Twin supergravities from (Yang-Mills)^2

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based on

[arXiv:1301.4176 arXiv:1309.0546 arXiv:1312.6523
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arXiv:1602.08267 arXiv:1610.07192
A. Anastasiou, L. Borsten, M. J. Duff, M. Hughes,
A. Marrani, S. Nagy and M. Zoccali]

1.0 Basic idea

- Strong nuclear, Weak nuclear and Electromagnetic forces described by Yang-Mills gauge theory (non-abelian generalisation of Maxwell). Gluons, W, Z and photons have spin 1.

- Gravitational force described by Einstein's general relativity. Gravitons have spin 2.

- But maybe $(spin\ 2) = (spin\ 1)^2$. If so:

  1) Do global gravitational symmetries follow from flat-space Yang-Mills symmetries?

  2) Do local gravitational symmetries and Bianchi identities follow from flat-space Yang-Mills symmetries?

  3) What about twin supergravities with same bosonic lagrangian but different fermions?

1.1 Gravity as square of Yang-Mills

- A recurring theme in attempts to understand the quantum theory of gravity and appears in several different forms:
- Closed states from products of open states and KLT relations in string theory [Kawai, Lewellen, Tye:1985, Siegel:1988],
- On-shell $D = 10$ Type IIA and IIB supergravity representations from on-shell $D = 10$ super Yang-Mills representations [Green, Schwarz and Witten:1987],
- Vector theory of gravity [Svidzinsky 2009]
- Supergravity scattering amplitudes from those of super Yang-Mills in various dimensions, [Bern, Carrasco, Johanson:2008, 2010; Bern, Huang, Kiermaier, 2010, 2012,Montiero, O'Connell, White 2011, 2014, Bianchi:2008,Elvang,Huang:2012,Cachazo:2013,Dolan:2013]
- Ambitwistor strings [Hodges:2011, Mason:2013, Geyer:2014]

1.2 Local and global symmetries from Yang-Mills

- LOCAL SYMMETRIES: general covariance, local lorentz invariance, local supersymmtry, local p-form gauge invariance
  [arXiv:1408.4434, Physica Scripta 90 (2015)]

- GLOBAL SYMMETRIES eg $G = E_7$ in $D = 4, \mathcal{N} = 8$ supergravity
  [arXiv:1301.4176 arXiv:1312.6523 arXiv:1402.4649 arXiv:1502.05359]

- TWIN SUPERGRAVITIES FROM (YANG-MILLS)^2

  [A. Anastasiou, L. Borsten, M. J. Duff, L. J. Hughes, A.Marrani, S. Nagy and M. Zoccali] [arXiv:1610.07192]

- LOCAL SYMMETRIES

2.1. Product?

- Most of the literature is concerned with products of momentum-space scattering amplitudes, but we are interested in products of off-shell left and right Yang-Mills field in coordinate-space

$$A_\mu(x)(L) \otimes A_\nu(x)(R)$$

  so it is hard to find a conventional field theory definition of the product.
- Where do the gauge indices go?
- Does it obey the Leibnitz rule

$$\partial_\mu(f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$

  If not, why not?

2.2 Convolution

- Here we present a $G_L \times G_R$ product rule :

$$[{A_\mu}^i(L) \star \Phi_{ii'} \star {A_\nu}^{i'}(R)](x)$$

where $\Phi_{ii'}$ is the "spectator" bi-adjoint scalar field introduced by Hodges [Hodges:2011] and Cachazo *et al* [Cachazo:2013] and where $\star$ denotes a convolution

$$[f \star g](x) = \int d^4 y f(y) g(x-y).$$

Note $f \star g = g \star f, (f \star g) \star h = f \star (g \star h)$, and, importantly obeys

$$\partial_\mu (f \star g) = (\partial_\mu f) \star g = f \star (\partial_\mu g)$$

and not Leibnitz

$$\partial_\mu (f \otimes g) = (\partial_\mu f) \otimes g + f \otimes (\partial_\mu g)$$

2.3 Gravity/Yang-Mills dictionary

For concreteness we focus on

- $\mathcal{N} = 1$ supergravity in $D = 4$, obtained by tensoring the $(4+4)$ off-shell $\mathcal{N}_L = 1$ Yang-Mills multiplet $(A_\mu(L), \chi(L), D(L))$ with the $(3+0)$ off-shell $\mathcal{N}_R = 0$ multiplet $A_\mu(R)$.
- Interestingly enough, this yields the new-minimal formulation of $\mathcal{N} = 1$ supergravity [Sohnius,West:1981] with its 12+12 multiplet $(h_{\mu\nu}, \psi_\mu, V_\mu, B_{\mu\nu})$
- The dictionary is,

$$
\begin{aligned}
Z_{\mu\nu} \equiv h_{\mu\nu} + B_{\mu\nu} &= A_\mu{}^i(L) \star \Phi_{ii'} \star A_\nu{}^{i'}(R) \\
\psi_\nu &= \chi^i(L) \star \Phi_{ii'} \star A_\nu{}^{i'}(R) \\
V_\nu &= D^i(L) \star \Phi_{ii'} \star A_\nu{}^{i'}(R),
\end{aligned}
$$

2.4 Yang-Mills symmetries

- The left supermultiplet is described by a vector superfield $V^i(L)$ transforming as

$$\delta V^i(L) = \Lambda^i(L) + \bar{\Lambda}^i(L) + f^i{}_{jk} V^j(L)\theta^k(L) + \delta_{(a,\lambda,\epsilon)} V^i(L).$$

Similarly the right Yang-Mills field $A_\nu{}^{i'}(R)$ transforms as

$$\delta A_\nu{}^{i'}(R) = \partial_\nu \sigma^{i'}(R) + f^{i'}{}_{j'k'} A_\nu{}^{j'}(R)\theta^{k'}(R) + \delta_{(a,\lambda)} A_\nu{}^{i'}(R).$$

and the spectator as

$$\delta\Phi_{ii'} = -f^j{}_{ik}\Phi_{ji'}\theta^k(L) - f^{j'}{}_{i'k'}\Phi_{ij'}\theta^{k'}(R) + \delta_a \Phi_{ii'}.$$

Plugging these into the dictionary gives the gravity transformation rules.

2.5 Gravitational symmetries

$$\begin{array}{ll} \delta Z_{\mu\nu} & = \partial_\nu \alpha_\mu(L) + \partial_\mu \alpha_\nu(R), \\ \delta \psi_\mu & = \partial_\mu \eta, \\ \delta V_\mu & = \partial_\mu \Lambda, \end{array}$$

where

$$\begin{array}{ccccccc} \alpha_\mu(L) & = & A_\mu{}^i(L) & \star & \Phi_{ii'} & \star & \sigma^{i'}(R), \\ \alpha_\nu(R) & = & \sigma^i(L) & \star & \Phi_{ii'} & \star & A_\nu{}^{i'}(R), \\ \eta & = & \chi^i(L) & \star & \Phi_{ii'} & \star & \sigma^{i'}(R), \\ \Lambda & = & D^i(L) & \star & \Phi_{ii'} & \star & \sigma^{i'}(R), \end{array}$$

illustrating how the local gravitational symmetries of general covariance, 2-form gauge invariance, local supersymmetry and local chiral symmetry follow from those of Yang-Mills.

2.6 Lorentz multiplet

New minimal supergravity also admits an off-shell Lorentz multiplet $(\Omega_{\mu ab}{}^{-}, \psi_{ab}, -2V_{ab}{}^{+})$ transforming as

$$\delta\mathcal{V}^{ab} = \Lambda^{ab} + \bar{\Lambda}^{ab} + \delta_{(a,\lambda,\epsilon)}\mathcal{V}^{ab}. \tag{1}$$

This may also be derived by tensoring the left Yang-Mills superfield $V^{i}(L)$ with the right Yang-Mills field strength $F^{abi'}(R)$ using the dictionary

$$\mathcal{V}^{ab} = V^{i}(L) \star \Phi_{ii'} \star F^{abi'}(R),$$

$$\Lambda^{ab} = \Lambda^{i}(L) \star \Phi_{ii'} \star F^{abi'}(R).$$

2.7 Bianchi identities

- The corresponding Riemann and Torsion tensors are given by

$$R^{+}_{\mu\nu\rho\sigma} = -F_{\mu\nu}{}^{i}(L) \star \Phi_{ii'} \star F_{\rho\sigma}{}^{i'}(R) = R^{-}_{\rho\sigma\mu\nu}.$$

$$T^{+}_{\mu\nu\rho} = -F_{[\mu\nu}{}^{i}(L)\star\Phi_{ii'}\star A_{\rho]}{}^{i'}(R) = -A_{[\rho}{}^{i}(L)\star\Phi_{ii'}\star F_{\mu\nu]}{}^{i'}(R) = -T^{-}_{\mu\nu\rho}$$

- One can show that (to linearised order) both the gravitational Bianchi identities

$$DT = R \wedge e \tag{2}$$

$$DR = 0 \tag{3}$$

follow from those of Yang-Mills

$$D_{[\mu}(L)F_{\nu\rho]}{}^{I}(L) = 0 = D_{[\mu}(R)F_{\nu\rho]}{}^{I'}(R)$$

2.9 To do

- Convoluting the off-shell Yang-Mills multiplets $(4+4, \mathcal{N}_L = 1)$ and $(3+0, \mathcal{N}_R = 0)$ yields the $12+12$ new-minimal off-shell $\mathcal{N} = 1$ supergravity.
- Clearly two important improvements would be to generalise our results to the full non-linear transformation rules and to address the issue of dynamics as well as symmetries.

3.0 Global symmetries

- GLOBAL SYMMETRIES

3.1 Triality Algebra

- Second, the triality algebra $\mathfrak{tri}(\mathbb{A})$

$$\mathfrak{tri}(\mathbb{A}) \equiv \{(A, B, C) | A(xy) = B(x)y + xC(y)\}, \quad A, B, C \in \mathfrak{so}(n), \quad x, y \in \mathbb{A}.$$

$$\begin{aligned}
\mathfrak{tri}(\mathbb{R}) &= 0 \\
\mathfrak{tri}(\mathbb{C}) &= \mathfrak{so}(2) + \mathfrak{so}(2) \\
\mathfrak{tri}(\mathbb{H}) &= \mathfrak{so}(3) + \mathfrak{so}(3) + \mathfrak{so}(3) \\
\mathfrak{tri}(\mathbb{O}) &= \mathfrak{so}(8)
\end{aligned}$$

[Barton and Sudbery:2003]:

3.2 Global symmetries of supergravity in D=3

- MATHEMATICS: Division algebras: R, C, H, O

  $(DIVISION\ ALGEBRAS)^2 = MAGIC\ SQUARE\ OF\ LIE\ ALGEBRAS$

- PHYSICS: $N = 1, 2, 4, 8\ D = 3\ Yang - Mills$

  $(YANG - MILLS)^2 = MAGIC\ SQUARE\ OF\ SUPERGRAVITIES$

- CONNECTION: $N = 1, 2, 4, 8 \sim R, C, H, O$

  $MATHEMATICS\ MAGIC\ SQUARE = PHYSICS\ MAGIC\ SQUARE$

- The $D = 3\ G/H\ grav$ symmetries are given by ym symmetries

$$G(grav) = \mathfrak{tri}\ ym(L) + \mathfrak{tri}\ ym(R) + 3[ym(L) \times ym(R)].$$

eg

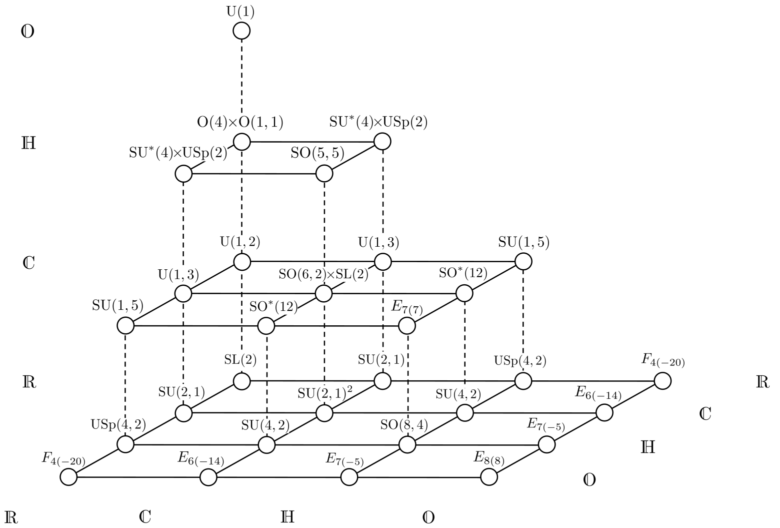
$$E_{8(8)} = SO(8) + SO(8) + 3(\mathbb{O} \times \mathbb{O})$$
$$248 = 28 + 28 + (8_v, 8_v) + (8_s, 8_s) + (8_c, 8_c)$$

3.3 Final result

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathcal{N}=2, f=4$ $G=\mathrm{SL}(2,\mathbb{R})$, dim 3 $H=\mathrm{SO}(2)$, dim 1	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36
$\mathbb{C}$	$\mathcal{N}=3, f=8$ $G=\mathrm{SU}(2,1)$, dim 8 $H=\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 4	$\mathcal{N}=4, f=16$ $G=\mathrm{SU}(2,1)^2$, dim 16 $H=\mathrm{SU}(2)^2\times\mathrm{SO}(2)^2$, dim 8	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46
$\mathbb{H}$	$\mathcal{N}=5, f=16$ $G=\mathrm{USp}(4,2)$, dim 21 $H=\mathrm{USp}(4)\times\mathrm{USp}(2)$, dim 13	$\mathcal{N}=6, f=32$ $G=\mathrm{SU}(4,2)$, dim 35 $H=\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{SO}(2)$, dim 19	$\mathcal{N}=8, f=64$ $G=\mathrm{SO}(8,4)$, dim 66 $H=\mathrm{SO}(8)\times\mathrm{SO}(4)$, dim 34	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69
$\mathbb{O}$	$\mathcal{N}=9, f=32$ $G=F_{4(-20)}$, dim 52 $H=\mathrm{SO}(9)$, dim 36	$\mathcal{N}=10, f=64$ $G=E_{6(-14)}$, dim 78 $H=\mathrm{SO}(10)\times\mathrm{SO}(2)$, dim 46	$\mathcal{N}=12, f=128$ $G=E_{7(-5)}$, dim 133 $H=\mathrm{SO}(12)\times\mathrm{SO}(3)$, dim 69	$\mathcal{N}=16, f=256$ $G=E_{8(8)}$, dim 248 $H=\mathrm{SO}(16)$, dim 120

- The $\mathcal{N}>8$ supergravities in $D=3$ are unique, all fields belonging to the gravity multiplet, while those with $\mathcal{N}\leq 8$ may be coupled to k additional matter multiplets [Marcus and Schwarz:1983; deWit, Tollsten and Nicolai:1992]. The real miracle is that tensoring left and right YM multiplets yields the field content of $\mathcal{N}=2,3,4,5,6,8$ supergravity with $k=1,1,2,1,2,4$: just the right matter content to produce the U-duality groups appearing in the magic square.

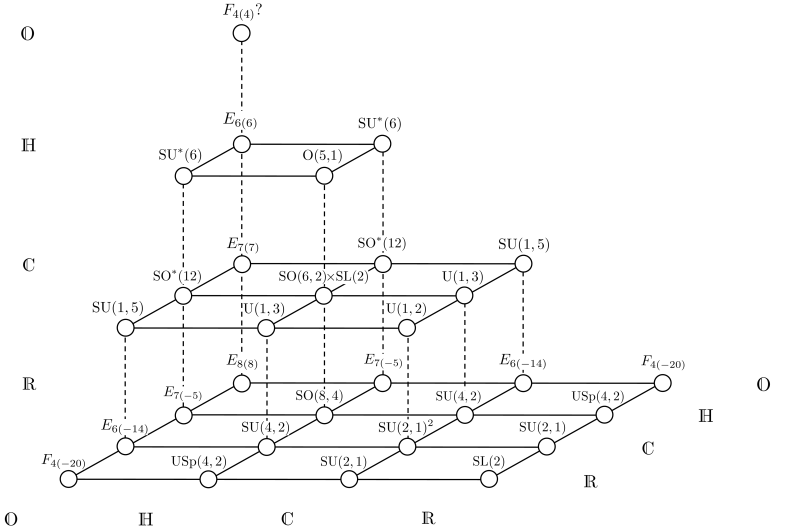
3.4 Magic Pyramid: G symmetries



4.7 Summary Gravity: Conformal Magic Pyramid

- We also construct a *conformal* magic pyramid by tensoring conformal supermultiplets in D = 3, 4, 6.
- The missing entry in D = 10 is suggestive of an exotic theory with G/H duality structure $F_{4(4)}/Sp(3) \times Sp(1)$.

3.5 Conformal Magic Pyramid: G symmetries

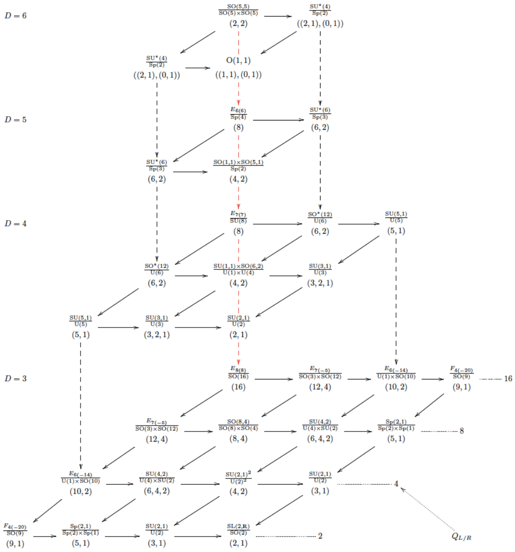


- TWIN SUPERGRAVITIES

4.1. Twins?

- We consider so-called 'twin supergravities' - pairs of supergravities with $\mathcal{N}_+$ and $\mathcal{N}_-$ supersymmetries, $\mathcal{N}_+ > \mathcal{N}_-$, with identical bosonic sectors - in the context of tensoring super Yang-Mills multiplets.
  [Gunaydin, Sierra and Townsend Dolivet, Julia and Kounnas Bianchi and Ferrara]
- Classified in [Roest and Samtleben Duff and Ferrara]

4.2 Pyramid of twins



4.3 Example: $\mathcal{N}_+ = 6$ and $\mathcal{N}_- = 2$ twin supergravities

- The $D = 4, \mathcal{N} = 6$ supergravity theory is unique and determined by supersymmetry. The multiplet consists of

$$\mathbf{G}_6 = \{g_{\mu\nu}, 16A_\mu, 30\phi; 6\Psi_\mu, 26\chi\}$$

- Its twin theory is the magic $\mathcal{N} = 2$ supergravity coupled to 15 vector multiplets based on the Jordan algebra of 3×3 Hermitian quaternionic matrices $\mathfrak{J}_3(\mathbb{H})$. The multiplet consists of

$$\mathbf{G}_2 \oplus 15\mathbf{V}_2 = \{g_{\mu\nu}, 2\Psi_\mu, A_\mu\} \oplus 15\{A_\mu, 2\chi, 2\phi\}$$

- In both cases the 30 scalars parametrise the coset manifold

$$\frac{\mathrm{SO}^\star(12)}{\mathrm{U}(6)}$$

and the 16 Maxwell field strengths and their duals transform as the **32** of $\mathrm{SO}^\star(12)$ where $\mathrm{SO}^\star(2n) = O(n, H)$

4.4 Yang-Mills origin twin supergravities

- Key idea: reduce the degree of supersymmetry by using 'fundamental' matter multiplets

$$\chi^{\mathsf{adj}} \longrightarrow \chi^{\mathsf{fund}}$$

- Twin supergravities are systematically related through this process

- Generates new from old (supergravities that previously did not have a Yang-Mills origin)

$\mathcal{N} = 6$

- The $\mathcal{N} = 6$ multiplet is the product of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ vector multiplets,

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^{\rho}] \otimes \tilde{\mathbf{V}}_4 = \mathbf{G}_6,$$

- $\mathbf{G}_{\mathcal{N}}, \mathbf{V}_{\mathcal{N}}$ and $\mathbf{C}_{\mathcal{N}}$ denote the $\mathcal{N}$-extended gravity, vector, and spinor multiplets

- The hypermultiplet $\mathbf{C}_2^{\rho}$ carries a non-adjoint representation ρ of G

- $\mathbf{C}_2^{\rho}$ does not 'talk' to the right adjoint valued multiplet $\tilde{\mathbf{V}}_4$

5.8. Yang-Mills origin of $(6, 2)$ twin supergravities

To generate the twin $\mathcal{N}_- = 2$ theory:

- Replace the right $\mathcal{N} = 4$ Yang-Mills by an $\mathcal{N} = 0$ multiplet

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^{\rho}] \otimes \tilde{\mathbf{V}}_4 \quad \longrightarrow \quad [\mathbf{V}_2 \oplus \mathbf{C}_2^{\rho}] \otimes \left[\tilde{A} \oplus \tilde{\chi}^{\rho\alpha} \oplus \tilde{\phi}^{[\alpha\beta]}\right]$$

- Here $\tilde{\chi}^{\alpha}$ in the adjoint of $\tilde{G}$ and **4** of $SU(4)$ is replaced by $\tilde{\chi}^{\rho\alpha}$ in a non-adjoint representation of $\tilde{G}$

- $\tilde{\chi}^{\rho\alpha}$ does not 'talk' to the right adjoint valued multiplet $\mathbf{V}_2$, but does with $\mathbf{C}_2^{\rho}$

- Gives a "sum of squares"

$$[\mathbf{V}_2 \oplus \mathbf{C}_2^{\rho}] \otimes \left[\tilde{A} \oplus \tilde{\chi}^{\rho\alpha} \oplus \tilde{\phi}^{[\alpha\beta]}\right] = \mathbf{V}_2 \otimes \left[\tilde{A} \oplus \tilde{\phi}^{[\alpha\beta]}\right] \oplus [\mathbf{C}_2^{\rho} \otimes \tilde{\chi}^{\rho\alpha}] = \mathbf{G}_2 \oplus 15\mathbf{V}_2$$

5.8. Sum of squares

Introduce bi-fundamental scalar $\Phi^{a\tilde{a}}$ to obtain sum of squares off-shell:

- Block-diagonal spectator field Φ with bi-adjoint and bi-fundamental sectors

$$\Phi = \begin{pmatrix} \Phi^{i\tilde{i}} & 0 \\ 0 & \Phi^{a\tilde{a}} \end{pmatrix}.$$

- The off-shell dictionary correctly captures the sum-of-squares rule:

$$[\mathbf{V}_{\mathcal{N}_L} \oplus \mathbf{C}^{\rho}_{\mathcal{N}_L}] \circ \Phi \circ [\tilde{\mathbf{V}}_{\mathcal{N}_R} \oplus \tilde{\mathbf{C}}^{\tilde{\rho}}_{\mathcal{N}_R}] = \mathbf{V}^{i}_{\mathcal{N}_L} \circ \Phi_{i\tilde{i}} \circ \tilde{\mathbf{V}}^{\tilde{i}}_{\mathcal{N}_R} \oplus \mathbf{C}^{a}_{\mathcal{N}_L} \circ \Phi_{a\tilde{a}} \circ \tilde{\mathbf{C}}^{\tilde{a}}_{\mathcal{N}_R}$$

- Crucially, the gravitational symmetries are correctly generated by those of the Yang-Mills-matter factors via $\star$ and Φ.

5.10 Universal rule

- This construction generalises: all pairs of twin supergravity theories in the pyramid are related in this way

Super Yang-Mills factors

$$
\begin{array}{ccc}
\left[\mathbf{V}_{\mathcal{N}_L} \oplus \mathbf{C}_{\mathcal{N}_L}\right] \otimes \tilde{\mathbf{V}}_{\mathcal{N}_R} & \longrightarrow & \left[\mathbf{V}_{\mathcal{N}_L} \oplus \mathbf{C}_{\mathcal{N}_L}\right] \otimes \left[\tilde{A} \oplus \tilde{\chi} \oplus \tilde{\phi}\right] \\
\downarrow & & \downarrow \\
\mathbf{G}_{\mathcal{N}_+} + \mathit{matter} & \underset{\text{twin}}{\longrightarrow} & \mathbf{G}_{\mathcal{N}_-} + \mathit{matter}
\end{array}
$$

Twin supergravities

5.12. Remarks

- Twin relations relations gives new from old
- Raises the question: what class of gravitational theories are double-copy constructible?
- What about supergravity coupled to the MSSM: is it a double-copy?

- “Supergravity is very compelling but it has yet to prove its worth by experiment”

  MJD ”What’s up with gravity?” New Scientist 1977

- “...a remark still unfortunately true at Supergravity@25. Let us hope that by the Supergravity@50 conference, or before, we can say something different.”

  MJD ”M-theory on manifolds of G_2 holonomy” Supergravity@25 2001