\rho_{\mathrm{de}})$$

Assume no matter entropy perturbation.

The pressure perturbation is then written in terms of the entropy perturbation

$$w\Gamma = \frac{\delta\hat{P}}{\rho} - \frac{dP}{d\rho}\left(\Delta - \hat{\Theta}\right)$$

Recall the definition of χ to eliminate W: $\hat{\chi} = \frac{f'_{\mathcal{R}}}{\bar{\epsilon}_{\mathrm{H}}}\left\{W + 4X - \tfrac{1}{3}\mathrm{K}^2(Y - 2Z)\right\}$

To eliminate Y, use the previous expression linking χ to Y and Z: $Y = Z - \frac{1}{1+f_{\mathcal{R}}}\hat{\chi}$

Finally, thanks to the equation of state of the dark anisotropic stress, and the field equations, χ, X and Z are expressed in terms of the perturbed fluid variables, yielding the equation of state for the entropy perturbation in the dark sector.

Equations of state for f(R) perturbations

Anisotropic stress

$$w_{\rm de}\Pi^{\rm S}_{\rm de} = \frac{1}{3g_{\rm K}\epsilon_{\rm H}}{\rm K}^2\left\{\Delta_{\rm de} - \frac{f'_{\mathcal{R}}}{2(1+f_{\mathcal{R}})}\hat{\Theta}_{\rm de} + \frac{\Omega_{\rm m}}{\Omega_{\rm de}}\frac{f_{\mathcal{R}}}{1+f_{\mathcal{R}}}\Delta_{\rm m} - \frac{\Omega_{\rm m}}{\Omega_{\rm de}}\frac{f'_{\mathcal{R}}}{2(1+f_{\mathcal{R}})}\hat{\Theta}_{\rm m}\right\}$$

Entropy perturbation

$$\begin{aligned} w_{\rm de}\Gamma_{\rm de} = & \left[\zeta_{\rm de} - \frac{\bar{\epsilon}_{\rm H}}{3g_{\rm K}\epsilon_{\rm H}}\frac{2(1+f_{\mathcal{R}})-f'_{\mathcal{R}}}{f'_{\mathcal{R}}}\right]\Delta_{\rm de} - \zeta_{\rm de}\hat{\Theta}_{\rm de} \\ & + \frac{\Omega_{\rm m}}{\Omega_{\rm de}}\left[\zeta_{\rm m} - \frac{\bar{\epsilon}_{\rm H}}{3g_{\rm K}\epsilon_{\rm H}}\frac{2f_{\mathcal{R}}-f'_{\mathcal{R}}}{f'_{\mathcal{R}}}\right]\Delta_{\rm m} - \frac{\Omega_{\rm m}}{\Omega_{\rm de}}\zeta_{\rm m}\hat{\Theta}_{\rm m} \end{aligned}$$

Notations

$$\zeta_{\rm i} \equiv \frac{g_{\rm K}\epsilon_{\rm H} - \bar{\epsilon}_{\rm H}}{3g_{\rm K}\epsilon_{\rm H}} - \frac{dP_{\rm i}}{d\rho_{\rm i}} \qquad g_{\rm K} \equiv 1 + \frac{{\rm K}^2}{3\epsilon_{\rm H}} \qquad {\rm K} \equiv \frac{k}{aH}$$

$$\epsilon_{\rm H} \equiv -\frac{H'}{H} \qquad \bar{\epsilon}_{\rm H} = -\frac{\mathcal{R}'}{6H^2}$$

Consider a dark sector f(R) fluid with constant equation of state at the background level:

$$w_{\rm de} = {\rm cste}$$

Assume a dust like matter fluid: $w_{\rm m} = 0$

This determines all background functions:

$$\rho_{\rm i} = \rho_{\rm i0} a^{-3(1+w_{\rm i})}$$

$$H^2 = H_0^2 \left\{ \Omega_{\rm de0} a^{-3(1+w_{\rm de})} + \Omega_{\rm m0} a^{-3} \right\}$$

$$\Omega_{\rm i}(a) = \frac{\rho_{\rm i}}{3H^2}$$

$$\epsilon_{\rm H}(a) = \tfrac{3}{2}(1 + w_{\rm de}\Omega_{\rm de})$$

$$\mathcal{R}(a) = 12H^2(1 - \tfrac{1}{2}\epsilon_{\rm H})$$

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$$\longrightarrow$$

$$\Omega_{\rm i}(a) = \frac{\rho_{\rm i}}{3H^2}$$

$$\epsilon_{\rm H}(a) = \tfrac{3}{2}(1 + w_{\rm de}\Omega_{\rm de})$$

$$\mathcal{R}(a) = 12H^2(1 - \tfrac{1}{2}\epsilon_{\rm H})$$

As we saw, the time-time projection of the stress-energy tensor of the dark sector gives:

$$\Omega_{\rm de} = -\frac{f}{6H^2} + (1 - \epsilon_{\rm H}) f_{\mathcal{R}} - f'_{\mathcal{R}}$$

which is a second order differential equation that completely determines f(R),

$$f'' + \left(\epsilon_{\rm H} - 1 - \frac{\mathcal{R}''}{\mathcal{R}'} \right) f' + \frac{\mathcal{R}'}{6H^2} f = -\mathcal{R}' \Omega_{\rm de}$$

once the initial conditions are specified for f and f'.

Song-Hu-Sawicki [arXiv:0610532]

$$f'' + \left(\epsilon_{\mathrm{H}} - 1 - \frac{\mathcal{R}''}{\mathcal{R}'}\right) f' + \frac{\mathcal{R}'}{6H^2} f = -\mathcal{R}' \Omega_{\mathrm{de}}$$

$$f'' + \left(\epsilon_{\mathrm{H}} - 1 - \frac{\mathcal{R}''}{\mathcal{R}'}\right) f' + \frac{\mathcal{R}'}{6H^2} f = -\mathcal{R}'\Omega_{\mathrm{de}}$$

The particular solution is: $$f_{\mathrm{part}} = -6H^2\Omega_{\mathrm{de}}$$

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The particular solution is: $f_{\rm part} = -6H^2\Omega_{\rm de}$

In the matter domination era, the differential equation without r.h.s reduces to

$2f'' + 7f' - 3f = 0$ leading to $f = A_+ a^{p_+} + A_- a^{p_-}$ with $p_\pm \equiv \dfrac{-7 \pm \sqrt{73}}{4}$

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Due to tight observational constraint in the high curvature regime ,the decaying mode is unacceptable so we set its amplitude to zero:

$$A_- = 0$$

Hence, initial conditions are specified in the matter domination era as

$$f(a_{\mathrm{init}}) = f_{\mathrm{part}}(a_{\mathrm{init}}) + A_+ a_{\mathrm{init}}^{p+}$$
$$f'(a_{\mathrm{init}}) = f'_{\mathrm{part}}(a_{\mathrm{init}}) + p_+ A_+ a_{\mathrm{init}}^{p+}$$

$$f'' + \left(\epsilon_{\rm H} - 1 - \frac{\mathcal{R}''}{\mathcal{R}'}\right) f' + \frac{\mathcal{R}'}{6H^2} f = -\mathcal{R}'\Omega_{\rm de}$$

The particular solution is: $f_{\rm part} = -6H^2\Omega_{\rm de}$

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$$f'(a_{\rm init}) = f'_{\rm part}(a_{\rm init}) + p_+ A_+ a_{\rm init}^{p+}$$

Different f(R) function are parametrized by a single number, A_+, or equivalently

$$B_0 \equiv -\frac{f'_{\mathcal{R}}}{\epsilon_{\rm H}(1+f_{\mathcal{R}})}\Big|_{a=a_0}$$

Song-Hu-Sawicki [arXiv:0610532]

Solving the dynamics

Perturbed fluid equations $$\delta(\nabla^\mu D_{\mu\nu}) = 0$$

$$\Delta' - 3w\Delta - 2w\Pi^{\mathrm{S}} + g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta} = 3\,(1+w)\,X$$

$$\hat{\Theta}' + 3(\tfrac{dP}{d\rho} - w + \tfrac{1}{3}\epsilon_{\mathrm{H}})\hat{\Theta} - 3\tfrac{dP}{d\rho}\Delta - 2w\Pi^{\mathrm{S}} - 3w\Gamma = 3\,(1+w)\,Y$$

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The two perturbed fluid equations of the dark sector

$$\Delta'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}}$$

$$\hat{\Theta}'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - \epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}} - 3\Gamma_{\mathrm{de}}$$

The two perturbed fluid equations of the standard matter fluid

$$\Delta'_{\mathrm{m}} = -g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3X$$

$$\hat{\Theta}'_{\mathrm{m}} = -\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3Y$$

Solving the dynamics

Perturbed fluid equations $\delta(\nabla^\mu D_{\mu\nu}) = 0$

$$\Delta' - 3w\Delta - 2w\Pi^{\mathrm{S}} + g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta} = 3\,(1+w)\,X$$

$$\hat{\Theta}' + 3(\tfrac{dP}{d\rho} - w + \tfrac{1}{3}\epsilon_{\mathrm{H}})\hat{\Theta} - 3\tfrac{dP}{d\rho}\Delta - 2w\Pi^{\mathrm{S}} - 3w\Gamma = 3\,(1+w)\,Y$$

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$$\Delta'_{\mathrm{m}} = -g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3X$$

$$\hat{\Theta}'_{\mathrm{m}} = -\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3Y$$

And one evolution equation for the metric perturbations $Z' = X - Y$

(coming from the definition of the gauge invariant notation: X = Z' + Y)

X,Y are then replaced by their expression in terms of the fluid variables (and Z), according to the analysis made in the previous slides.

Background evolution determined by: $\Omega_{\mathrm{de0}}, \quad w_{\mathrm{m}}, \quad w_{\mathrm{de}}, \quad B_0$

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System of first order differential equations for the evolution of perturbations

$$\Delta'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}}$$

$$\hat{\Theta}'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - \epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}} - 3\Gamma_{\mathrm{de}}$$

$$\Delta'_{\mathrm{m}} = -g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3X$$

$$\hat{\Theta}'_{\mathrm{m}} = -\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3Y$$

$$Z' = X - Y$$

$$X = \tfrac{1}{2}\Omega_{\mathrm{de}}\hat{\Theta}_{\mathrm{de}} + \tfrac{1}{2}\Omega_{\mathrm{m}}\hat{\Theta}_{\mathrm{m}}$$

$$Y = Y(Z, \Delta_{\mathrm{de}}, \hat{\Theta}_{\mathrm{de}}, \Delta_{\mathrm{m}}, \hat{\Theta}_{\mathrm{m}})$$

$$\Pi^{\mathrm{S}}_{\mathrm{de}} = \Pi^{\mathrm{S}}_{\mathrm{de}}(\Delta_{\mathrm{de}}, \hat{\Theta}_{\mathrm{de}}, \Delta_{\mathrm{m}}, \hat{\Theta}_{\mathrm{m}})$$

$$\Gamma_{\mathrm{de}} = \Gamma_{\mathrm{de}}(\Delta_{\mathrm{de}}, \hat{\Theta}_{\mathrm{de}}, \Delta_{\mathrm{m}}, \hat{\Theta}_{\mathrm{m}})$$

Background evolution determined by: $\Omega_{\mathrm{de0}}, \quad w_{\mathrm{m}}, \quad w_{\mathrm{de}}, \quad B_0$

System of first order differential equations for the evolution of perturbations

Battye-Bolliet-Pearson [arXiv:1508.04569]

$$\Delta'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}}$$
$$\hat{\Theta}'_{\mathrm{de}} = -3\Delta_{\mathrm{de}} - \epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{de}} - 2\Pi^{\mathrm{S}}_{\mathrm{de}} - 3\Gamma_{\mathrm{de}}$$
$$\Delta'_{\mathrm{m}} = -g_{\mathrm{K}}\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3X$$
$$\hat{\Theta}'_{\mathrm{m}} = -\epsilon_{\mathrm{H}}\hat{\Theta}_{\mathrm{m}} + 3Y$$
$$Z' = X - Y$$

$$X = \tfrac{1}{2}\Omega_{\mathrm{de}}\hat{\Theta}_{\mathrm{de}} + \tfrac{1}{2}\Omega_{\mathrm{m}}\hat{\Theta}_{\mathrm{m}}$$
$$Y = Y(Z, \Delta_{\mathrm{de}}, \hat{\Theta}_{\mathrm{de}}, \Delta_{\mathrm{m}}, \hat{\Theta}_{\mathrm{m}})$$
$$\Pi^{\mathrm{S}}_{\mathrm{de}} = \Pi^{\mathrm{S}}_{\mathrm{de}}(\Delta_{\mathrm{de}}, \hat{\Theta}_{\mathrm{de}}, \Delta_{\mathrm{m}}, \hat{\Theta}_{\mathrm{m}})$$
$$\Gamma_{\mathrm{de}} = \Gamma_{\mathrm{de}}(\Delta_{\mathrm{de}}, \hat{\Theta}_{\mathrm{de}}, \Delta_{\mathrm{m}}, \hat{\Theta}_{\mathrm{m}})$$

Initial conditions for perturbations are set in the matter dominated era, when departure from General Relativity are expected to be negligible (observational constraints)

$$\Delta_{\mathrm{de}}(a_{\mathrm{init}}) = 0$$
$$\hat{\Theta}_{\mathrm{de}}(a_{\mathrm{init}}) = 0$$

$$\Omega_{\mathrm{m}}\Delta_{\mathrm{m}}(a_{\mathrm{init}}) = -\tfrac{2}{3}\mathrm{K}^2 Z_{\mathrm{init}}$$
$$\Omega_{\mathrm{m}}\hat{\Theta}_{\mathrm{m}}(a_{\mathrm{init}}) = 2Z_{\mathrm{init}}$$

$$\delta g_{00} = -2a^2\psi$$
$$\delta g_{ij} = -2a^2\phi\delta_{ij} + h^{\mathrm{V}} \cdot v_{ij} + h^{\mathrm{T}} \cdot e_{ij}$$
$$Y = \psi$$
$$Z = \phi$$

Evolution of the gravitational potentials

From the equation of state of the anisoptropic stress and the definition of χ, one gets the following relation:

$$BK^2(Y-2Z) = 3BW + 12BX + 3\frac{\bar{\epsilon}_H}{\epsilon_H}(Y-Z)$$

$$\delta g_{00} = -2a^2\psi$$
$$\delta g_{ij} = -2a^2\phi\delta_{ij} + h^V \cdot v_{ij} + h^T \cdot e_{ij}$$

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$$B_0 \equiv -\frac{f'_\mathcal{R}}{\epsilon_H(1+f_\mathcal{R})}\Big|_{a=a_0}$$

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$$Y = \psi \qquad B_0 \equiv -\frac{f'_{\mathcal{R}}}{\epsilon_{\mathrm{H}}(1+f_{\mathcal{R}})}\Big|_{a=a_0}$$
$$Z = \phi$$

$$B\mathrm{K}^2(Y - 2Z) = 3BW + 12BX + 3\frac{\bar{\epsilon}_{\mathrm{H}}}{\epsilon_{\mathrm{H}}}(Y - Z)$$

Transition scale: $B\mathrm{K}^2 \simeq 1 \Rightarrow Z \to \frac{1}{2}Y$

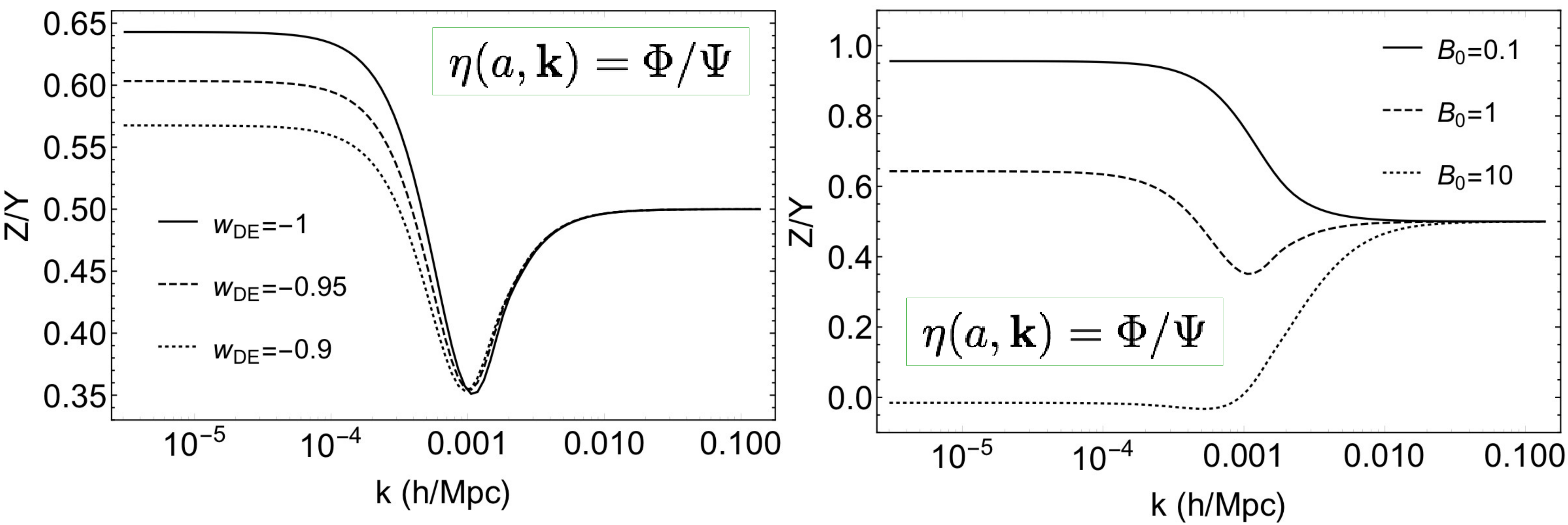

Figure: Spectrum of the ratio Z/Y (or −Φ/Ψ) for different values of the equation of state parameter when B_0=1 (left) and different designer f(R) scenarios parametrized by B_0=1 and with w_{de} = −1 (right). On the x-axis, the wavenumber is written in units ‘h/Mpc’, where h = 0.73 is the reduced Hubble constant.

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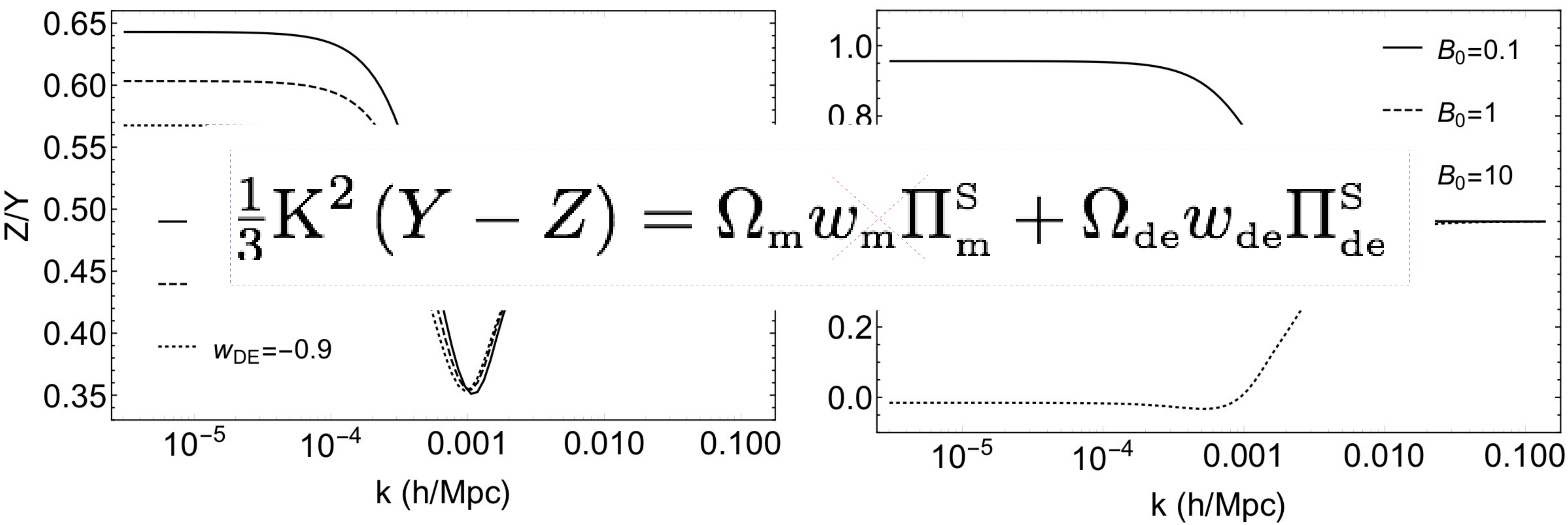
Transition scale: $B\mathrm{K}^2 \simeq 1 \Rightarrow Z \to \frac{1}{2}Y$

$$\frac{1}{3}\mathrm{K}^2\,(Y-Z) = \Omega_{\mathrm{m}}\cancel{w_{\mathrm{m}}}\Pi^{\mathrm{S}}_{\mathrm{m}} + \Omega_{\mathrm{de}}w_{\mathrm{de}}\Pi^{\mathrm{S}}_{\mathrm{de}}$$



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Modification to the Poisson equation

$$-\frac{2}{3}\mathrm{K}^2 Z = \Omega_{\mathrm{m}}\Delta_{\mathrm{m}} + \Omega_{\mathrm{de}}\Delta_{\mathrm{de}} = \Omega_{\mathrm{m}}\Delta_{\mathrm{m}}\left(1 + \frac{\Omega_{\mathrm{de}}\Delta_{\mathrm{de}}}{\Omega_{\mathrm{m}}\Delta_{\mathrm{m}}}\right)$$

$$\delta g_{00} = -2a^2\psi$$
$$\delta g_{ij} = -2a^2\phi\delta_{ij} + h^{\mathrm{V}} \cdot v_{ij} + h^{\mathrm{T}} \cdot e_{ij}$$
$$Y = \psi$$
$$Z = \phi$$

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$$-\tfrac{2}{3}\mathrm{K}^2 Z = \Omega_{\mathrm{m}}\Delta_{\mathrm{m}} + \Omega_{\mathrm{de}}\Delta_{\mathrm{de}} = \Omega_{\mathrm{m}}\Delta_{\mathrm{m}}\left(1 + \frac{\Omega_{\mathrm{de}}\Delta_{\mathrm{de}}}{\Omega_{\mathrm{m}}\Delta_{\mathrm{m}}}\right)$$

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$$\delta g_{ij} = -2a^2\phi\delta_{ij} + h^{\mathrm{V}}\cdot v_{ij} + h^{\mathrm{T}}\cdot e_{ij}$$
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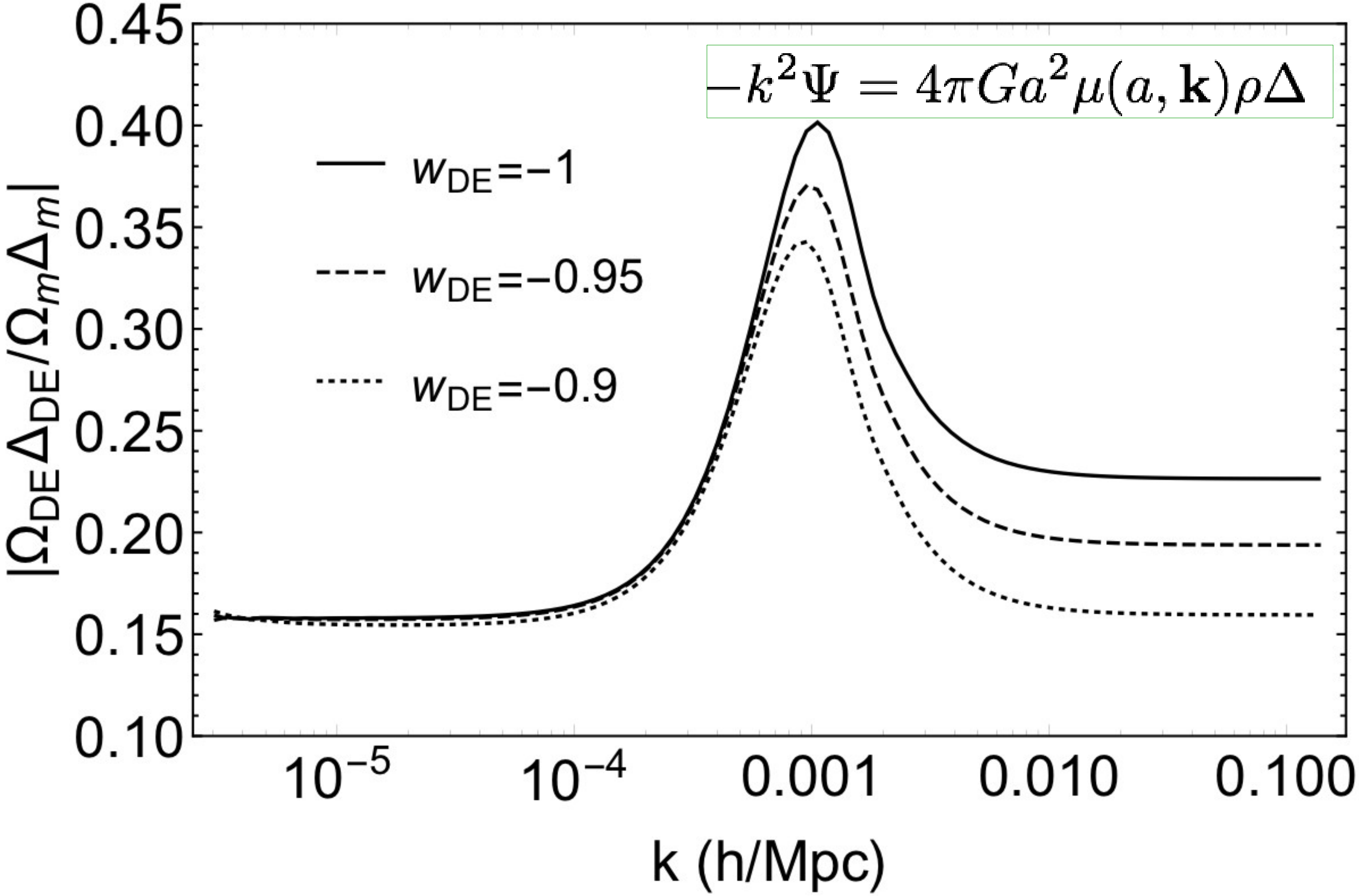


Figure: Scale dependence of the correcting term to the Poisson equation, evaluated today for different (constant) equation of state parameter and B_0=1.

For recent constraints on B0
See : Song et al arXIv:1507.01592

Concluding remarks

The Equation of State approach is an elegant formalism for studying the phenomenology of cosmological perturbations in f(R) gravity.

The f(R) modifications can be implemented by simply adding a new fluid specie at the perturbed level, rather than modifying the whole set of equations for the metric perturbations.

Battye, Bolliet, Pearson, f(R) as a dark energy fluid, *PRD 2016*.

We have illustrated the EoS approach with f(R) gravity, but it should apply to any modified gravity theory.

Perspectives

1) Implementation of the EoS approach in CLASS [Blas-Lesgourgues-Tram arXiv:1104.2933]

2) *Classification of modified gravity theories in terms of their equations of state for dark sector perturbations.*

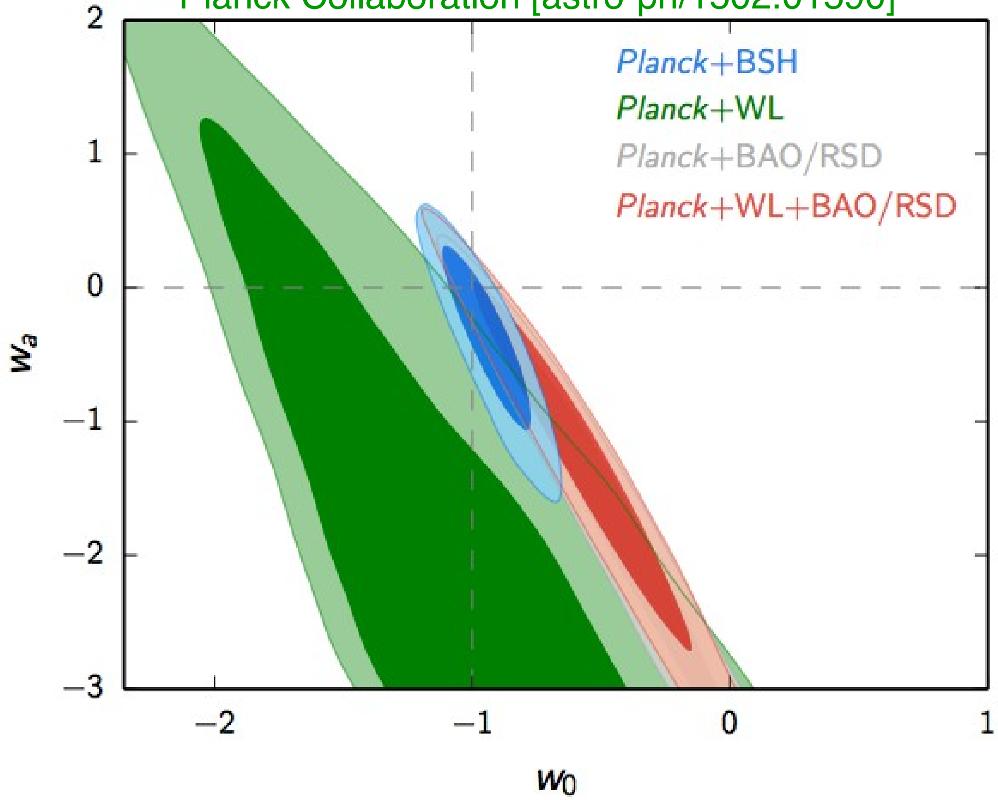
The equations of state for dark energy

The equation of state at background level

Kunz [astro-ph/1204.5482]

Model	equation of state	Dark sector parameters
Constant w	$w = w_0$	1
Linear (CPL)	$w = w_0 + (1-a)w_a$	2
Quadratic	$w = w_0 + w_1(1-a) + w_2(1-a)^2$	3
Cubic	$w = w_0 + w_1(1-a) + w_2(1-a)^2 + w_3(1-a)^3$	4
ΛCDM	$w = -(1-\Omega_m)/[\Omega_m a^{-3} + (1-\Omega_m)]$	1

Planck Collaboration [astro-ph/1502.01590]



But constraints on the background dark sector equation of state parameter are not sufficient to distinguish between different dark energy (DE) and modified gravity (MG) models.

Constraints on the perturbative degrees of freedom of the dark sector are essential.