# Conformal Anomalies and Gravitational Waves

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**Work based on:**

*K. Meissner and H.N.:* `arXiv:1607.07312`

*H. Godazgar, K. Meissner and H.N.: in progress*

# Executive Summary

Is the cancellation of conformal anomalies required

- **Quantum mechanically:** to ensure quantum consistency of perturbative quantum gravity?

... in analogy with cancellation of gauge anomalies for Standard Model (where they are required to maintain renormalizability), and/or

- **already at classical level:** corrections from induced anomalous non-local action to Einstein Field Equations may potentially overwhelm smallness of Planck scale $\ell_{\rm PL}$ $\Rightarrow$ huge corrections to any solution?

If so, cancellation requirement could lead to *very strong* restrictions on admissible theories!

# Conformal Symmetry

Conformal symmetry comes in two versions:

1. Global conformal symmetry = extension of Poincaré group by dilatations $D$ and conformal boosts $K^\mu$

2. Local dilatations = Weyl transformations

$$g_{\mu\nu}(x) \rightarrow e^{2\sigma(x)} g_{\mu\nu}(x)$$

Important consequence: flat space limit of Weyl and diffeomorphism invariant theories exhibits full (global) conformal symmetry (via conformal Killing vectors)

$\rightarrow$ important restrictions on effective actions $\Gamma = \Gamma[g]$.

# Conformal Anomaly $\equiv$ Trace Anomaly

**Conformal anomaly ($\equiv$ trace anomaly)** [Deser,Duff,Isham(1976)]

$$T_\mu{}^\mu(x) = a\,\mathbf{E}_2(x) \equiv aR(x) \qquad\qquad (D = 2)$$

$$T_\mu{}^\mu(x) = \mathcal{A}(x) \equiv a\,\mathbf{E}_4(x) + c\,C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}(x) \quad (D = 4)$$

**where $\mathbf{E}_4(x) \equiv$ Euler number density**

$$\mathbf{E}_4 \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2$$

$$C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 2R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2$$

**Coefficients $c_s$ and $a_s$ for fields of spin $s$ (with $s \leq 2$) were computed already long ago.**

[Duff(1977);Christenses,Duff(1978);Fradkin,Tseytlin(1982); Tseytlin(2013);

see also: Eguchi,Gilkey,Hanson, Phys.Rep.66(1980)213]

# Anomalous Effective Action

Anomaly can be obtained by varying anomalous effective action $\Gamma_{\text{anom}} = \Gamma_{\text{anom}}[g]$

$$\mathcal{A}(x) = -\frac{2}{\sqrt{-g(x)}} g_{\mu\nu}(x) \frac{\delta\Gamma_{\text{anom}}[g]}{\delta g_{\mu\nu}(x)}$$

but this effective action is necessarily *non-local.*

Simplest example: string theory in *non-critical* dimension has a trace anomaly $T^{\mu}{}_{\mu} \propto R \Rightarrow$ leads to anomalous effective action = Liouville theory. [Polyakov(1981)]

$$\Gamma_{\text{anom}}^{D=2} \propto \int d^2x \sqrt{-g} R \,\square_g^{-1} R$$

- new propagating degree of freedom (longitudinal mode of world sheet metric = Liouville field)

$\Rightarrow$ changes physics in dramatic ways!

Analog for gravity in $D = 4$: non-local actions that give anomaly *exactly* are known, for instance [Riegert(1984)]

$$\Gamma_{\text{anom}}[g] = \int d^4x\, d^4y\, \sqrt{-g(x)}\sqrt{-g(y)} \left(\mathbf{E}_4 - \frac{2}{3}\Box_g R\right)(x) G^P(x,y) \left(\mathbf{E}_4 - \frac{2}{3}\Box_g R\right)(y)$$

with $\triangle^P G^P(x) = \delta^{(4)}(x)$, and the 4th order operator

$$\triangle^P \equiv \Box_g \Box_g + 2\nabla_\mu \left(R^{\mu\nu} - \frac{1}{3}g^{\mu\nu}R\right)\nabla_\nu$$

However, no closed form actions are known that have the correct conformal properties (as would be obtained from Feynman diagrams), despite many efforts.

[Deser,Schwimmer(1993);Erdmenger,Osborn(1998);Deser(2000);Barvinsky et al.(1998);

Mazur,Mottola(2001);...]

In lowest order

$$\Gamma_{\text{anom}}^{D=4} \propto \int d^4x\, \sqrt{-g}\,\mathbf{E}_4\,\Box_g^{-1}R + \cdots$$

where $\cdots$ stands for *infinitely many* (non-local) terms.

While

$$\mathcal{A}(x) = -\frac{2}{\sqrt{-g(x)}} g_{\mu\nu}(x) \frac{\delta\Gamma_{\mathrm{anom}}[g]}{\delta g_{\mu\nu}(x)}$$

*is* local, contribution to Einstein equations

$$\ell_{PL}^{-2}\left[R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right] = -\frac{2}{\sqrt{-g(x)}} \frac{\delta\Gamma_{\mathrm{anom}}[g]}{\delta g^{\mu\nu}(x)} + \cdots$$

in general *remains non-local* for non-scalar modes.

Claim: non-localities from $\Box_g^{-1}$ in $\Gamma_{\mathrm{anom}}[g]$ can 'overwhelm' smallness of Planck scale and produce observable deviations for Einstein's equations!

Typical correction is (symmetrized traceless part of)

$$\propto \nabla_\mu\big(G^{\mathrm{ret}} \star \mathbf{E}_4\big)\,\nabla_\nu\big(G^{\mathrm{ret}} \star R\big) + \cdots$$

with retarded propagator $G^{\mathrm{ret}}$ in space-time background given by metric $g_{\mu\nu}$ solving classical Einstein equations.

For order of magnitude estimate, evaluate this integral for a (conformally flat) cosmological background

$$ds^2 = a(\eta)^2(-d\eta^2 + d\mathbf{x}^2)$$

by integrating from end of radiation era $(= t_{\mathrm{rad}})$ back to $t_0 = n_*\ell_{\mathrm{PL}}$, with $a(\eta) = \eta/(2t_{\mathrm{rad}})$ and $\eta = 2\sqrt{tt_{\mathrm{rad}}}$ and with retarded Green's function [Waylen(1978)]

$$G^{\mathrm{ret}}(\eta, \mathbf{x}; \eta', \mathbf{y}) = \frac{1}{4\pi|\mathbf{x} - \mathbf{y}|} \cdot \frac{\delta(\eta - \eta' - |\mathbf{x} - \mathbf{y}|)}{a(\eta)a(\eta')}$$

Resulting correction on r.h.s. of Einstein's equations

$$T_{00}^{\mathrm{anom}} \sim 10^{-5}\, t_{\mathrm{rad}}^{-1}\, (n_*\ell_{\mathrm{PL}})^{-3}$$

'beats' factor $\sim (t_{\mathrm{rad}}\ell_{\mathrm{PL}})^{-2}$ on l.h.s. even for $n_* \sim 10^8$ !

Similar results from evaluating contribution of Riegert action $\rightarrow$ could be a generic phenomenon, and thus affect *any* solution of Einstein equations. [Godazgar,Meissner,HN]

# Cancelling conformal anomalies

| | massless | | massive | |
|---|---|---|---|---|
| | $c_s$ | $a_s$ | $\bar{c}_s$ | $\bar{a}_s$ |
| $0(0^*)$ | $\frac{3}{2}(\frac{3}{2})$ | $-\frac{1}{2}(\frac{179}{2})$ | $\frac{3}{2}(\varnothing)$ | $-\frac{1}{2}(\varnothing)$ |
| $\frac{1}{2}$ | $\frac{9}{2}$ | $-\frac{11}{4}$ | $\frac{9}{2}$ | $-\frac{11}{4}$ |
| $1$ | $18$ | $-31$ | $\frac{39}{2}$ | $-\frac{63}{2}$ |
| $\frac{3}{2}$ | $-\frac{411}{2}$ | $\frac{589}{4}$ | $-201$ | $\frac{289}{2}$ |
| $2$ | $783$ | $-571$ | $\frac{1605}{2}$ | $-\frac{1205}{2}$ |

- $\bar{c}_s$ and $\bar{a}_s$ include lower helicities: $\bar{c}_1 = c_1 + c_0$, *etc.*

- Gravitinos and supergravity needed for cancellation

- No cancellation possible for $N \leq 4$ supergravities

NB: gravitino contribution may evade positivity properties because there does not exist a gauge invariant traceless energy momentum tensor for $s = \frac{3}{2}$. [A.Schwimmer]

$$c_2 + 5c_{\frac{3}{2}} + 10c_1 + 11c_{\frac{1}{2}} + 10c_0 = 0 \qquad (N = 5)$$

$$c_2 + 6c_{\frac{3}{2}} + 16c_1 + 26c_{\frac{1}{2}} + 30c_0 = 0 \qquad (N = 6)$$

$$c_2 + 8c_{\frac{3}{2}} + 28c_1 + 56c_{\frac{1}{2}} + 70c_0 = 0 \qquad (N = 8)$$

Old result: combined contribution $\sum_s (c_s + a_s)$ vanishes for all $N \geq 3$ theories with appropriate choice of field representations for spin zero fields [Townsend,HN(1981)].

**Thus: conformal anomalies for $\sum_s a_s$ and $\sum_s c_s$ cancel only for $N \geq 5$ supergravities!** [Meissner,HN]

... as they do for 'composite' U(5), U(6) and SU(8) R-symmetry anomalies. [Marcus(1985)]

Implications for finiteness of $N \geq 5$ supergravities?

[Cf. Carrasco,Kallosh,Roiban,Tseytlin(2013);Bern,Davies,Dennen(2014)]

# Idem for D=11 SUGRA compactified $\mathbf{AdS}_4 \times S^7$

| | $SO(8)$ representations |
|---|---|
| 0 | $[n{+}2\ 0\ 0\ 0]$, $[n\ 0\ 2\ 0]$, $[n{-}2\ 2\ 0\ 0]$, $[n{-}2\ 0\ 0\ 2]$, $[n{-}2\ 0\ 0\ 0]$ |
| $\frac{1}{2}$ | $[n{+}1\ 0\ 1\ 0]$, $n{-}1\ 1\ 1\ 0]$, $[n{-}2\ 1\ 0\ 1]$, $[n{-}2\ 0\ 0\ 1]$ |
| 1 | $[n\ 1\ 0\ 0]$, $[n{-}1\ 0\ 1\ 1]$ , $[n{-}2\ 1\ 0\ 0]$ |
| $\frac{3}{2}$ | $[n\ 0\ 0\ 1]$ , $[n{-}1\ 0\ 1\ 0]$ |
| 2 | $[n\ 0\ 0\ 0]$ |

**'Floor-by-floor' cancellation** [Cf.Gibbons,HN(1985)]**: for all** $n$

$$\bar{c}_2 f_2(n) + \bar{c}_{\frac{3}{2}} f_{\frac{3}{2}}(n) + \bar{c}_1 f_1(n) + \bar{c}_{\frac{1}{2}} f_{\frac{1}{2}}(n) + \bar{c}_0 f_0(n) = 0$$

where $f_s(n) \equiv \sum$ (dimensions of **SO(8)** spin-$s$ irreps)
at **Kaluza-Klein** level $n$ (no anomalies for odd $D$).

# Conceptual Issues

Why worry about conformal anomalies in theories that are not even classically conformally invariant?

HOWEVER: recall axial anomaly and anomalous conservation of axial current

$$\partial^\mu J_\mu^A = 2im\bar{\psi}\gamma^5\psi + \frac{\alpha}{8\pi}F^{\mu\nu}\tilde{F}_{\mu\nu}$$

$\rightarrow$ anomaly is crucial *even in presence of explicitly broken axial symmetry ($m \neq 0$).*

Idem for gauge anomalies in Standard Model: these must cancel even when quarks and leptons acquire masses via spontaneous symmetry breaking.

Is there a hidden conformal structure behind $N \geq 5$ supergravities (and M Theory)? But cannot be conformal supergravity in any conventional sense...

# Outlook

*V. Mukhanov: "You cannot figure out the fundamental theory by simply looking at the sky!"*

*But maybe there is a way...*